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# HOW MANY TRIANGLES ?

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Abstract. A result known as Roberts' Theorem states that $n$ lines in general position in the plane determine at least $n–2$ triangular regions. An interesting proof of this statement, due to Belov, is presented.

This is an exposition of an elementary but nontrivial fact of elementary geometry. Throughout, we denote by $F$ a *simple family* of $n \geq 3$ (infinite) straight lines in the plane, that is, a family in which no two lines are parallel and no three meet at a common point. It is easily proved by induction that the lines of $F$ intersect in $\frac{n(n-1)}{2}$ different points ("vertices" of $F$) which in turn determine a total of $n(n-2)$ finite segments of the lines ("edges" of $F$). The lines also determine $\frac{(n-1)(n-2)}{2}$ convex polygonal regions ("faces" of $F$). This is illustrated for $n = 5$ by two examples in Figure 1.

While the total number of faces determined by a simple family $F$ of $n$ lines is independent of the positions of the lines, the kind of

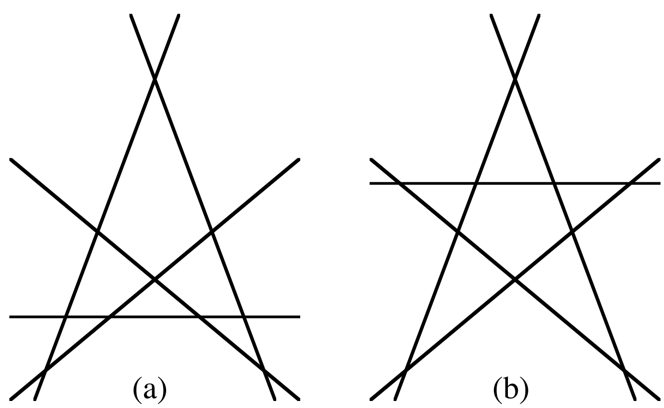

Figure 1.

faces and the number of each kind are not the same in all cases. For example, with $n = 5$, the family in Figure 1(a) has three triangles and three quadrangles as faces, while the family in Figure 1(b) has five triangles and one pentagon.

Before continuing, the reader is encouraged to experiment a little, and try to form an opinion concerning the number of triangles possible for simple families $F$ consisting of $n$ lines.

This question seems to have been first considered by Samuel Roberts [R] more than a century ago. He reached the conclusion:

*Simple families* $F$ *of* $n$ *lines always determine at least* $n-2$ *triangles.*

It is easy to see that simple families with precisely $n-2$ triangles exist for every $n \geq 3$; Figure 2 illustrates one systematic way of constructing such families. However, to establish that for general $n$ no simple family of $n$ lines can determine only $n-3$ or fewer triangles is not straightforward. In particular, as mentioned in [G], Roberts' arguments are not convincing at all. If the reader now makes a few experiments and attempts to prove what is known as "Roberts' theorem", it will become apparent that one of the difficulties arises from the fact that although adding a line to an existing simple family

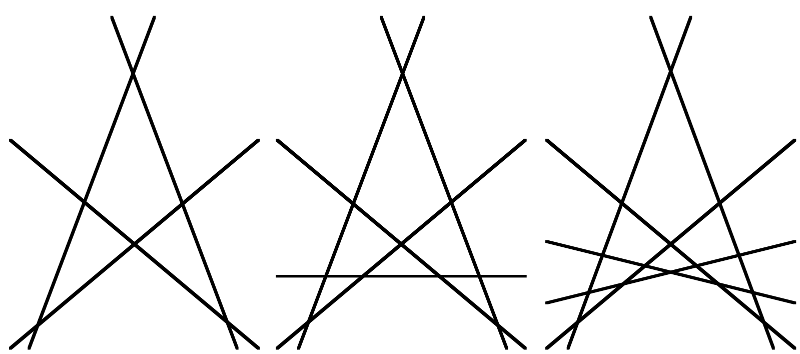

Figure 2.

cannot decrease the number of triangles, it is possible that the number of triangles does not increase. An example to this effect is shown in Figure 3. (It is easy to show that one can always increase the number of triangles by adding a suitable line, but this is not relevant to the question considered here.)

The first published proof of "Roberts' theorem" was contained in a paper by Shannon [S] in 1979. However, Shannon's proof is not elementary, since in it Roberts' theorem is a corollary of results dealing with families of hyperplanes in Euclidean spaces of arbitrarily high dimensions, without the possibility to restrict the arguments to the planar case. The following proof is taken from a note by Belov [B]. As the reader will see, the idea of this proof is very nice, and can be used in various generalizations and in other contexts. Unfortunately, the formulation of Belov's short note is so confused that at first I could not decide whether it was brilliant or wrong; the translator of the English version did an excellent job, but could not eliminate the confusion in the original. I am greatly indebted to my colleague Boris Solomyak for his patience in explaining to me the idea behind the proof.

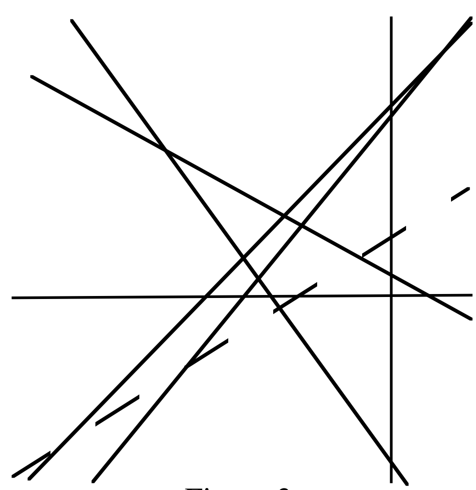

Figure 3.

