SELFINTERSECTIONS OF POLYGONS

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A polygon $P = [V_1, V_2, \dots, V_n]$ is a set of vertices $V_1, \dots, V_n$, together with the n edges (closed segments) $E_1 = [V_1,V_2]$, $E_2 = [V_2,V_3]$, … , $E_n = [V_n,V_1]$. Such a polygon will often be called an n-gon, and we shall assume that $n \geq 3$. Polygons need not be convex, but we restrict attention to polygons with the property that no point of the plane belongs to three or more sides; the main question discussed here concerns the number $x(P)$ of selfintersection points that an n-gon P may have. We are interested in $x(n)$, the maximal number of selfintersections possible in an n-gon, and in $X(n)$, the set of possible numbers of selfintersections of n-gons.

The investigation of $x(P)$, $x(n)$ and $X(n)$ appears worthwhile because, on the one hand, it shows how even simple and natural questions can have complicated answers; on the other hand, the history of the result is quite intricate and can serve as a warning about the frequency with which writers practice sloppy thinking — and so sometimes "establish" fallacies as facts. In a sequel to this note we shall consider the analogous questions for n-arcs, which are objects similar to n-gons but in which instead of straight edges we may use arcs of curves satisfying certain natural conditions.

It is not hard to verify that $X(3) = \{0\}$, $X(4) = \{0, 1\}$, $X(5) = \{0, 1, 2, 3, 5\}$ and $X(6) = \{0, 1, 2, 3, 4, 5, 6, 7\}$, hence $x(3) = 0$, $x(4) = 1$, $x(5) = 5$, $x(6) = 7$. These assertions are illustrated in Figure 1. The complete characterization of $x(n)$ and $X(n)$ is given by the following result.

Theorem 1. If n is odd then $x(n) = \frac{1}{2}n(n-3)$ and $X(n)$ is the set of all integers from 0 to $x(n)$ except $x(n) - 1$; if n is even then $x(n) = \frac{1}{2}n(n-4) + 1$ and $X(n)$ is the set of all integers between 0 and $x(n)$.

Before proving this theorem, here is a survey of its history. The first author (as far as I know) to consider the problem in print was

Baltzer [1] in 1885; he found the correct value of $x(n)$ for odd n, but he mistakenly thought that $x(n) = \binom{n-2}{2} = \frac{1}{2}(n^2 - 5n + 6)$ for even n, and gave a description of $X(n)$ that was incorrect for all $n \geq 6$. Baltzer's error was caused by his belief that in order to obtain an n-gon P with $x(P) = x(n)$, when drawing P each new edge must intersect all previously drawn ones. This mistake was noted by Brunel [4] in 1894, who stated (without proof) the correct value of $x(n)$ and described the construction of polygons yielding all values in $X(n)$ for $n \equiv 2 \pmod 4$, saying that similar constructions apply in the other cases. Brückner [3, pp. 10 – 12] in 1900 gives constructions for all values in $X(n)$ and states the correct values of $x(n)$ for all n; however, he does not seem to feel there is need to prove that the given value of $x(n)$ for even n is indeed maximal. This incompleteness in Brückner's treatment was observed by Steinitz [9, Section 4] in 1916. In 1923 Steinitz [10] finally gave the complete solution, including both the characterization of $X(n)$ and the constructions. Later chapters in

