Convex Drawings of Intersecting Families of Simple Closed Curves

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Abstract

A FISC or *family of intersecting simple closed curves* is a collection of simple closed curves in the plane with the properties that there is some open region common to the interiors of all the curves, and that every two curves intersect in finitely many points.

Let $\mathcal{F}$ be a FISC. Intersections of the curves represent the vertices of a plane graph, $G(\mathcal{F})$, whose edges are the curve arcs between vertices. The directed dual of $G(\mathcal{F})$, denoted $\vec{D}(\mathcal{F})$, is the dual graph of $G(\mathcal{F})$, but with edges oriented to indicate inclusion in fewer interiors of the curves. A convex drawing of $G(\mathcal{F})$ is one in which every curve is convex. The graph $G(\mathcal{F})$ has a convex drawing if there is some FISC $\mathcal{C}$ whose curves are all convex and where $\mathcal{F}$ can be transformed into $\mathcal{C}$ by a continuous transformation of the plane. We prove that $G(\mathcal{F})$ has a convex drawing if and only if $\vec{D}(\mathcal{F})$ contains only one source and only one sink. This means that we can determine in $O(v)$ time, where v is the number of vertices in $G(\mathcal{F})$, whether $\mathcal{F}$ admits a convex drawing in the plane.

1 Results

When does a family of simple Jordan curves have the property that, by a continuous transformation of the plane, the curves become convex? The curves of Figure 1 admit no such transformation, but clearly some families can be drawn with convex curves.

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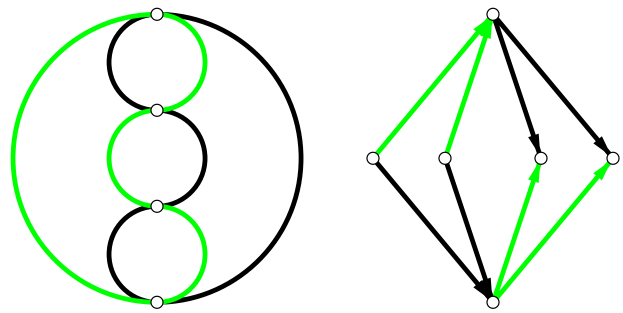

Figure 1: (a) An example of two intersecting curves that do not have a convex drawing. (b) The corresponding dual graph, which has two sources and two sinks.

We say that a collection of curves is *finitely intersecting* if every pair of curves intersect in finitely many points; at each intersection point, two curves may cross or not cross. Several curves may meet at the same point. A *family of intersecting simple closed curves* (or a FISC) is a collection of finitely-intersecting simple closed curves in the plane, with the property that there is some open region common to the interiors of all the curves. A FISC is *simple* if at every point of intersection exactly two curves meet, and they cross each other. A FISC is *convex* if each curve is convex.

We say that two FISCs are *isomorphic* if, by continuous transformation of the plane, one of them can be changed into the other or its mirror image [6]. If the FISC contains n curves then it is an n-FISC. Since the main result of this paper can be deduced easily for all FISCs once it is proved for those FISCs in which every curve intersects some other curve, we shall assume throughout the sequel that this condition is satisfied.

An n-FISC $\mathcal{F} = C_1, C_2, \ldots, C_n$ can be regarded as a plane graph $G = G(\mathcal{F})$, whose vertices are the intersections of the curves. The labelled edges of G are of the form $C(v, w)$, where there is a arc on curve C with intersection points v and w, and no intersection points between them on this arc. The label of the edge is i if $C = C_i$. Each face, including the outer infinite face, is called a *region* when referring to G. The *weight* of a region is the number of curves that contain it. A k-*region* is one of weight k. The unbounded 0-region is called the *outer region*. Any n-region is called an *inner region*.

The dual graph $D(\mathcal{F})$, of a FISC $\mathcal{F}$ is constructed by placing a vertex within each region of $\mathcal{F}$. For each edge of $\mathcal{F}$ that borders two regions, an edge of the dual graph is drawn which connects the vertices within these regions. Note that each of the *dual vertices* corresponds to a face in $\mathcal{F}$, and each of the *FISC vertices* corresponds to a face in $D(\mathcal{F})$. We can associate with each of the dual vertices the subset of curve interiors of the corresponding region of $\mathcal{F}$. We define the *directed* dual graph, $\vec{D}(\mathcal{F})$, by imposing a direction on each edge that indicates whether one dual vertex contains the other's subset [7]. Clearly, a

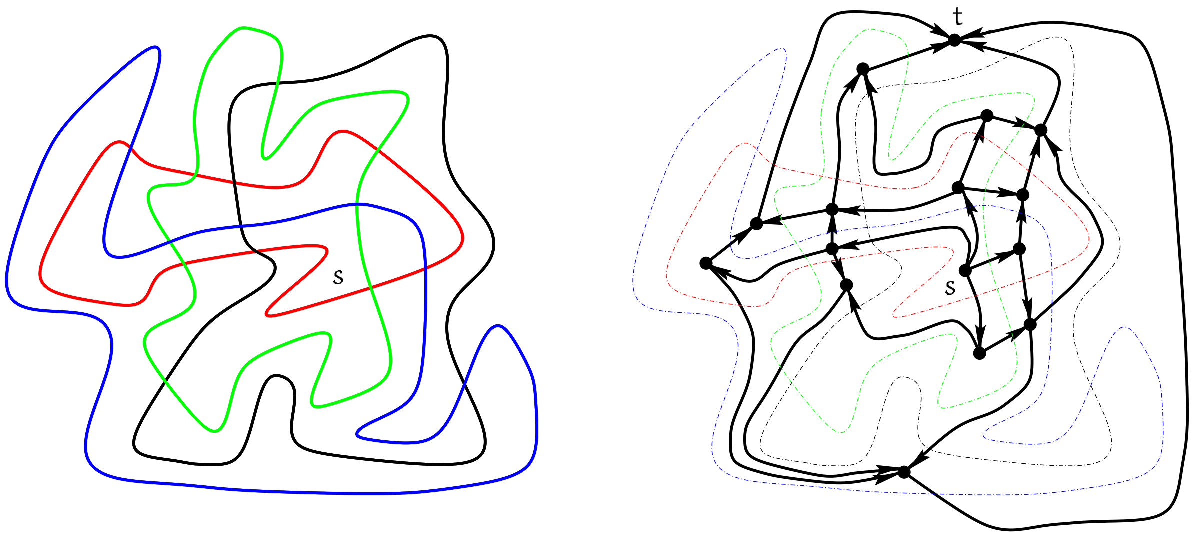

Figure 2: (a) Four curves in the plane with common region s. (b) The corresponding directed dual graph with source s and sink t.

directed dual graph cannot contain a directed circuit.

A FISC is *monotone* if $\vec{\mathrm{D}}(\mathcal{F})$ has a unique source (no incoming edges) and a unique sink (no out-going edges). The FISC shown in Figure 2(a) is monotone, as can be inferred by considering the directed dual graph shown in Figure 2(b). We will use the FISC of Figure 2 as a running example throughout this paper. It is a simple FISC.

Our main result is the following theorem. Since a source or sink in a planar graph can be found in time proportional to the number of vertices in the graph, the theorem implies an efficient algorithm for determining whether a FISC can be drawn with all curves convex.

Theorem 1.1 *A FISC is isomorphic to a convex FISC if and only if it is monotone.*

Our preliminary motivation was the drawing of Venn diagrams, a particular type of FISC, and our references reflect that motivation.

References

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