Let $F$ be a simple family of $n$ lines in the Euclidean plane. Assume that the lines are given by equations $L_i : a_i x + b_i y = c_i$, for $i = 1,2,...,n$; we consider a variable family $F(t)$ of lines $M_i(t) : a_i x + b_i y = c_i + d_i t$, where $d_i$'s are real numbers to be determined, and $t$ is a real parameter. Clearly, $L_i = M_i(0)$, and $M_i(t)$ is parallel to $L_i$ for all $i$ and $t$. We consider now three lines $L_i, L_j, L_k$ that determine a triangular face $T$ of $F$, and the corresponding triangles $T(t)$ determined by the lines $M_i(t), M_j(t), M_k(t)$. By using the determinantal formula for the area of a triangle, after a somewhat lengthy but straightforward calculation we find that the area of the triangle $T(t)$ is, up to a constant factor, given by

$$\left( \begin{vmatrix} a_i & b_i & d_i \\ a_j & b_j & d_j \\ a_k & b_k & d_k \end{vmatrix} t + \begin{vmatrix} a_i & b_i & c_i \\ a_j & b_j & c_j \\ a_k & b_k & c_k \end{vmatrix} \right)^2 .$$

Thus the area is a quadratic function of $t$, and it will be independent of $t$ (that is, be constant) if and only if the coefficient of $t$ in the above expression vanishes, that is, if

$$\begin{vmatrix} a_i & b_i & d_i \\ a_j & b_j & d_j \\ a_k & b_k & d_k \end{vmatrix} = 0.$$

This means that the area of $T(t)$ is constant if and only if $d_i, d_j$ and $d_k$ satisfy a homogeneous linear equation with coefficients determined by the coefficients of the equations of $L_i, L_j, L_k$. Therefore, if $d_1, \ldots, d_n$ are determined so as to satisfy the linear equations appropriate to all the triangles of $F$, then all the corresponding triangles generated by the lines $M_i(t)$ will have areas that are independent of $t$. Now assume that $F$ has fewer than $n-2$ triangles; then the $n$ variables $d_1, \ldots, d_n$ are required to satisfy a system of at most $n-3$ equations. Enlarging the system by the addition of the equations $d_1 = 0$ and $d_2 = 0$, we obtain a system of $n-1$ or fewer homogeneous linear equations in $n$ variables. By the basic theorem of linear algebra, it follows that the system has nontrivial solutions; we choose one such solution $d_1, \ldots, d_n$ and use it in the remaining part of the argument. Note that if $t$ is sufficiently close to $0$ then $F(t)$ is isomorphic to $F$; the combinatorial type of $F(t)$ will change, as $t$ varies, only at a values $t^*$ of $t$ for which $T(t^*)$ has a vertex of multiplicity $m$ greater than or equal to $3$. Moreover, for all $t$ in a sufficiently small neighborhood of such a value $t^*$, the faces of $F(t)$ resulting from the separation of the $m$ lines concurrent for $t^*$ will include at least one triangle (in fact, by an inductive assumption, at least $m–2$ triangles), and this triangle will disappear at $t^*$. Observe that such a multiple point must occur for some value of $t$; indeed, the choice $d_1 = d_2 = 0$ means that the intersection point of the lines $M_1(t) = L_1$ and $M_2(t) = L_2$ is invariant, and hence some value of $t$ must move each line $M_i(t)$ with a nonzero $d_i$ to concurrence with $M_1(t)$ and $M_2(t)$; since $d_1, \ldots, d_n$ was nontrivial, there is at least one such $d_i$, and without loss of generality we can assume that the $t$ in question is positive. Now consider $t_0$, the smallest positive $t$ for which $F(t)$ has a multiple point. Then, for all $t$ such that $0 \le t < t_0$, the family $F(t)$ is isomorphic to $F(t)$, hence has at most $n–3$ triangles, and the area of each triangle in $F(t)$ is constant as $t$ varies in that interval; but by the above, for $t$ sufficiently close to $t_0$ there is at least one triangle of variable area which tends to $0$ as $t$ approaches $t_0$. The contradiction was reached by assuming that the number of triangles in $F$ is at most $n–3$; hence it must be at least $n–2$, and Roberts' Theorem is proved.

It is easily seen that Belov's proof works equally well in all dimensions, establishing that simple families of $n$ hyperplanes in the d-dimensional Euclidean space determine at least $n–d$ n-dimensional simplices. In fact, modifications of Belov's proof may be used to establish the conjecture from [G] that any planar family of $n$ lines, not all concurrent, determines at least n–2 triangles. This, and an appropriate generalization to higher dimensions, has been first established in [S]. Belov [B] mentions a number of other applications of the method used in the above proof.

Another remarkable proof of Roberts' Theorem has been found recently by Felsner and Kriegel [F]. It is purely combinatorial, and establishes the extension of the result to all simple families of pseudolines. (A family of curves is a "simple family of pseudolines" if each curve is obtained from a straight line by continuous deformation of a finite part of the line, every two curves have precisely one point in common at which they cross each other, and no three curves have a common point.)

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