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# Some New Transversality Properties

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**Abstract.** The theorems of Ceva and Menelaus are concerned with cyclic products of ratios of lengths of collinear segments of triangles or more general polygons. These segments have one endpoint at a vertex of the polygon and one at the intersection point of a side with a suitable line. To these classical results we have recently added a 'selftransversality theorem' in which the 'suitable line' is determined by two other vertices. Here we present additional 'transversality' properties in which the 'suitable line' is determined either by a vertex and the intersection point of two diagonals, or by the intersection points of two pairs of such diagonals. Unexpectedly it turns out that besides several infinite families of systematic cases there are also a few 'sporadic' cases.

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**Key words:** Ceva, Menelaus, selftransversality, transversal, polygon, cyclic product.

## 1. Introduction

One of the most familiar theorems of affine geometry is Menelaus' theorem that states (see Figure 1) that if a transversal cuts the sides $BC, CA, AB$ (extended if necessary) of a triangle $[A, B, C]$ in the points $D, E, F$ respectively, then

$$\left\|\frac{BD}{DC}\right\| \cdot \left\|\frac{CE}{EA}\right\| \cdot \left\|\frac{AF}{FB}\right\| = -1.$$

Each term is the ratio of the lengths of the stated line segments, and the double lines indicate that *signed lengths* are to be used – thus if $X, Y, Z$ are distinct collinear points, then $\|XY/YZ\|$ is positive if $Y$ lies between $X$ and $Z$, and is negative otherwise.

*Menelaus' Theorem* is the first, and one of the simplest, of a large class of theorems concerning polygons or higher-dimensional objects in affine spaces. For one of these, let $P = [V_0, V_1, \ldots, V_{n-1}]$ be a polygon in $\mathbb{A}^d$, affine space of $d$ dimensions. Suppose that on each edge or chord $[V_i, V_{i+m}]$ of $P$ a point $W_i$ is

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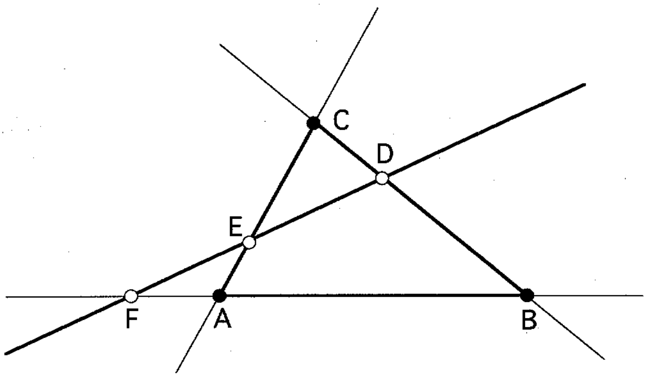


*Figure 1.* The theorem of Menelaus: the product of the ratios of signed lengths $\|\frac{BD}{DC}\| \cdot \|\frac{CE}{EA}\| \cdot \|\frac{AF}{FB}\|$ equals $-1$.

defined in some geometrically significant way. Then, under suitable conditions that need to be made explicit,

$$\prod \left\| \frac{V_i W_i}{W_i V_{i+m}} \right\| = \text{constant.} \tag{1}$$

Here and throughout, we adhere to the following conventions: the product is from $i = 0$ to $i = n - 1$, all subscripts are to be reduced modulo $n$, all the sides and chords are to be extended as necessary so the required intersections exist, and all the points in the construction must be in sufficiently general position so that the left side of (1) is defined. The constant will always turn out to have the value $+1$ or $-1$.

For example, let $P$ be an $n$-gon ($n \geqslant 3$) in $\mathbb{A}^d (d \geqslant 2)$. If F is a $(d-1)$-flat (affine subspace of dimension $d - 1$), and $F$ meets the edge $V_i V_{i+1}$ of the polygon $P$ in $W_i$, then (1) holds with constant equal to $(-1)^n$. This is a generalization of Menelaus' Theorem to $n$-gons in $A^d$ due to Carnot [C1] for $d = 3$. As another example suppose $n = 5$, the pentagon $P = [V_0, V_1, V_2, V_3, V_4]$ lies in $\mathbb{A}^3$, and $W_i$ is defined as the intersection of the edge $V_i V_{i+1}$ with the plane containing $V_{i+2}$, $V_{i+3}$, $V_{i+4}$; then (1) holds. Additional examples of theorems of this kind can be found in [GS1], [GS2], [GS3].

Here we present two new results of which typical instances are are shown in Figure 2. Our *First Transversality Theorem* is illustrated in Figure 2(a) for a pentagon. For each $i$, if $Z_i$ is defined as the intersection of the chords $V_{i+1}V_{i+3}$ and $V_{i+2}V_{i+4}$, and $W_{i+2}$ as the intersection of $V_i Z_i$ and $V_{i+2}V_{i+3}$, then (1) holds with constant equal to 1.

Figure 2(b) illustrates the *Second Transversality Theorem* also for a pentagon. For each $i$, if $Z_i$ is the intersection of $V_i V_{i+2}$ with $V_{i+1}V_{i+4}$, $Y_i$ is the intersection of $V_i V_{i+4}$ with $V_{i+1}V_{i+2}$, and $W_i$ is the intersection of $Y_i Z_i$ with $V_i V_{i+1}$, then (1) holds with constant equal to 1.

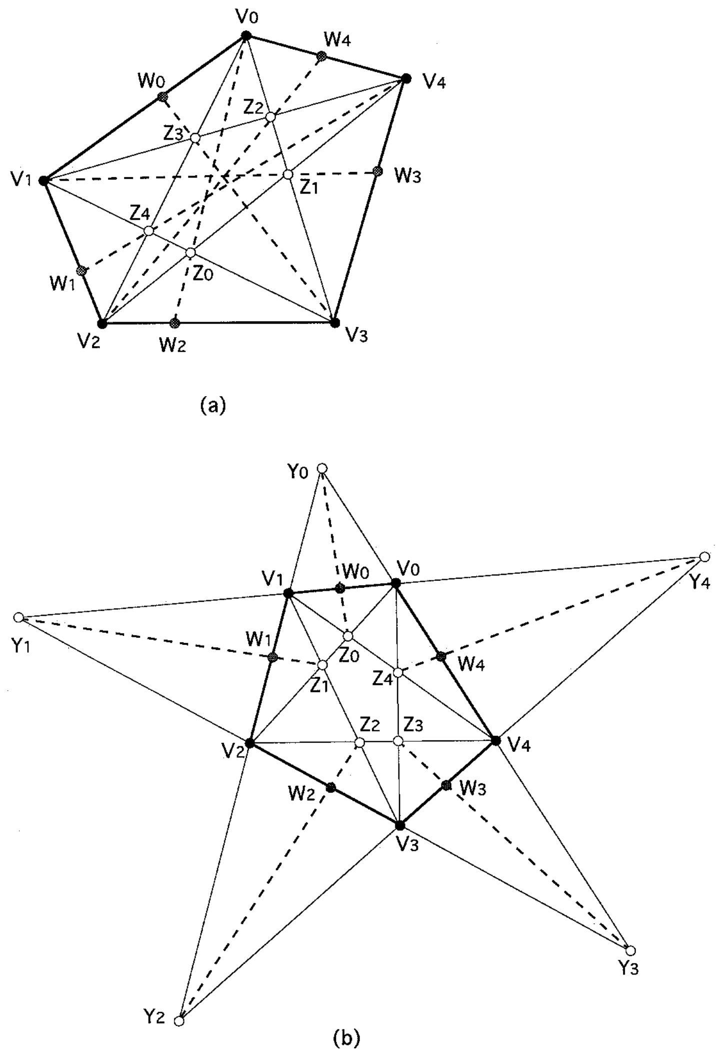


*Figure 2.* (a) The First Transversal Theorem 5(2; 1, $-2$) $=$ 5(3; 3, 1). Here $\Pi\|\frac{V_iW_i}{W_iV_{i+1}}\| = 1$. (b) The Second Transversal Theorem 5(1; 1, $-1$; 0, 2) $=$ 5(1; 0, $-2$; 0, $-4$), Type 2(i). Here the same relation $\Pi\|\frac{V_iW_i}{W_iV_{i+1}}\| = 1$ holds.

The two transversality theorems, together with the Selftransversality Theorem of [GS1], form a natural sequence. In each case two points define a transversal which cuts the side or chord $V_iV_{i+m}$ of $P$ in $W_i$. In the case of the Selftransversality Theorem the two points are vertices of $P$; in the First Transversality Theorem one point is a vertex of $P$ and the other is the point of intersection of two chords of $P$; in the Second Transversality Theorem both points are defined as points of intersection of chords of $P$. All these theorems relate to polygons in the affine plane $\mathbb{A}^2$.

As all the results of this paper belong to plane affine geometry, at first sight there seems no good reason why they were not discovered years, or even centuries, ago. The explanation is simple: we have the advantage, denied to earlier investigators, of sophisticated computer programs, in particular Mathematica® software on a Macintosh computer. Use of these programs led to large numbers of empirical results, some of which are listed in the table at the end of this paper. Examination of these suggested that, apart from a few anomalous or sporadic cases that arise for small values of $n$, all are covered by four theorems, each with several cases depending on one or two parameters. Because the computer search was exhaustive we can say even more: at least for $n \leqslant 20$ our theorems cover all possible cases, and there is heuristic evidence that this is true for all values of $n$. The anomalous cases that arise for small values of $n$ ('small' here means $n \leqslant 16$) seem to occur because of 'accidental' congruences between the parameters.

The proofs given below depend essentially on what we call the Area Principle. Although this method of argumentation is very old – according to Baptist [B, p. 61], A. L. Crelle used it in 1816 in his book [C3] – it has been reintroduced in [GS1], and independently by Chou *et al.* [C2]. In these and other publications the Area Principle and its variants have been found to be very useful mathematical techniques. Here we use them to prove a fundamental result, the *Elimination Lemma*, which enables many of our proofs, which are essentially manipulative in nature, to be made more concise.