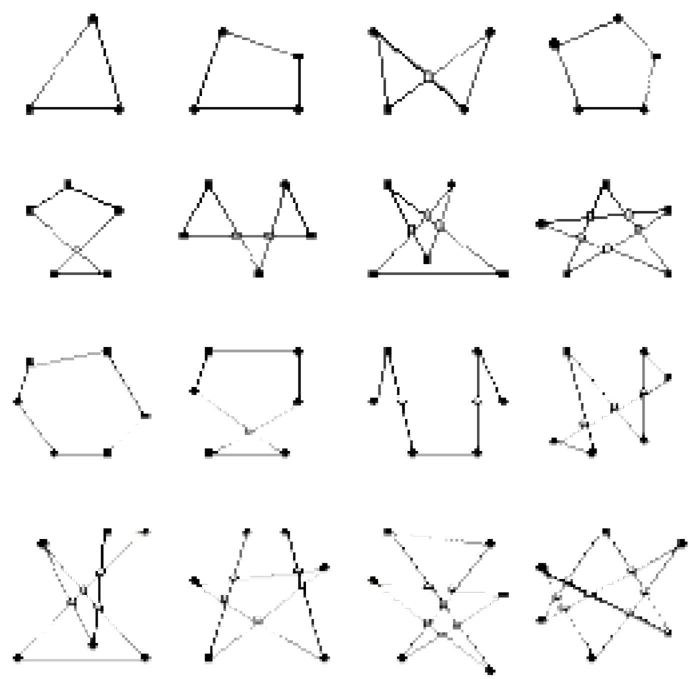

Figure 1. Examples of polygons with all possible values of $x(P)$, for n–gons with $n \leq 6$. The crossing points are indicated by open circles.

the history of this topic include: A brief account of the problem in Lietzmann [7, Chapter 5], which ignores Steinitz [10] and asserts that the result has not been established for even n. Unaware of Steinitz's result, Bergmann [2] and Furry & Kleitman [6] independently proved Theorem 1. Unaware of all previous work, Šklyarskii, Čencov & Yaglom [8, Problem 21] determined the value of $x(n)$. The proof below follows their arguments, which are much simpler than those of the other authors. The inductive construction of the polygons that establish the existential part of the theorem was inspired by the constructions of Woodall [11] in a somewhat different context.

Proof of Theorem 1. Since each edge of the n-gon P can intersect at most $n - 3$ other edges, we have $x(P) \leq n(n - 3)/2$. Thus $x(n) \leq n(n - 3)/2$ for all n , and since $x(P) = n(n - 3)/2$ if $n = 2k + 1$ is odd and if P is a suitable star–polygon (specifically, if P is the regular star-polygon usually denoted by $\{n/k\}$, see Figure 2), the value of $x(n)$ for odd n is as claimed.

Let now n be even; we shall prove by induction from n to $n + 2$ that $x(n) = 1 + n(n - 4)/2$; the assertion is obvious for $n = 4$. We shall assume that an $(n + 2)$–gon P is given which has more than $x(n + 2) = 1 + (n + 2)(n - 2)/2 = \frac{1}{2} n^2 - 1$ selfintersections, and we shall derive a contradiction to the inductive assumption. We consider each of the $n + 2$ "horns" of P, where a "horn" is the union of two adjacent edges of the polygon. At least one of the horns of P must contain at least $2n - 3$ of the selfintersections; indeed, otherwise P

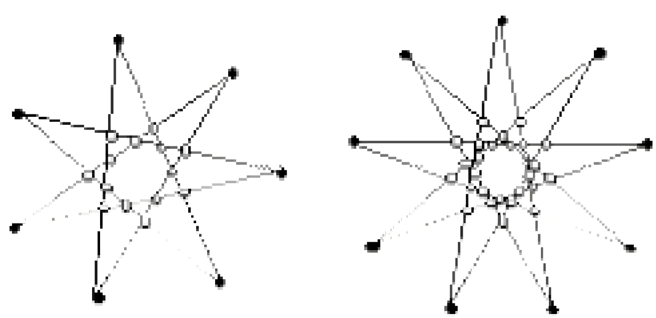

Figure 2. Regular polygons {7/3} and {9/4}, which are examples of polygons with odd numbers of vertices and maximal number of crossing points.

would have at most $\frac{1}{4}(n + 2)(2n - 4) = \frac{1}{2} n^2 - 2 < \frac{1}{2} n^2 - 1$ selfintersections. Let $H = E_1 \cup E_2$ be a horn containing at least $2n - 3$ selfintersections; then at least one of the edges forming the horn must contain at least $n - 1$ selfintersections of P. Let E_1 be such an edge. It must contain at most $(n + 2) - 3 = n - 1$ selfintersections of P; hence it contains precisely $n - 1$. This means that it intersects all edges that are not adjacent to it, and in that case, since $n + 2$ is even, it follows that the edges adjacent to E_1 are on opposite sides of the line determined by E_1. If E_1 contains $n - 1$ selfintersections, it follows that E_2 can contain at most $n - 2$ selfintersections since E_2 cannot meet the other edge adjacent to E_1. Thus the horn H contains precisely $2n - 3$ selfintersections of P, and without loss of generality we may assume that H and the neighboring edges (labelled E_0 and E_3) of P have mutual relationship as indicated in either Figure 3(a) or in Figure 3(c). The two cases differ in the relation of the line L, determined by the edge E_0 (which precedes E_1), to the edge E_2: in the first case L meets E_2, in the second it misses E_2. (Since the number of selfintersections of an ordinary polygon is not changed by sufficiently small displacements of the vertices of P, we may assume that no line determined by two vertices of P passes through a third vertex of P.)

