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THE SEARCH FOR SYMMETRIC VENN DIAGRAMS

by Branko Grünbaum
University of Washington, Box 354350, Seattle, WA 98195-4350
e-mail: grunbaum@math.washington.edu

Given a family $\mathbb{C} = \{C_1, C_2, ..., C_n\}$ of n simple (Jordan) curves which intersect pairwise in finitely many points, we say that it is an **independent family** if each of the 2^n sets

$$X_1 \cap X_2 \cap ... \cap X_n \qquad (*)$$

is not empty, where X_j denotes one of the two connected components of the complement of C_j (that is, each X_j is either the interior or the exterior of C_j). If, moreover, each of the sets in (*) is connected, we say that the independent family $\mathbb{C}$ is a **Venn diagram**. An independent family or Venn diagram is called **simple** if no three curves have a common point.

Introduced by the logician John Venn in 1880, Venn diagrams with $n \leq 3$ curves have been the staple of many finite mathematics and other courses. Over the last decade the interest in Venn diagrams for larger values of n has intensified (see, for example, Ruskey [9] and the many references given there). In particular, considerable attention has been devoted to symmetric Venn diagrams. A Venn diagram with n curves is said to be **symmetric** if rotations through $360/n$ degrees map the family of curves onto itself, so that the diagram is not changed by the rotation. This concept was introduced by Henderson [8], who provided two examples of non-simple symmetric Venn diagrams; one consists of pentagons, the other of quadrangles, but both can be modified to consist of triangles. A simple symmetric Venn diagram consisting of five ellipses was given in [6]. As noted by Henderson, symmetric Venn diagrams with n curves cannot exist for values of n that are composite. Hence $n = 7$ is the next value for which a symmetric Venn diagram might exist. Henderson stated in [8] that such a diagram has been found; however, at later inquiry he could not locate it, and it was conjectured in [6] that such diagrams do not exist.

In fact, this conjecture was disproved by the examples of simple Venn diagrams of seven curves given in [7], leading to the diametrically opposite conjecture that symmetric Venn diagrams exist for every prime n. By a curious coincidence, several additional examples of symmetric Venn diagrams with seven curves were produced shortly thereafter by other people (see, for example [2]). Details of the history of these discoveries can be found in the paper by Edwards [3] and the report by Ruskey [9]. The former presents a list of six different self-complementary simple symmetric Venn diagrams of seven sets, while that latter expands this and gives a list of 23 simple monotone symmetric Venn diagrams, as well as various other enumerations. (**Self-complementary** means that the Venn diagram is isomorphic to the one in which "inclusion" and "exclusion" are interchanged; by the result of [1], **monotone** is equivalent to saying that the Venn diagram is isomorphic to one with convex curves.) These results were obtained by exhaustive computer searches.

The next step towards clarifying the conjecture would be to investigate whether there exist any symmetric Venn diagrams of 11 curves. Despite claims (like the one in [2]; all such claims were later withdrawn) by several people of having found diagrams of this kind, none are known at this time. The sheer size of the problem for 11 curves puts it beyond the reach of the available approaches through exhaustive computer searches. Hence it may be worthwhile to investigate a more general problem which may be solvable for one or two values beyond $n = 7$, in hope that new ideas will appear that may be applicable to the elusive case of $n = 11$.

Henderson's argument that symmetric Venn diagrams cannot exist if the number of curves is a composite integer is based on the following fact from number theory: if $n = r s$, where r and s are integers greater than 1 and r is a prime number, then the binomial coefficient $\binom{n}{r}$ is not divisible by n. On the other hand, this obstacle disappears if instead of Venn diagrams one is considering independent families of n sets — however, such families seems to be of little interest since it is very easy to generate them for every n; examples for $n = 4$ and 6 appear in [6]. But while it may seem, on number-theoretical or combinatorial grounds, that such families must have a very large number of regions, a closer investigation shows that as far as combinatorics and number theory are concerned, the number of regions could be not too much larger than in a Venn diagram. This happens because many of the types of regions occur in n-tuples, and only few require duplication in order to accommodate rotational invariance.

Let us denote by (a,b,...,f) a selection of the elements a, b, ... , f, from the family of labels of the members of the independent family of curves. All selections that can be transformed into each other by cyclic permutations of the labels are said to constitute a **type** of selections. Clearly, in a symmetric independent family of n curves, each type (except the selections of none, or of all labels) must be represented by n or a multiple of n regions. A discussion of the case $n = 6$ may illustrate this contention. The 12 relevant selections here are (a), (a,a+1), (a,a+2), (a,a+3), (a,a+1,a+2), (a,a+1,a+3), (a,a+1,a+4), (a,a+2,a+4), (a,a+1,a+2,a+3), (a,a+1,a+2,a+4), (a,a+1, a+3, a+4), (a,a+1,a+2,a+3,a+4). Hence there must be at least $12 \cdot 6 + 2 = 74$ regions in any symmetric independent family of six curves, instead of the 64 regions in a Venn diagram of 6 curves.