The paper is organized as follows. In Section 2 we give precise definitions of the transversality properties, and introduce appropriate notation. Section 3 presents the Area Principle and the Elimination Lemma. The main results are formulated and proved in Section 4 and Section 5 is devoted to remarks and comments.

Further examples of the First Transversality Theorem are shown in Figure 7, and of the Second Transversality Theorem in Figures 3, 4, 5 and 6. The notation used in the captions to these figures will be explained in Section 2.

## 2. The Transversality Properties

THE FIRST TRANSVERSALITY PROPERTY $n(m; r, s)$, see Figure 8. *Let* $P = [V_0, V_1, \dots, V_{n-1}]$ *be a given* $n$*-gon in the affine plane and* $m, r, s$ *be given integers* (*parameters*)*. For each* $i = 0, 1, \dots, n-1$ *let the intersection of the chords* $V_{i-r}V_{i-s}$ *and* $V_{i+r}V_{i+s}$ *be denoted by* $Z_i$*, and let the transversal* $V_iZ_i$ *meet the base chord* $V_{i-m}V_{i+m}$ *in* $W_i$*. Then, for suitable values of the parameters,*

$$\prod \left\| \frac{V_{i-m}W_i}{W_iV_{i+m}} \right\| = 1. \tag{2}$$

In Theorem 1 we state explicitly the values of the parameters $m, r$ and $s$ for which the transversal property $n(m; r, s)$ is true, that is, (2) holds. An example of the property 7(2; 1, 2) is shown in Figure 7.

THE SECOND TRANSVERSALITY PROPERTY $n(m; r, s; t, u)$, see Figure 9. *Let* $P = [V_0, V_1, \dots V_{n-1}]$ *be a given* $n$*-gon in the affine plane and* $m, r, s, t, u$ *be given integers* (*parameters*). *For each* $i = 0, 1, \dots, n-1$ *let the intersection of the chords* $V_{i-r}V_{i-s}$ *and* $V_{i+m+r}V_{i+m+s}$ *be denoted by* $Z_i$, *and the intersection of the chords* $V_{i-t}V_{i-u}$ *and* $V_{i+m+t}V_{i+m+u}$ *be denoted by* $Y_i$. *Let the transversal* $Y_iZ_i$ *meet the base chord* $V_iV_{i+m}$ *in* $W_i$. *Then, for suitable values of the parameters,*

$$\prod \left\| \frac{V_iW_i}{W_iV_{i+m}} \right\| = 1. \tag{3}$$

As before, we place no restrictions on $P$ other than that the lines in the figure and the fractions in (3) are well-defined. In Theorems 2, 3, 4 and 5 we state explicitly the values of the parameters $m, r, s, t, u$ for which the second transversal property $n(m; r, s; t, u)$ is true, that is, (3) holds. Examples 5(1; 0, −2; 0, −4), 11(1; 0, 1; 3, 4), 7(1; 0, 2; 0, 4), 8(1; 0, 1; 2, 3), and 9(3; 0, −1; −1, 1) are shown in Figures 2b, 3, 4, 5, 6.

In both transversality properties the polygons are quite general: vertices may coincide and edges may cross or overlap. The only restrictions are that each line or chord under consideration must be specified by two distinct points, and that the denominators in (2) and (3) must not vanish. The 'appearance' of the figures illustrating the properties depend, of course, on the choice of initial polygon. For example, Figure 10 illustrates the second transversal property 5(1; 0, -2; 0, -4) which is the same as that shown in Figure 2(b).

All the results of this paper properly belong to affine geometry since they are invariant under *affinities*, that is, nonsingular linear transformations combined with translations. Equivalently, affinities may be defined as projective transformations which leave the line at infinity fixed.

The notations $n(m; r, s)$ and $n(m; r, s; t, u)$ for the transversality properties are very useful, but suffer from the disadvantage that they are not unique. For example, as noted in the caption, the second transversality property shown in Figure 2(b) may be denoted by 5(1; 1, −1; 0, 1) or 5(1; 0, −2; 0, −4). But there are many other possibilities also such as 5(1; 0, −4; 1, −1), 5(1; 0, −2; −1, −2) and so on. Also 5(2; 0, 1; 1, −2) represents the same property for the pentagram $[V_0, V_2, V_4, V_1, V_3]$ and therefore is not considered distinct. Checking for repetitions is therefore not trivial, and is probably best achieved by making a sketch of the configuration. In the table at the end of the paper all repetitions have been excluded, as have *non-primitive* cases. We say that a property is *primitive* if the HCF (GCD) of the parameters ($n, m, r, s$ in $n(m; r, s)$ or $n, m, r, s, t, u$ in $n(m; r, s; t, u)$) is 1. A nonprimitive assertion, for which the HCF of the parameters is $d > 1$, is really a trivial consequence of the corresponding property for an $(n/d)$-gon.

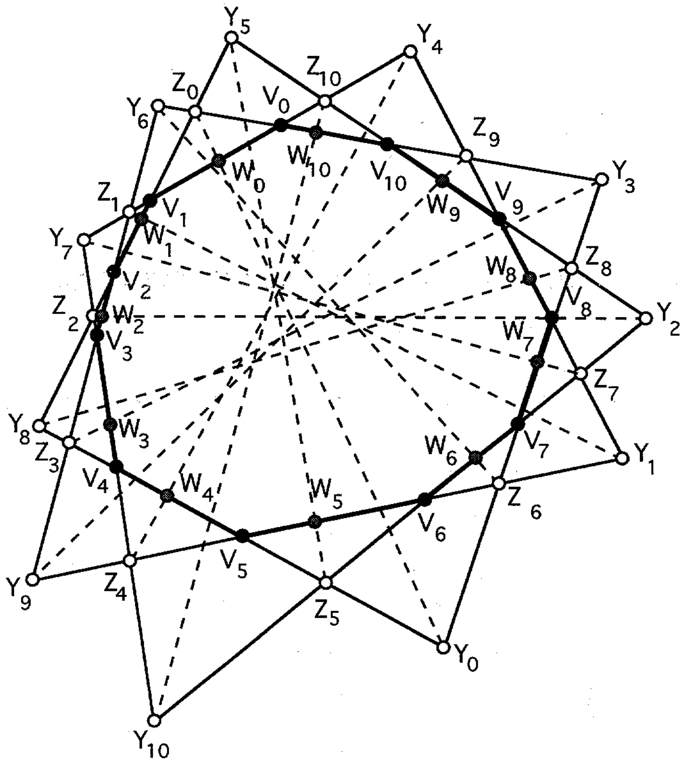


*Figure 3.* The Second Transversality Theorem 11(1; 0, 1; 3, 4), Type 2(iv).

In Figure 2, and all subsequent figures illustrating the transversality properties, we follow the convention that the transversal ($V_iZ_i$ for the first and $Y_iZ_i$ for the second transversal property) is indicated by a dotted line, and the base chord ($V_{i-m}V_{i+m}$ for the first and $V_iV_{i+m}$ for the second transversal property) is indicated by a heavy line.

## 3. Preliminaries

We make use of two basic results which are variants of the area principle [GS1].

APB: *The Area Principle for triangles with equal bases.* This states that *the areas of two triangles with equal bases are in the same ratio as their heights*. For example, in Figure 11, the shaded triangles have the same base [BC]; APB asserts that

$$\left\| \frac{A_1P}{A_2P} \right\| = \left\| \frac{A_1BC}{A_2BC} \right\| .$$

Since we are using signed lengths for $A_1P, A_2P$ and also signed areas for the triangles (the area of a triangle is positive if it is oriented counterclockwise, and

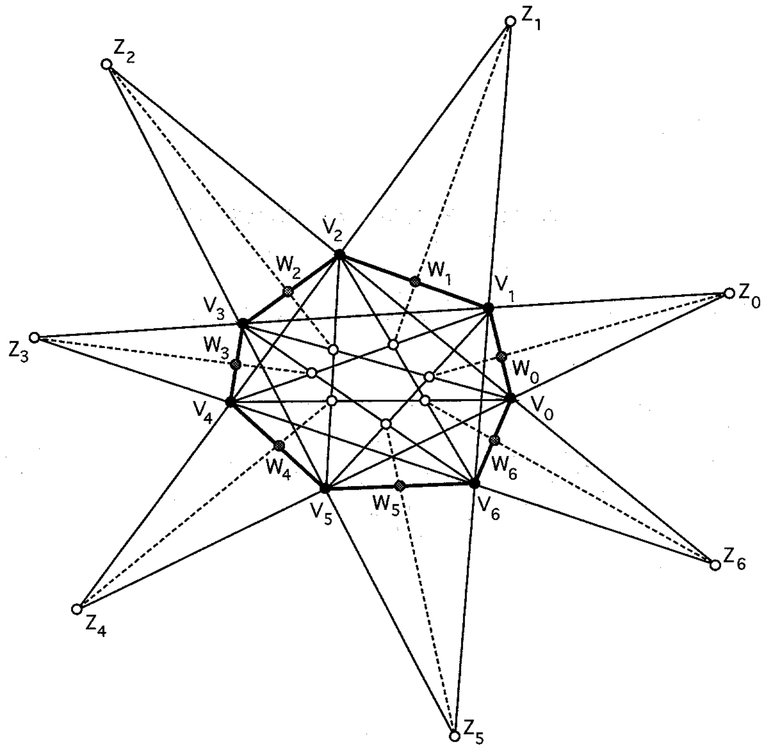


*Figure 4*. The Second Transversality Theorem 7(1; 0, 2; 0, 4), Type 2(i).