In the first case (see Figure 3(a)) we delete from P the vertices V_1, V_2, V_3 and the edges incident with them, and from the remaining parts of P we form an n-gon P* by adjoining the vertex W and the edges $[V_0, W]$ and $[W, V_1]$ (see Figure 3(b)). To estimate the loss in the number of selfintersection points arising from the replacement of P by P* we note that besides the loss of the $2n - 3$ intersections with H we only have to take into account those intersections that occur on the part of E_3 that is between W and V_3. But each edge of P that intersects $[W, V_3]$ must also intersect both E_1 and E_2, and therefore it must intersect L in $[V_0, W]$, hence it does not contribute to the net loss of selfintersections. Thus the n–gon P* satisfies $x(P^*) = x(P) - (2n - 3) > \frac{1}{2} n^2 - 1 - (2n - 3) > \frac{1}{2} n(n - 4) + 1 = x(n)$, contradicting the definition of $x(n)$; hence this case cannot arise.

In the second case (see Figure 3(c)), the vertices V_0, V_1, V_3, V_2 (taken in this order) determine a convex quadrangle; hence each edge that intersects E_0 and E_2 must also intersect the diagonal $[V_0, V_3]$ of the quadrangle. Therefore, replacing $E_0, V_1, E_1, V_2, E_2,$ by the edge $[V_0, V_3]$ (see Figure 3(d)) we obtain an n-gon P^* for which $x(P^*) \geq x(P) - (2n - 3) - 1 > \frac{1}{2} n^2 - 1 - 2n + 2 = \frac{1}{2} n(n - 4) + 1 = c_n$; the -1 in the second step is to account for the possible loss of an intersection point of E_0 and E_3. The contradiction reached shows that $x(n) \leq \frac{1}{2} n(n-4) + 1$ for all even n. To establish equality we shall now show that for each even $n \geq 4$ there exist n–gons with $\frac{1}{2} n(n-4) + 1$ selfintersections.

Actually, we shall show, by induction from n to n + 2, that for even n and for each integer k with $0 \leq k \leq x(n)$, there exists an