The above example can be generalized to obtain a lower bound on the number of regions that must be present in any symmetric independent family of n curves. The resulting lower bound is $M(n) = 2 + n \cdot (C_n - 2)$, where C_n is the number of distinct 2-colored necklaces of n beads, provided rotationally equivalent necklaces are not distinguished. The numbers C_n have been studied by several authors (see [4] Table I, [5] page 139, or [10] sequence M0564, where additional references can be found). From explicit formulas for the numbers C_n it it may be shown that the rate of growth of $M(n)$ is about 2^n for all n, and that if n is prime then $M(n) = 2^n$.

Thus one may reasonably pose the following question:

Is there for every n a symmetric independent family of n curves with only $M(n)$ regions ?

This clearly generalizes the question about the existence of symmetric Venn diagrams with prime numbers of curves. The advantage of the new question is that it can be answered affirmatively for $n = 4$ and $n = 6$ (see Figures 1 and 2), and the first open case, $n = 8$, with $C_8 = 36$ and $M(8) = 274$, would seem not to require prohibitively large computational effort. We venture the following conjecture:

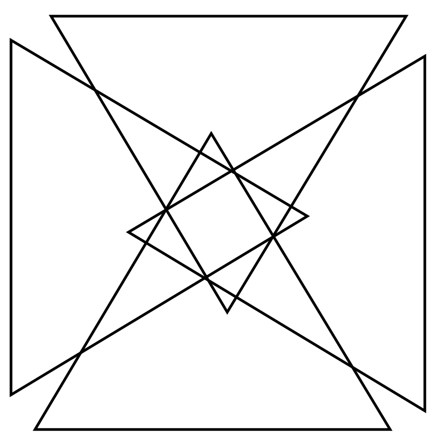

Figure 1. A symmetric independent family of four equilateral triangles, with $M(4) = 18$ regions.

Conjecture 1. For every integer n there exists a symmetric independent family of n curves with only M(n) regions.

A curious property of the known examples of minimal symmetric independent families for composite n is that none is simple. While for $n = 4$ it can be shown that no such family can be simple, it is not clear whether the same is true for $n = 6$ or higher values of n.

Conjecture 2. If n is not a prime, every symmetric independent family with M(n) regions is non-simple.

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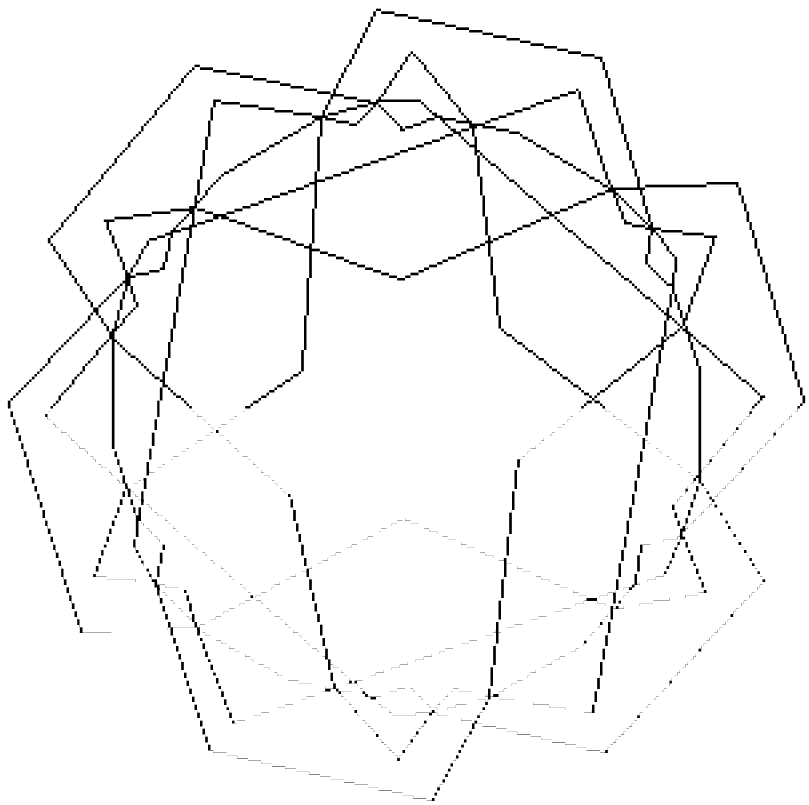

Figure 2. A symmetric independent family of six polygons with M(6) = 74 regions. At each intersection point any two polygons cross each other. The polygons could have been selected to be convex, but then many of the regions would have been very small. The existence of a convex representation is a consequence of a general result established in [1].