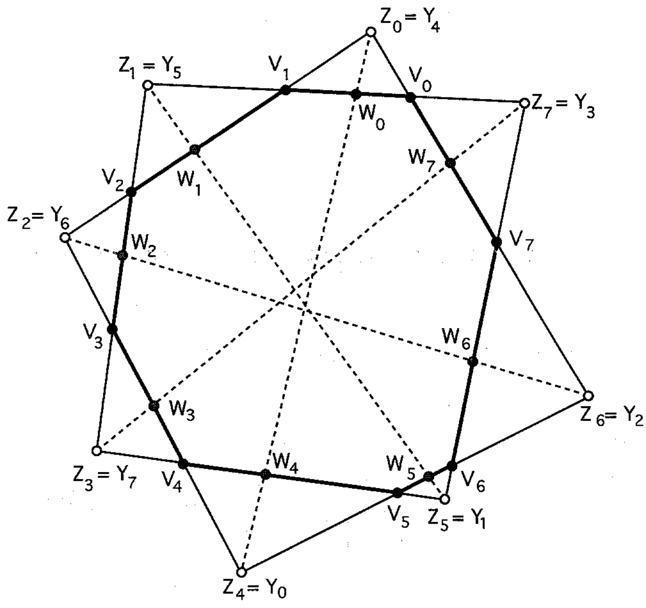


*Figure 5*. The Second Transversality Theorem 8(1; 0, 1; 2, 3) Type 2(iv).

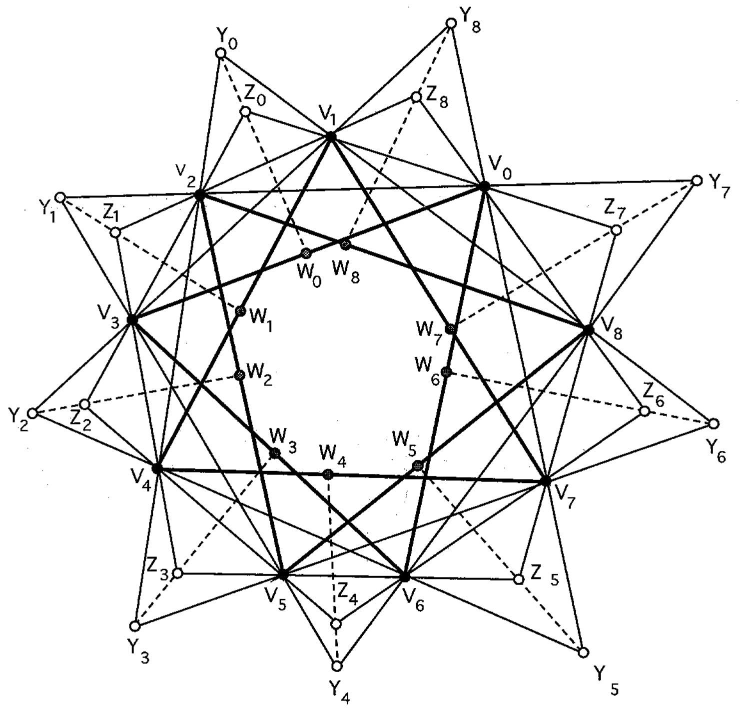


*Figure 6*. The Second Transversality Theorem 9(3; 0, $-1$; $-1$, 1) Type 2(ii).

negative if oriented clockwise) as is indicated by the double lines, APB is true whether $A_1, A_2$ lie on the same, or opposite sides of BC.

APH: *The Area Principle for triangles with equal heights.* This states that *the areas of two triangles with equal heights are in the same ratio as the lengths of their bases*. For example, in Figure 12, the shaded triangles have the same apex $A$, and so have equal heights; APH asserts that

$$\left\| \frac{B_1C_1}{B_2C_2} \right\| = \left\| \frac{AB_1C_1}{AB_2C_2} \right\| .$$

Using APB and APH we can now prove the following fundamental lemma:

THE ELIMINATION LEMMA. *Let* $P = [V_0, V_1, \ldots, V_{n-1}]$ *be an* $n$*-gon,* $A = V_dV_e \cap V_fV_g$ *and* $B, C$ *be any points not on* $V_dV_e$ *or* $V_fV_g$ (*see* Figure 13). *Then*

$$\left\| \frac{V_dAB}{V_fAC} \right\| = \left\| \frac{V_dV_eB}{V_fV_gC} \right\| \cdot \left\| \frac{V_fV_gV_d}{V_eV_dV_f} \right\| .$$

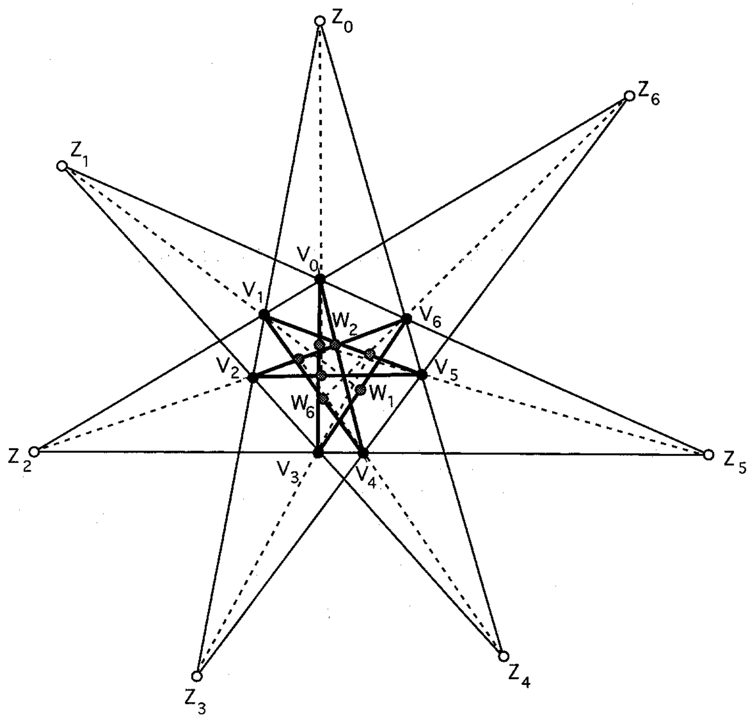


*Figure 7.* The First Transversality Theorem 7 (2; 1, 2) (sporadic), To avoid clutter not all the points $W_i$ are labelled.

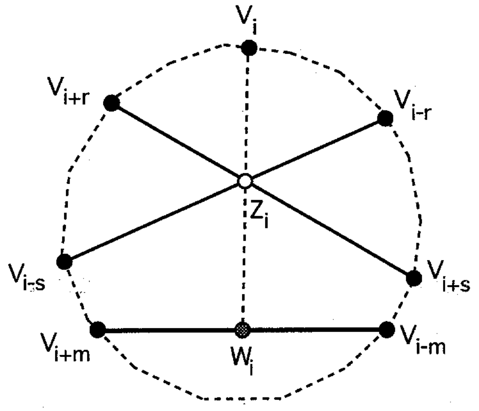


*Figure 8.* Explanation of the notation $n(m; r, s)$ for the First Transversality Property.

*Proof.* By APB applied to triangles with base $[V_f V_g]$ and apexes $V_e, V_d$,

$$\left\| \frac{V_g V_f V_e}{V_f V_g V_d} \right\| = \left\| \frac{A V_e}{V_d A} \right\| .$$

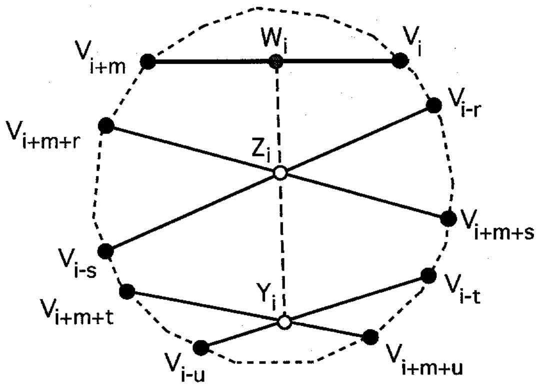


*Figure 9.* Explanation of the notation $n(m; r, s; t, u)$ for the Second Transversality Property.

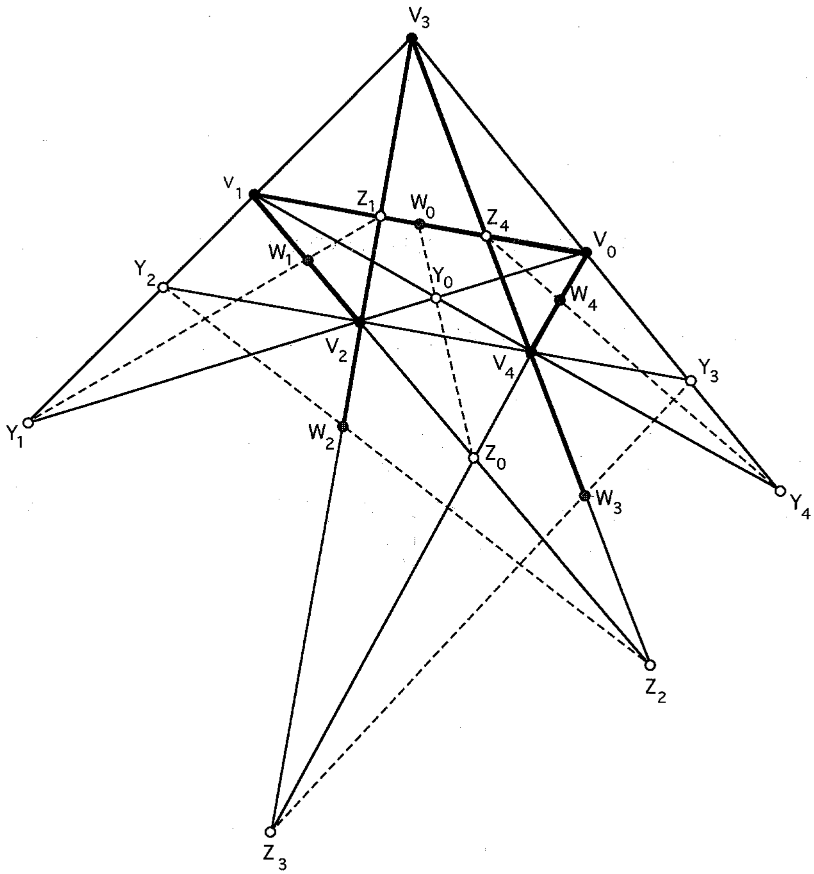


*Figure 10.* The Second Transversality Property 5(1; 0, −2; 0, −4), Type 2(i). This is the same case as that shown in Figure 2(b). It illustrates the different appearance of the diagram when one starts from a different polygon.