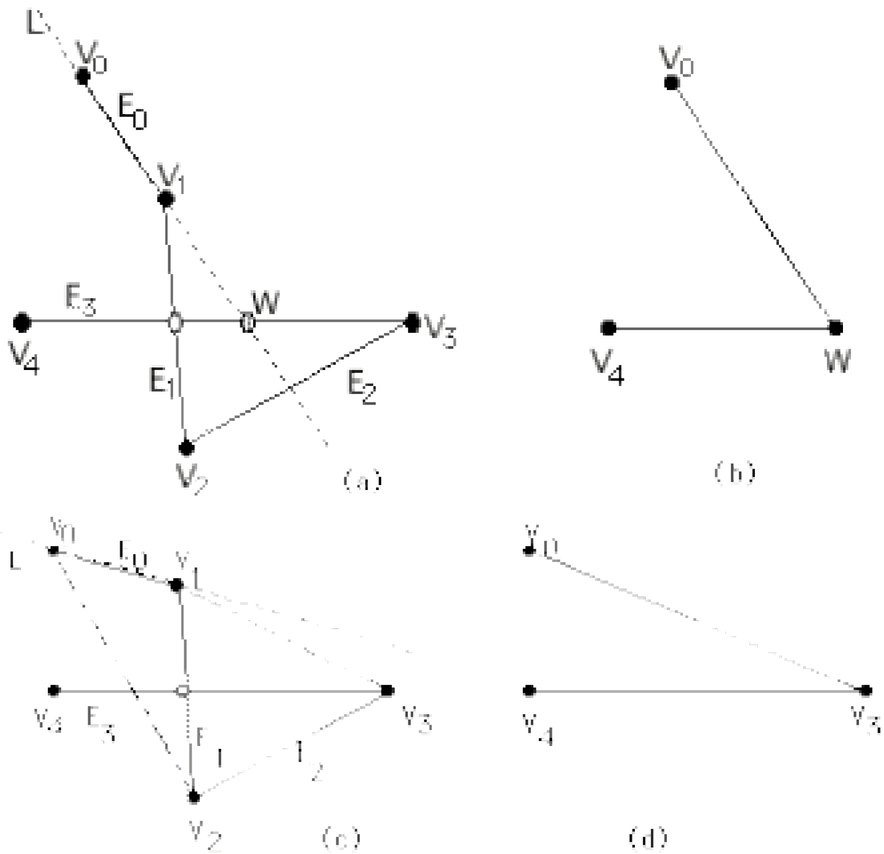

Figure 3. Illustration of the argument used in the proof of the upper bound in Theorem 1 for even n; (a) and (b) refer to the first case, (c) and (d) to the second case.

n-gon P with $x(P) = k$. As shown by easy examples (compare Figure 1), this is certainly true for $n = 4$ and $n = 6$. Moreover, we shall show that there exists an n-gon P_n such that $x(P_n) = x(n)$, and that an edge E of P_n has the properties:

(i) the edges of P_n adjacent to E are on opposite sides of E;

(ii) E crosses the maximal possible number $n - 3$ of edges of P_n.

The polygons P_4 and P_6, and an edge of each that can be chosen as E, are shown in Figure 1; the special edge is heavily drawn. Now the inductive step from n to $n + 2$ is very simple (compare Figure 4): The edge $E = [V_1,V_2]$ (in Figure 4(a)) may be replaced, arbitrarily close to E, by a polygonal path formed by new vertices W_1 and W_2 and three edges E_1, E_2, E_3, such that each of the new edges intersects all the edges that were intersected by E, and introduce either four (Figure 4(b)) or three (Figure 4(c)) additional intersections. Applied to P_n, the first operation yields an $(n + 2)$–gon P_{n+2} with $x(P_{n+2}) = x(P_n) - (n - 3) + 3(n - 3) + 4 = 1 + n(n-4)/2 + 2n - 2 = \frac{1}{2} n^2 - 1 = x(n+2)$, in which the edge $E_2 = [W_1,W_2]$ may be chosen as E. This completely establishes the bound $x(n)$. We note that the net gain in the number of selfintersections is $2n - 2$, and observe that by suitably "lowering" W_2 and/or "raising" W_1, we can reduce this added number of selfintersection by an arbitrary even integer. Similarly, the procedure indicated in Figure 4(c) increases the number of selfintersections by $2n - 3$, and its modifications increase it by $2n - 5, ..., 3$, or 1. Hence we can obtain $(n + 2)$–gons having

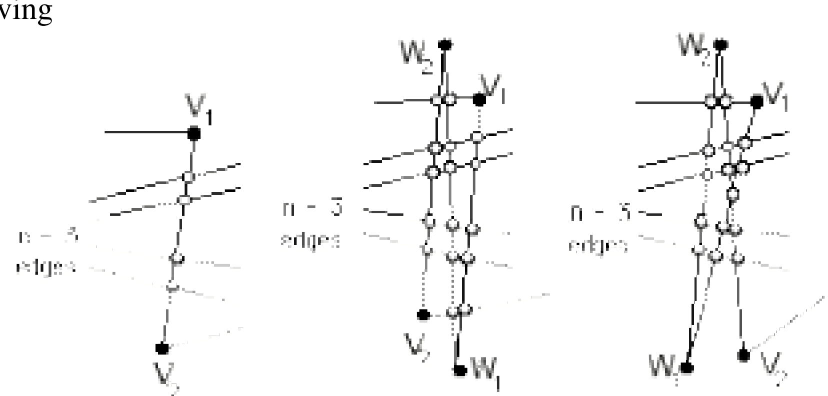

Figure 4. Illustration of the construction used in the existence part of Theorem 1 for even n.

any desired number of selfintersections between x(n) and x(n+2). To obtain (n + 2)–gons with any number of selfintersections that does not exceed x(n) we can take an appropriate n–gon and replace one of its edges by a suitable chain of three edges. Thus all assertions of Theorem 1 concerning even n are proved.