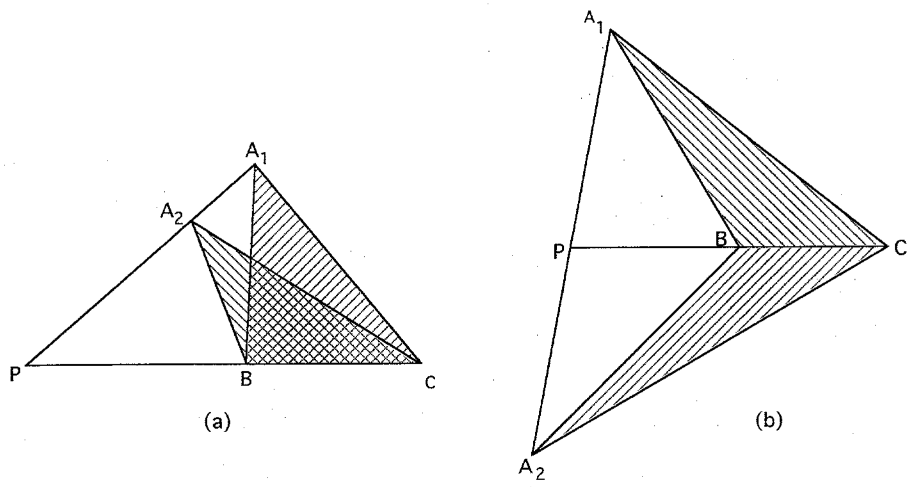


*Figure 11*. The Area Principle APB.

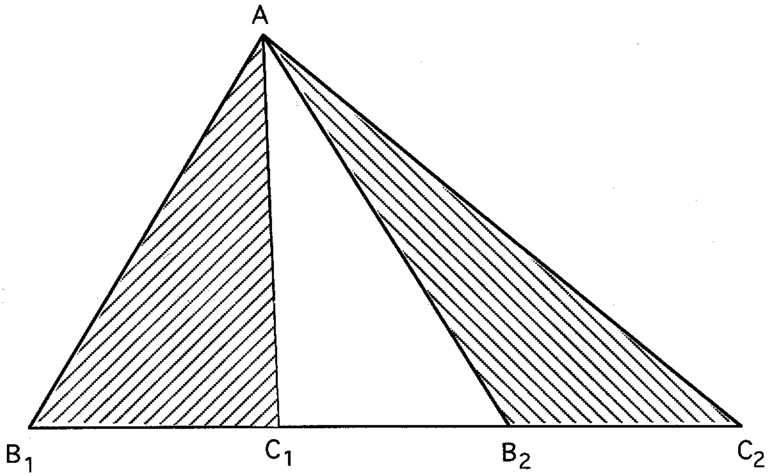


*Figure 12*. The Area Principle APH.

Adding 1 to each side we obtain

$$\left\|\frac{V_g V_d V_f V_e}{V_f V_g V_d}\right\| = \left\|\frac{V_d V_e}{V_d A}\right\|,$$

where the numerator on the left is the (signed) area of the quadrangle $[V_g, V_d, V_f, V_e] = [V_g, V_f, V_e] \cup [V_f, V_g, V_d]$, and so

$$\left\|\frac{V_g V_d V_f V_e}{V_f V_g V_d}\right\| = \left\|\frac{V_d V_e B}{V_d A B}\right\| \tag{4}$$

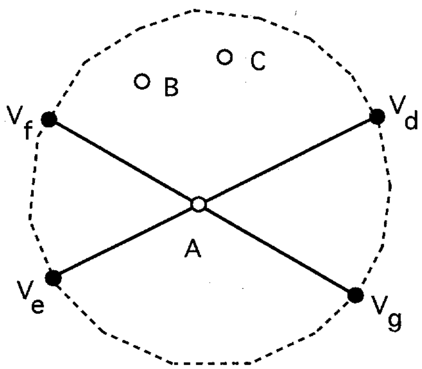


*Figure 13.* The Elimination Lemma.

by APH applied to triangles with bases $[V_d, V_e]$ and $[V_d, A]$ with the same vertex B.

In an exactly similar manner we obtain

$$\left\| \frac{V_d V_f V_e V_g}{V_e V_d V_f} \right\| = \left\| \frac{V_f V_g}{V_f A} \right\| = \left\| \frac{V_f V_g C}{V_f A C} \right\|. \tag{5}$$

Dividing (5) by (4) the areas of the quadrangles cancel, and this yields

$$\left\| \frac{V_d A B}{V_f A C} \right\| = \left\| \frac{V_d V_e B}{V_f V_g C} \right\| \cdot \left\| \frac{V_f V_g V_d}{V_e V_d V_f} \right\|.$$

as required. This completes the proof of the Elimination Lemma.

The purpose of the lemma is to reduce the quotient on the left, which involves $As, Bs, Cs$ and $Vs$ to the expression on the right which involves $Bs, Cs$ and $Vs$ only. In other words, the As have been eliminated. Repeated use of this lemma enables us to express products of ratios involving points which are intersections of chords in terms of $Vs$ (the vertices of the polygon) only.

## 4. The Main Theorems

THEOREM 1. *The first transversality property $n(m; r, s)$ is true in at least the following cases:*

(i) $n(3; 1, -2) (n = 5$ *or* $n \geqslant 7))$
(ii) 5(2; 1, 2), 7(2; 1, 2) *and* 8(2; 1, 5).

The three cases in (ii) are sporadic; that in (i) is the general case which holds for the values of $n$ indicated. As stated in the Introduction, we believe that Theorem 1 covers all cases in which the first transversal theorem is true. However due to the fact that the notation is not unique, as indicated earlier, great care must be taken

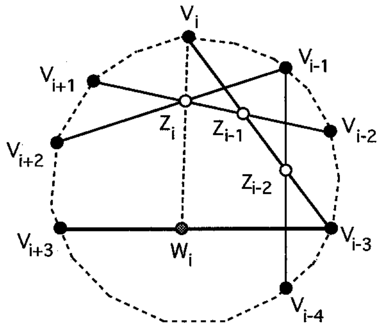


*Figure 14.* Theorem 1.

in checking individual cases to avoid repetitions. For example, a computer search for $n = 9$ yielded three cases, namely 9(3; 1, 4), 9(3; 1, −2) and 9(3; 2, 5). In fact, these are all the same as the general case 9(3; 1, −2). For the first, rename the vertex $V_i$ as $V'_{2i}$ – this is equivalent to applying the transversal property 9(3; 1, 4) to the (9/2)-gon $P' = [V'_0, \ldots, V'_{n-1}]$. Then 9(3; 1, 4) becomes 9(6; 2, 8) which is the same as 9(3; 2, −1) or 9(3; 1, −2). For the third case 9(3; 2, 5) rename $V_i$ as $V'_{5i}$; then 9(3; 2, 5) becomes 9(15; 10, 25) which is the same as 9(3; 1, −2). Exactly similar considerations apply to all cases $n \geqslant 7$.

*Proof.* First consider the regular case (i). Since

$$Z_i = V_{i-1}V_{i+2} \cap V_{i+1}V_{i-2}$$

it follows that $Z_{i-1}$ and $Z_{i-2}$ lie on the line $V_iV_{i-3}$ (see Figure 14). Then, trivially,

$$\begin{aligned} 1 &= \prod \left\| \frac{V_iV_{i-3}}{Z_{i-1}Z_{i-1}} \right\| \cdot \prod \left\| \frac{Z_{i-1}Z_{i-2}}{V_iV_{i-3}} \right\|, \\ &= \prod \left\| \frac{V_iV_{i-3}Z_i}{Z_{i-1}Z_{i-2}Z_i} \right\| \cdot \prod \left\| \frac{Z_{i-1}Z_{i-2}Z_{i-3}}{V_iV_{i-3}Z_{i-3}} \right\| \end{aligned}$$

by APH applied to triangles with apexes $Z_i$ and $Z_{i-3}$ and bases on the line $V_iV_{i-3}$. Rearranging,

$$1 = \prod \left\| \frac{V_iV_{i-3}Z_i}{V_iV_{i-3}Z_{i-3}} \right\| \cdot \prod \left\| \frac{Z_{i-1}Z_{i-2}Z_{1-3}}{Z_{i-1}Z_{i-2}Z_i} \right\|. \tag{6}$$

The second product clearly takes the value 1 since the same triangles occur in the numerator and denominator as one can see by making the substitution $i \to i + 1$ in the numerator. For the first product we obtain, by making the substitution $i \to i + 3$ in the denominator

$$1 = \prod \left\| \frac{V_iV_{i-3}Z_i}{V_{i+3}V_iZ_i} \right\| = \prod \left\| \frac{V_{i-3}W_i}{W_iV_{i+3}} \right\|,$$

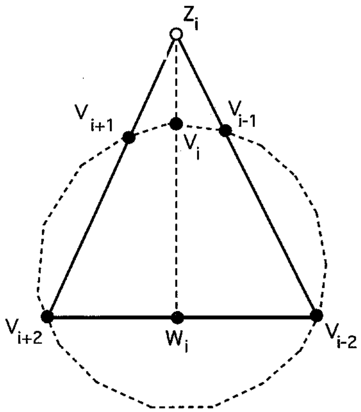


*Figure 15*. Theorem 1.

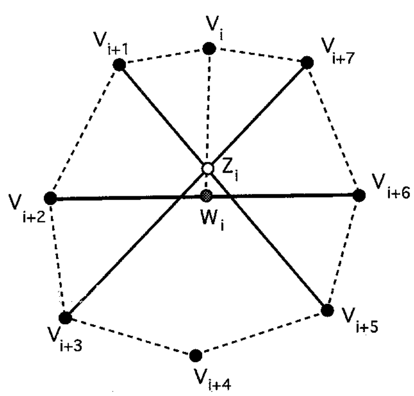


*Figure 16*. Theorem 1.

where the second equality follows by APB applied to triangles with base $[V_i, Z_i]$ and apexes $V_{i-3}, V_{i+3}$. This completes the proof of (i). Notice that the value of $n$ played no part in the calculation, except that $n = 3, n = 4$ are clearly excluded, and $n = 6$ is excluded because then the chord $[V_{i-3}, V_{i+3}]$ is not defined.

For the first two sporadic cases in (ii), namely 5(2; 1, 2) and 7(2; 1, 2),

$$\left\| \frac{V_{i-2}W_i}{W_iV_{i+2}} \right\| = - \left\| \frac{V_iZ_iV_{i-2}}{V_iZ_iV_{i+2}} \right\|$$

by APB applied to triangles with base $[V_iZ_i]$ and apexes $V_{i-2}, V_{i+2}$ (see Figure 15). Now apply the Elimination Lemma with $d = i-2, e = i-1, f = i+2, g = i+1$, $A = Z_i, B = C = V_i$, to obtain,

$$\left\| \frac{V_{i-2}Z_iV_i}{V_{i+2}Z_iV_i} \right\| = \left\| \frac{V_{i-2}V_{i-1}V_i}{V_{i+2}V_{i+1}V_i} \right\| \cdot \left\| \frac{V_{i+2}V_{i+1}V_{i-2}}{V_{i-1}V_{i-2}V_{i+2}} \right\| .$$

Hence

$$\prod \left\| \frac{V_{i-2}W_i}{W_iV_{i+2}} \right\| = \prod \left\| \frac{V_{i-2}V_{i-1}V_i}{V_iV_{i+1}V_{i+2}} \right\| \cdot \prod \left\| \frac{V_{i+1}V_{i-2}V_{i+2}}{V_{i+2}V_{i-1}V_{i-2}} \right\|.$$

The first term on the right has the value 1 (substitute $i \to i+2$ in the numerator) and the second term on the right has the value 1 if $n = 5$ or 7 (make the substitutions $i \to i+1$ or $i \to i+4$ in the numerators, respectively).

Finally, for the third sporadic case 8(2; 1, 5)

$$\prod \left\| \frac{V_{i-2}W_i}{W_iV_{i+2}} \right\| = \prod \left\| \frac{V_{i-2}Z_iV_i}{V_{i+2}V_iZ_i} \right\|$$

by APB applied to triangles with bases $[V_iZ_i]$ and apexes $V_{i-2} = V_{i+6}, V_{i+2}$ (see Figure 16). Now

$$\prod \left\| \frac{V_{i-2}Z_iV_i}{V_{i+2}V_iZ_i} \right\| = \prod \left\| \frac{V_iZ_{i+2}V_{i+2}}{V_{i+2}V_iZ_i} \right\| = \prod \left\| \frac{V_iZ_iV_{i+2}}{V_iZ_iV_{i+2}} \right\| = 1,$$

where the first equality rises from the substitution $i \to i+2$ in the numerator, and the second by rearrangement and noticing that $Z_i = Z_{i+2}$. This completes the proof of Theorem 1.

THEOREM 2. *The second transversality property* $n(m; r, s; t, u)$ *is true in at least the following cases:*

(i) $n(m; 0, a; 0, 2a)$ *where* $m + 3a \equiv 0$,
(ii) $n(m; 0, a; a, -a)$ *where* $m + 3a \equiv 0$,
(iii) $n(m; 0, a; a, 2a)$ *where* $2m + 3a \equiv 0$,
(iv) $n(m; 0, m; b, m+b)$ *where* $2m + 3b \equiv 0$,
(v) $n(m; 0, -2m; b, b-2m)$ *where* $m \equiv 3b$,
(vi) 7(1; 0, 1; 1, 4), 7(1; 0, −2; −2, 2), 8(1; 0, −2; −2, 3), 10(1; 0, −2; −2, 4), 11(1; 0, 1; 1, 6),

*where all the congruences are modulo* $n$.

Hence there are five general cases, (i) – (v) and five sporadic cases listed in (vi). Additional cases arise if $n$ is divisible by 2 (Theorem 3), by 4 (Theorem 4) or if $n = 16$ (Theorem 5). The same remarks about checking cases, made after the statement of Theorem 1, apply here also.

The theorem is true for all $n \geqslant 4$ whenever the statements are meaningful, but for small values of n $(n < 10)$ a number of anomalous cases arise. For example, when $n = 4$, case 2(i) with $m = 1, a = 1$ is the same as case 2(ii) with the same values of $m$ and $a$. When $n = 5$, case 2(ii) with $m = 1, a = 3$ reduces to the first transversality result since, for all $i$, the intersection point $Y_i$ coincides with the vertex $V_{i+3}$. For $n = 6$, in case 2(iv) with $m = 3, b = -2$, the point $Z_i$ (and

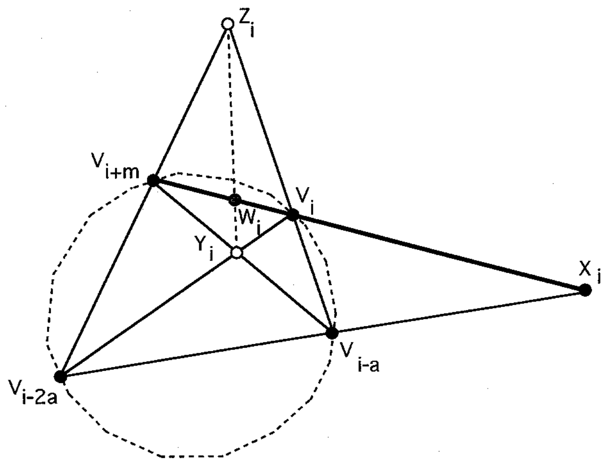


*Figure 17.* Theorem 2(i).

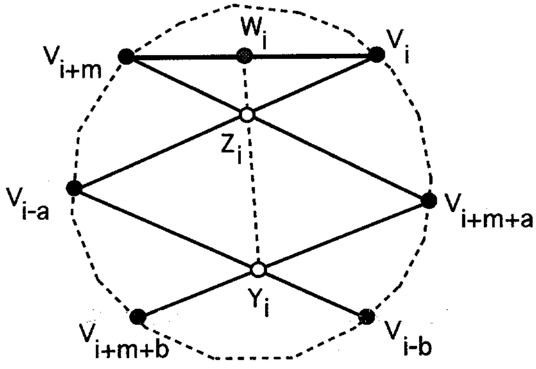


*Figure 18.* Theorem 2(ii), (iii) and (vi).

therefore the transversal $Y_iZ_i$) is not defined. For $n \geqslant 10$ each of the first five parts of the theorem leads to at least one non-trivial assertion. Full details can be found in the table at the end of the paper.

*Proof.* (i) The congruence $m + 3a \equiv 0$ implies that $V_i, V_{i-a}, V_{i-2a}, V_{i+m} = V_{i-3a}$ are the vertices of a complete quadrangle $Q$ whose diagonal points are $Y_i$, $Z_i$ and $X_i$ (see Figure 17). Hence

$$\left\| \frac{V_iW_i}{W_iV_{i+m}} \right\| = \left\| \frac{V_iX_i}{V_{i+m}X_i} \right\| = \left\| \frac{V_iV_{i-a}V_{i-b}}{V_{i+m}V_{i-a}V_{i-b}} \right\|,$$

where the first equality follows by the harmonic properties of $Q$, and the second by APB applied to triangles with bases $[V_{i-a}, V_{i-2a}]$ and apexes $V_i, V_{i+m}$. Hence,

$$\prod \left\| \frac{V_i W_i}{W_i V_{i+m}} \right\| = \prod \left\| \frac{V_i V_{i-a} V_{i-2a}}{V_{i-3a} V_{i-a} V_{i-2a}} \right\| = 1,$$

where the second equality follows from the substitution $i \to i-a$ in the numerator.

(ii), (iii) *and* (vi). Consider $n(m;0,a;a,b)$ (see Figure 18). By APB for triangles with bases $[Y_i, Z_i]$ and apexes $V_i, V_{i+m}$:

$$\begin{aligned} \left\| \frac{V_i W_i}{W_i V_{i+m}} \right\| &= -\left\| \frac{V_i Z_i Y_i}{V_{i+m} Z_i Y_i} \right\| \\ &= -\left\| \frac{V_i V_{i-a} Y_i}{V_{i+m} V_{i+m+a} Y_i} \right\| \cdot \left\| \frac{V_{i+m} V_{i+m+a} V_i}{V_{i-a} V_i V_{i+m}} \right\|, \end{aligned} \tag{6}$$

where the second equality follows from the Elimination Lemma with $d = i, e = i-a$, $f = i+m, g = i+m+a$, $A = Z_i$, $B = C = Y_i$. Again by the Elimination Lemma with $d = i-a, e = i-b, f = i+m+a$, $g = i+m+b, A = Y_i$, $B = V_i$ and $C = V_{i+m}$,

$$\left\| \frac{V_{i-a} Y_i V_i}{V_{i+m+a} Y_i V_{i+m}} \right\| = \left\| \frac{V_{i-a} V_{i-b} V_i}{V_{i+m+a} V_{i+m+b} V_{i+m}} \right\| \cdot \left\| \frac{V_{i+m+a} V_{i+m+b} V_{i-a}}{V_{i-b} V_{i-a} V_{i+m+a}} \right\|.$$

Substituting this expression for the first factor of the last term in (6) yields

$$\begin{aligned} \left\| \frac{V_i W_i}{W_i V_{i+m}} \right\| &= -\left\| \frac{V_{i+m} V_{i+m+a} V_i}{V_{i-a} V_i V_{i+m}} \right\| \cdot \left\| \frac{V_{i-a} V_{i-b} V_i}{V_{i+m+a} V_{i+m+b} V_{i+m}} \right\| \cdot \\ &\quad \cdot \left\| \frac{V_{i+m+b} V_{i+m+a} V_{i-a}}{V_{i-a} V_{i-b} V_{i+m+a}} \right\| \end{aligned}$$

which, upon taking products and rearranging leads to

$$\begin{aligned} \prod \left\| \frac{V_i W_i}{W_i V_{i+m}} \right\| &= \prod \left\| \frac{V_i V_{i+m} V_{i+m+a}}{V_{i-a} V_{i-b} V V_{i+m+a}} \right\| \cdot \prod \left\| \frac{V_{i+m+b} V_{i+m+a} V_{i-a}}{V_i V_{i+m} V_{i-a}} \right\| \cdot \\ &\quad \cdot \prod \left\| \frac{V_{i-a} V_{i-b} V_i}{V_{i+m+b} V_{i+m+a} V_{i+m}} \right\|. \end{aligned} \tag{7}$$

For (ii) we substitute $b \equiv -a$ and $m \equiv -3a$ leading to

$$\begin{aligned} \prod \left\| \frac{V_i W_i}{W_i V_{i+m}} \right\| &= \prod \left\| \frac{V_i V_{i-3a} V_{i-2a}}{V_{i-a} V_{i+a} V_{i-2a}} \right\| \cdot \prod \left\| \frac{V_{i-4a} V_{i-2a} V_{i-a}}{V_i V_{i-3a} V_{i-a}} \right\| \cdot \\ &\quad \cdot \prod \left\| \frac{V_{i-a} V_{i+a} V_i}{V_{i-4a} V_{i-2a} V_{i-3a}} \right\|. \end{aligned}$$

Each term on the right takes the value 1 as can be seen by making the substitutions $i \to i+a, i \to i+a, i \to i-3a$ respectively in the numerators of the three fractions. This establishes (ii).