We return now to the case of odd n = 2k+1. As mentioned earlier, the star-polygon {n/k} can be taken as a polygon P_n that attains the bound x(n). In order to obtain from it the (n + 2)–gons required in the theorem we proceed in analogy to the even case, but with a modification caused by the fact that now P_n has no edge E of the type used above. Instead, P_n has an edge E that intersects n – 3 other edges of P_n, for which the edges adjacent to it are on the same side of E (see Figure 2). We shall again replace such an E by a chain of three edges, each of which is intersected by all edges of P that intersect E, and which introduce either 5 or 2 additional selfintersections (see Figure 5). By suitably "lowering" W_2 and/or "raising" W_1 we can obtain (n + 2)–gons with any number of selfintersections between x(n) and x(n) – (n – 3) + 3(n – 3) + 5 = n(n – 3)/2 + 2n – 1 = (n + 2)(n – 1)/2 = x(n+2), except x(n+2) – 1. Moreover, the edge $E_2 = [W_1, W_2]$ can be chosen as E for P_{n+2}. As in the previous case, (n + 2)–gons with any number of selfintersections up to x(n) (except x(n) – 1) can be obtained from n-gons with the same number of selfintersections by adding vertices, while an (n + 2)-gon with x(n) – 1 selfintersections results from an

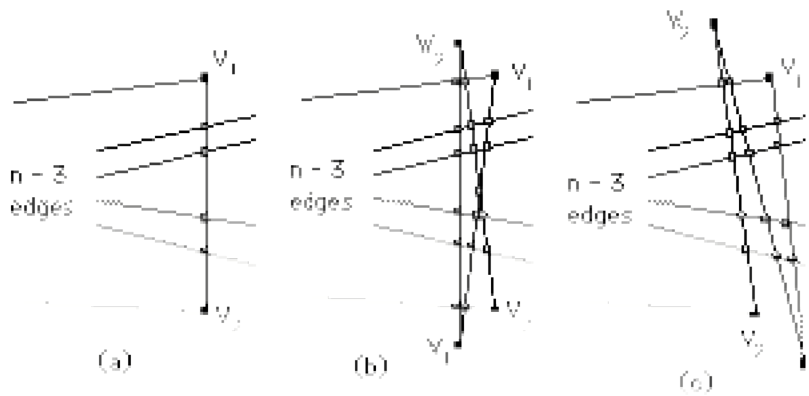

Figure 5. Illustration of the construction used in the existence part of Theorem 1 for odd n.

n–gon with $x(n) - 2$ selfintersection on replacing an edge by the chain of three edges shown in Figure 6. This completes the proof of the theorem for odd n, except for the assertion that for such n no n–gon has precisely $x(n) - 1$ selfintersections.

Most writers (in particular, Baltzer, Brunel, Brückner and Steinitz) consider that last assertion obvious — although I cannot say why (except possibly because one cannot obtain such a polygon by reducing by 1 the number of selfintersections in a star–n–gon P_n with $x(n)$ selfintersections). A proof may be derived from the observation that if P^* is an n–gon (with odd n) such that $x(P^*) = x(n) - 1$ then P^* contains precisely one pair of non-adjacent edges E', E" that do not cross each other. Then, similarly to the arguments in the main part of the proof, we have to consider two cases: either E' and E" are opposite edges of a convex quadrangle, or the line determined by one (say E') crosses the other. In either case, the line L determined by E" does not meet E'. Since n is odd, one of the two polygonal arcs that connect E' to E" in P^* contains an even number of edges, say $[V_0, V_1]$, $[V_1, V_2]$, ... , $[V_{2j-1}, V_{2j}]$, where V_0 is an endpoint of E" and V_{2j} of E'. Since all but the last of these edges intersect E', the vertices $V_1, V_3, ...,$ are on the same side of L as E', and $V_2, V_4, ..., V_{2j}$ on the opposite side — contradicting the choice of V_{2j} as being an endpoint of E'.

This completes the proof of Theorem 1.

A recent result of Cairns & King [5] states that for each odd $n = 2k+1$ there is a unique type of polygons P with combinatorial symmetries that are compatible with the order of intersections on all the edges, and for which $x(P) = x(n)$; all such P are distortions of the regular polygon $\{n/k\}$. As is easily seen, without the combinatorial symmetry assumption there are, for each $n \geq 7$ and each $x \in X(n)$, $x \neq 0$, essentially different types of n-gons P with $x(P) = x$.

Figure 6. Illustration of the construction used in the existence part of Theorem 1 to obtain $(n + 2)$–gons with $x(n) - 1$ selfintersections, for odd n.

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