For (iii) we substitute $b \equiv 2a$ in (7). This yields

$$\prod \left\| \frac{V_i W_i}{W_i V_{i+m}} \right\| = \prod \left\| \frac{V_i V_{i+m} V_{i+m+a}}{V_{i-a} V_{i-2a} V_{i+m+a}} \right\| \cdot \prod \left\| \frac{V_{i+m+2a} V_{i+m+a} V_{i-a}}{V_i V_{i+m} V_{i-a}} \right\| \cdot \\ \cdot \prod \left\| \frac{V_{i-a} V_{i-2a} V_i}{V_{i+m+2a} V_{i+m+a} V_{i+m}} \right\| .$$

Each term on the right takes the value 1 as can be seen by making the substitutions $i \to i+m+a, i \to i+m+a, i \to i+m+2a$ respectively in the numerators of the three fractions and using $2m+3a \equiv 0$. This establishes (iii).

(vi) The five sporadic cases follow by another rearrangement of (7), namely

$$\prod \left\| \frac{V_i W_i}{W_i V_{i+m}} \right\| = \prod \left\| \frac{V_i V_{i+m} V_{i+m+a}}{V_i V_{i+m} V_{i-a}} \right\| \cdot \prod \left\| \frac{V_{i-a} V_{i-b} V_i}{V_{i+m+b} V_{i+m+a} V_{i+m}} \right\| \cdot \\ \cdot \prod \left\| \frac{V_{i+m+b} V_{i+m+a} V_{i-a}}{V_{i-a} V_{i-b} V_{i+m+a}} \right\| .$$

Substituting the values of $n, m, a, b$ for each of the five sporadic cases makes each of the three product terms in this expression equal to 1. We leave details to the reader. It will be observed that in showing that each term has the value 1 it is necessary to make use of the given value of $n$. This is precisely why these cases are sporadic – in the regular cases, the value of $n$ is essentially irrelevant. A similar remark applies to the sporadic cases in part (ii) of Theorem 1.

(iv) *and* (v). Consider $n(m; 0, a; b, a+b)$ (see Figure 19).

$$\left\| \frac{V_i W_i}{W_i V_{i+m}} \right\| = - \left\| \frac{V_i Z_i Y_i}{V_{i+m} Z_i Y_i} \right\|$$

by APB applied to triangles with bases $[Y_i, Z_i]$ and apexes $V_i, V_{i+m}$. Now

$$\left\| \frac{V_i Z_i Y_i}{V_{i+m} Z_i Y_i} \right\| = \left\| \frac{V_i V_{i-a} Y_i}{V_{i+m} V_{i+m+a} Y_i} \right\| \cdot \left\| \frac{V_{i+m} V_{i+m+a} V_i}{V_{i-a} V_i V_{i+m}} \right\|$$

by the Elimination Lemma with $d = i, e = i-a, f = i+m, g = i+m+a, A = Z_i$, $B = C = Y_i$. Hence

$$\prod \left\| \frac{V_i W_i}{W_i V_{i+m}} \right\| = \prod \left\| \frac{V_i V_{i-a} Y_i}{V_{i+m+a} V_{i+m} Y_i} \right\| \cdot \prod \left\| \frac{V_i V_{i+m} V_{i+m+a}}{V_i V_{i+m} V_{i-a}} \right\| ,$$

or, upon making the substitution $i \to i-m-a$ in the denominator of the first term,

$$\prod \left\| \frac{V_i W_i}{W_i V_{i+m}} \right\| = \prod \left\| \frac{V_i V_{i-a} Y_i}{V_i V_{i-a} Y_{i-m-a}} \right\| \cdot \prod \left\| \frac{V_i V_{i+m} V_{i+m+a}}{V_i V_{i+m} V_{i-a}} \right\| . \tag{8}$$

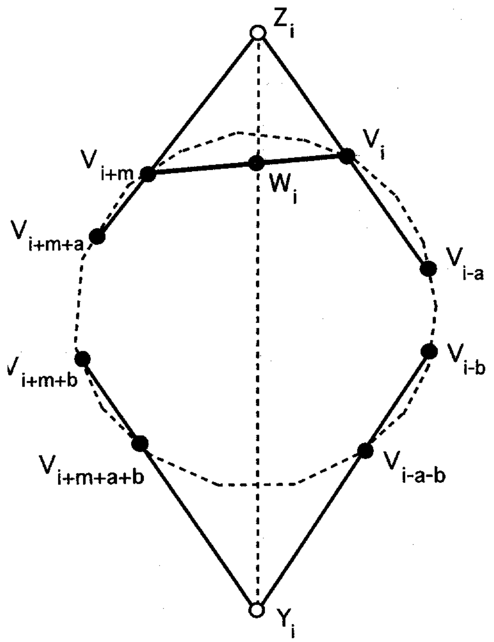


*Figure 19*. Theorem 2(iv) and (v).

Both $Y_{i+b}$ and $Y_{i-m-b-a}$ lie on the line $V_iV_{i-a}$, so, trivially,

$$1 = \left\| \frac{V_iV_{i-a}}{Y_{i-m-b-a}Y_{i+b}} \right\| \cdot \left\| \frac{Y_{i-m-b-a}Y_{i+b}}{V_iV_{i-a}} \right\|$$

and by two applications of APH to triangles with apexes $Y_i$ and $Y_{i-m-a}$,

$$1 = \left\| \frac{V_iV_{i-a}Y_i}{Y_{i-m-b-a}Y_{i+b}Y_i} \right\| \cdot \left\| \frac{Y_{i-m-b-a}Y_{i+b}Y_{i-m-a}}{V_iV_{i-a}Y_{i-m-a}} \right\|.$$

Taking products and interchanging denominators we obtain

$$1 = \prod \left\| \frac{V_iV_{i-a}Y_i}{V_iV_{i-a}Y_{i-m-a}} \right\| \cdot \prod \left\| \frac{Y_{i-m-b-a}Y_{i+b}Y_{i-m-a}}{Y_{i-m-b-a}Y_{i+b}Y_i} \right\|. \tag{9}$$

Divide (9) into (8) to eliminate the first product on the right side of each, and so obtain

$$\prod \left\| \frac{V_iW_i}{W_iV_{i+m}} \right\| = \prod \left\| \frac{V_iV_{i+m}V_{i+m+a}}{V_iV_{i+m}V_{i-a}} \right\| \cdot \prod \left\| \frac{Y_{i-m-b-a}Y_{i+b}Y_i}{Y_{i-m-b-a}Y_{i+b}Y_{i-m-a}} \right\|. \tag{10}$$

For (iv) substitute $a \equiv m$, then clearly the first factor has the value 1 as one can see from the substitution $i \to i - m$ in the numerator. If $2m + 3b \equiv 0$, the second factor becomes

$$\prod \left\| \frac{Y_{i+2b}Y_{i+b}Y_i}{Y_{i+2b}Y_{i+b}Y_{i+3b}} \right\|, \tag{11}$$

which has the value 1 as one can see from the substitution $i \to i + b$ in the numerator.

For (v) we substitute $a \equiv -2m$, then the first factor in (10) has the value 1 as one can see from the substitution $i \to i + m$ in the numerator. If also $m \equiv 3b$ the second factor is the same as (11) and so takes the value 1 as before. This proves (v) and completes the proof of the theorem.

It is worth remarking that whereas our theorem only states *sufficient* conditions on the parameters for the property to be true, in a case like that just considered, *necessity* is also easy to prove. The first factor on the right side of (10) takes the value 1 only if $m \equiv a$ or $a + 2m \equiv 0$ or $2a + m \equiv 0$ and this last congruence is clearly impossible for then the points $Z_i$ coincide with the vertices of the quadrangle. The second factor takes the value 1 only if $m + a \equiv 0$, or $2m + 3b + 2a \equiv 0$, or $m + 3b + a \equiv 0$. Combinations of these congruences are either inadmissible, or lead to the conditions given in the theorem. If $m \equiv a$ then either $2m \equiv 2a \equiv 0$ which is impossible, or $2m + 3b \equiv 0$ or $4m + 3b \equiv 0$, both of which lead to Theorem 2(iv). If, on the other hand, $a + 2m \equiv 0$ then either $m \equiv 0$ which is impossible, or $3b \equiv m$, or $3b \equiv 2m$, both of which lead to Theorem 2(v).

We now consider the case where $n$ is even and $n = 2m$.

THEOREM 3. *If $n = 2m$, then the second transversality property is true in at least the following cases:*

(i) $2m(m; a, b; -a, -b)$,
(ii) $2m(m; a, -a; b, -b)$,
(iii) $2m(m; a, -a; b, b + m)$, *and*
(iv) $2m(m; a, a + m; b, b + m)$.

*Here $a$ and $b$ may take any values subject to the condition that all points, lines and fractions are well-defined.*

As with Theorem 2, this theorem is true for all $n \geqslant 4$ whenever the assertions are meaningful. For $n = 4$ there are no cases, and for $n = 6, 8$ only cases 3(i) and 3(iii) lead to non-trivial assertions. For $n \geqslant 10$ all four parts of the theorem yield results; these are listed in the table at the end of the paper.

*Proof.* Since, in (i),

$$Z_i = V_{i-a}V_{i-b} \cap V_{i+m+a}V_{i+m+b}$$

and

$$Y_i = V_{i+a}V_{i+b} \cap V_{i+m-a}V_{i+m-b}$$

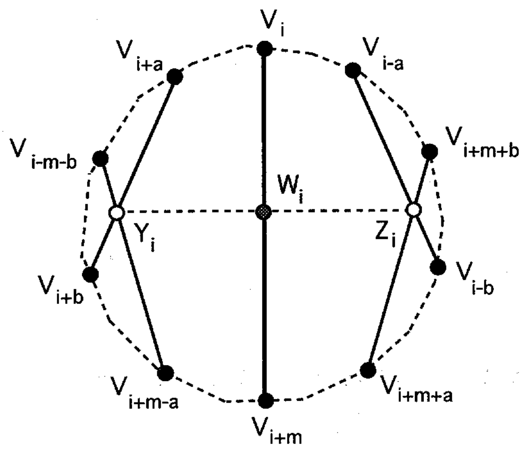


*Figure 20*. Theorem 3(i).

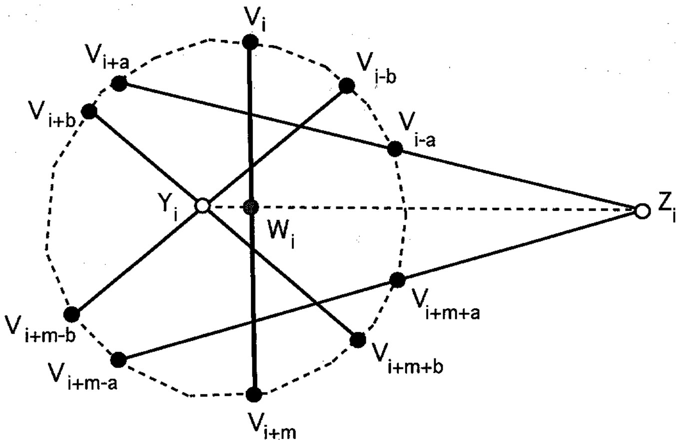


*Figure 21*. Theorem 3(ii), (iii) and (iv).

it follows that $Y_i$ and $Z_i$ are interchanged by the substitution $i \to i + m$, see Figure 20. In (ii), (iii) and (iv) if $r = a$ and $s = -a$, then

$$Z_i = V_{i-a}V_{i+a} \cap V_{i+m+a}V_{i+m-a},$$

and if $r = a, s = a + m$, then

$$Z_i = V_{i-a}V_{i-a-m} \cap V_{i+m+a}V_{i+a},$$

and so, in either cases, the substitution $i \to i+m$ leaves $Z_i$ invariant (see Figure 21). Exactly the same argument applies to $Y_i$ (with $b$ substituted for $a$). Hence in all four cases, both the line $Y_iZ_i$ and the chord $V_iV_{i+m}$ are left invariant under this substitution and therefore $W_i = W_{i+m}$. Consequently

$$\left\|\frac{V_iW_i}{W_iV_{i+m}}\right\| \cdot \left\|\frac{V_{i+m}W_{i+m}}{W_{i+m}V_i}\right\| = \left\|\frac{V_iW_i}{W_iV_{i+m}}\right\| \cdot \left\|\frac{V_{i+m}W_i}{W_iV_i}\right\| = 1. \tag{12}$$

Now

$$\prod \left\|\frac{V_iW_i}{W_iV_{i+m}}\right\|$$

is the product of $m$ pairs of factors like those on the left side of (12) and so this product has the value 1. This completes the proof of Theorem 3.

Examples of this type are easily constructed since $a$ and $b$ may take any values subject to the non-triviality conditions. Hence when $n = 2m$ is large, there are very many examples of this type.

We consider the case in which $n$ is divisible by 4 and $n = 4m$.

THEOREM 4. *If $n = 4m$, the second transversality property is true in at least the following cases:*

(i) $4m(m; a, 2m + a; -a, 2m - a)$,
(ii) $4m(m; 0, m; a, a + 2m)$, *and*
(iii) $4m(m; 0, m; a, m - a)$.

*Here a can take any value subject to the non-triviality conditions.*

The theorem leads to non-trivial assertions only if $n \geqslant 12$.

*Proof* (i). Since

$$Z_i = V_{i-a}V_{i-2m-a} \cap V_{i+m+a}V_{i-m+a}$$

and

$$Y_i = V_{i+a}V_{i-2m+a} \cap V_{i+m-a}V_{i+3m-a},$$

(see Figure 22) it follows that the substitution $i \to i + rm (r = 0, 1, 2, 3)$ leaves the line $Y_iZ_i$ invariant. Now $Y_iZ_i$ cuts the sides of the quadrangle $Q = [V_i, V_{i+m}, V_{i+2m}, V_{i+3m}]$ in $W_i$, $W_{i+m}$, $W_{i+2m}$, $W_{i+3m}$ respectively (see Figure 23). Hence by Menelaus' Theorem applied to $Q$ and the transversal $Y_iZ_i$,

$$\left\|\frac{V_iW_i}{W_iV_{i+m}}\right\| \cdot \left\|\frac{V_{i+m}W_{i+m}}{W_{i+m}V_{i+2m}}\right\| \cdot \left\|\frac{V_{i+2m}W_{i+2m}}{W_{i+2m}V_{i+3m}}\right\| \cdot \left\|\frac{V_{i+3m}W_{i+3m}}{W_{i+3m}V_i}\right\| \equiv 1. \tag{13}$$

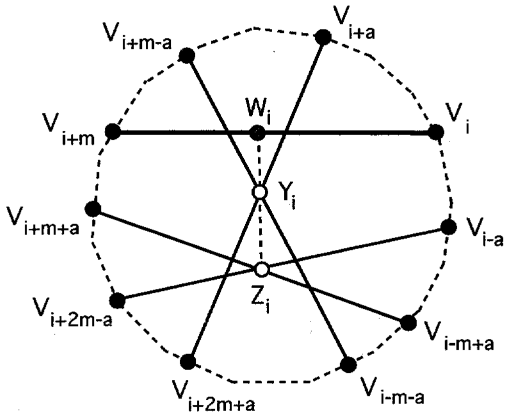


*Figure 22*. Theorem 4(i).

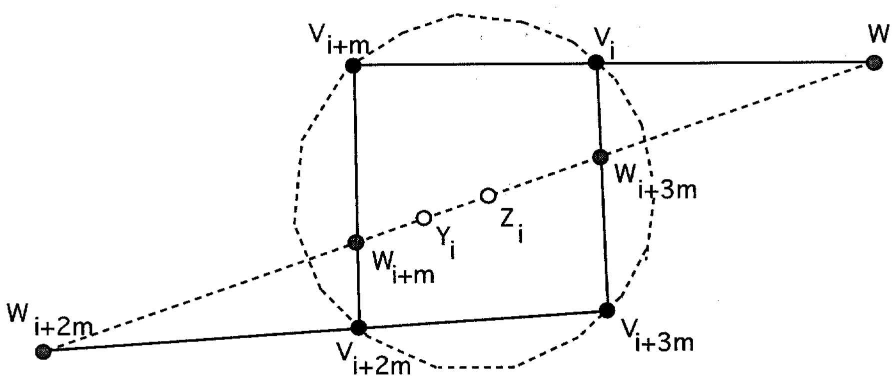


*Figure 23*. Theorem 4(i).

But

$$\prod \left\| \frac{V_i W_i}{W_i V_{i+m}} \right\|$$

is the product of $m$ sets of four factors as on the left side of (13) and so has the value 1. This proves (i).

(ii) and (iii). Here, since $Z_i = V_i V_{i-m} \cap V_{i+m} V_{i+2m}$, it follows that $Z_i = Z_{i+2m}$ and $Z_{i+m} = Z_{i+3m}$ are two of the diagonal points of the quadrangle $Q$ (see Figure 24). Since $Y_i = V_{i-a} V_{i-a-2m} \cap V_{i+m+a} V_{i+3m+a}$ in (ii), and $Y_i = V_{i-a} V_{i-m+a} \cap V_{i+m+a} V_{i+2m-a}$ in (iii), it follows that $Y_i = Y_{i+2m}$ and $Y_{i+m} = Y_{i+3m}$. Hence the line $Y_i Z_i$ meets the sides $V_i V_{i+m}$ and $V_{i+2m} V_{i+3m}$ of $Q$ in $W_i$ and $W_{i+2m}$ respectively. Similarly $Z_{i+m} Y_{i+m}$ meets the sides $V_{i+m} V_{i+2m}$ and

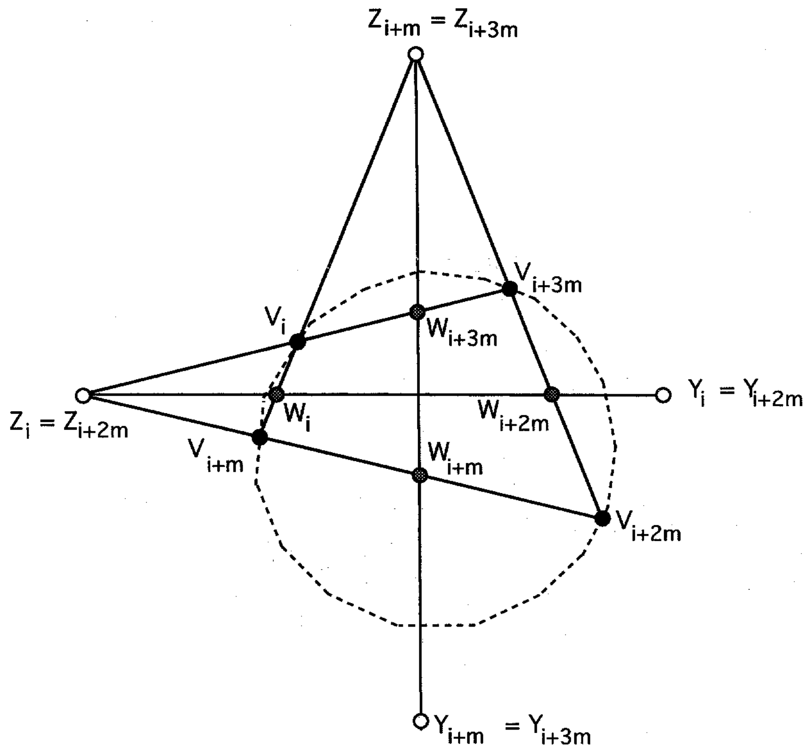


*Figure 24*. Theorem 4(ii) and (iii).

$V_{i+3m}V_i$ in $W_{i+m}$ and $W_{i+3m}$ respectively. Applying Menelaus' Theorem to $Q$ with transversal $Y_iZ_i$ we obtain

$$\left\| \frac{V_iW_i}{W_iV_{i+m}} \right\| \cdot \left\| \frac{V_{i+m}Z_i}{Z_iV_{i+2m}} \right\| \cdot \left\| \frac{V_{i+2m}W_{i+2m}}{W_{i+2m}V_{i+3m}} \right\| \cdot \left\| \frac{V_{i+3m}Z_i}{Z_iV_i} \right\| = 1. \tag{14}$$

Now $Z_i$, $V_i$, $W_{i+3m}$, $V_{i+3m}$ are in perspective from $Z_{i+m}$ with $Z_i$, $V_{i+m}$, $W_{i+m}$, and $V_{i+2m}$ respectively, and so, by the invariance of the cross ratio,

$$\left\| \frac{V_{i+m}W_{i+m}}{W_{i+m}V_{i+2m}} \right\| \Big/ \left\| \frac{V_{i+m}Z_i}{Z_iV_{i+2m}} \right\| = \left\| \frac{V_iW_{i+3m}}{W_{i+3m}V_{i+3m}} \right\| \Big/ \left\| \frac{V_iZ_i}{Z_iV_{i+3m}} \right\| = \lambda(\text{say}).$$

Hence

$$\left\| \frac{V_{i+m}Z_i}{Z_iV_{i+2m}} \right\| = \frac{1}{\lambda} \left\| \frac{V_{i+m}W_{i+m}}{W_{i+m}V_{i+2m}} \right\| \quad \text{and} \quad \left\| \frac{V_{i-m}Z_i}{Z_iV_i} \right\| = \lambda \left\| \frac{V_{i+3m}W_{i+3m}}{W_{i+3m}V_i} \right\|.$$

Substituting in (14) we obtain

$$\left\| \frac{V_iW_i}{W_iV_{i+m}} \right\| \cdot \left\| \frac{V_{i+m}W_{i+m}}{W_{i+m}V_{i+2m}} \right\| \cdot \left\| \frac{V_{i+2m}W_{i+2m}}{W_{i+2m}V_{i+3m}} \right\| \cdot \left\| \frac{V_{i+3m}W_{i+3m}}{W_{i+3m}V_i} \right\| = 1. \tag{15}$$

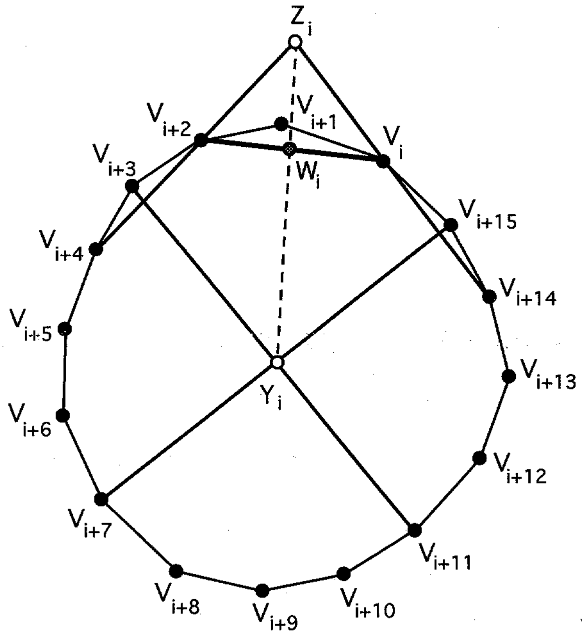


*Figure 25*. Theorem 5.

But

$$\prod \left\| \frac{V_i W_i}{W_i V_{i+m}} \right\|$$

is the product of $m$ sets of four factors as on the left side of (15), and so takes the value 1. This completes the proof of Theorem 4.

Finally we come to the one (sporadic) case of the second transversality property that is not covered by Theorems 2, 3 and 4.

THEOREM 5. *The second transversality property is true in the case*

$$16(2;0,2;1,9).$$

*Proof.* (see Figure 25). By APB for triangles with base $Y_i Z_i$,

$$\left\| \frac{V_i W_i}{W_i V_{i+2}} \right\| = - \left\| \frac{V_i Z_i Y_i}{V_{i+2} Z_i Y_i} \right\| = - \left\| \frac{V_i V_{i-2} Y_i}{V_{i+2} V_{i+4} Y_i} \right\| \cdot \left\| \frac{V_{i+2} V_{i+4} V_i}{V_{i-2} V_i V_{i+2}} \right\|,$$

where the second equality follows from the elimination lemma with $d = i$, $e = i-2$, $f = i+2$, $g = i+4$, $A = Z_i$ and $B = C = Y_i$. Taking products and rearranging

$$\prod \left\| \frac{V_i W_i}{W_i V_{i+2}} \right\| = \prod \left\| \frac{V_i V_{i-2} Y_i}{V_{i+4} V_{i+2} Y_i} \right\| \cdot \left\| \frac{V_i V_{i+2} V_{i+4}}{V_{i-2} V_i V_{i+2}} \right\|.$$

Now the second factor is clearly 1 (substitute $i \to i-2$ in the numerator). The first factor can be written

$$\left\| \frac{V_i V_{i-2} Y_i}{V_{i+4} V_{i+2} Y_{i+4}} \right\|$$

since $Y_i = Y_{i+4}$. The value of this product is 1 (substitute $i \to i+2$ in the numerator). Hence

$$\prod \left\| \frac{V_i W_i}{W_i V_{i+2}} \right\| = 1,$$

and Theorem 5 is proved.

## 5. Remarks and Comments

As remarked earlier, the completeness of our results has been checked empirically for all $n$ up to 20, and for some larger values. However, the fact that the running time of the program is proportional to the sixth power of $n$, makes exhaustive checking for large values of $n$ a practical impossibility with the computing facilities at our disposal. And this is true even with simplifications such as the following: we can delete all cases in which HCF $(n, m) = d$ and $m > d$ for these are repetitions of cases for which $m = d$.

In a few cases we have been able to back up our assertion that our enumeration is complete by theoretical considerations. For example, we have shown that the conditions in parts (iv) and (v) of Theorem 2 are necessary as well as sufficient, and there seems no way to extend the argumentation of Theorem 5 to values of n greater than 16.

The number of essentially distinct primitive cases of the second transversality property varies erratically with $n$. Apart from the sporadic cases which occur for $n = 7$ and 11, there are only five cases when $n \geqslant 7$ is prime, namely one given by each of the parts (i) to (v) of Theorem 2. But the number of cases can be quite large if $n$ is divisible by 4 or has several small factors. For example there are 72 cases when $n = 18$, and 61 when $n = 20$.

It is difficult to see any underlying pattern to our results. For example, see Figure 3. Here $n = 11$, $Z_i = V_i V_{i+10} \cap V_{i+1} V_{i+2}$, $Y_i = V_{i+4} V_{i+5} \cap V_{i+7} V_{i+8}$ and $W_i = V_i V_{i+1} \cap Y_i Z_i$. We know, by Theorem 2(iv) that the Second Transversal Theorem implies that (1) holds in this case; but it is hard to see, on general principles, why a similar result does *not* hold if, for example, we define $Y_i = V_{i+3} V_{i+4} \cap V_{i+8} V_{i+9}$ or $Y_i = V_{i+3} V_{i+5} \cap V_{i+7} V_{i+9}$, etc., instead of the above. In other words, a general explanation of the transversality phenomena, at least in the case of the Second Transversality Theorem, is still missing.

There probably exist many more configurations involving polygons in the affine plane in which a relation similar to (1) holds. We finish by giving one example

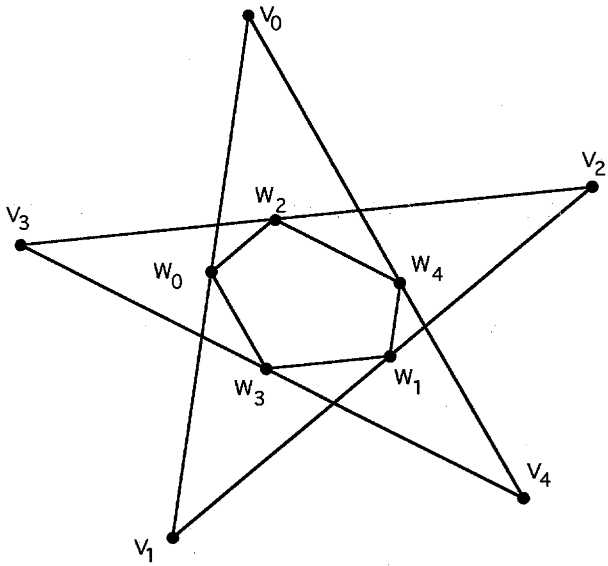


*Figure 26.* Another configuration in which $\Pi[V_i W - i/W_i V_i + 1] = 1$. Here $V_i V_{i+1}$ is parallel to $W_{i+4} W_{i+1}$.

(see Figure 26). Here the pentagram P = $[V_0, V_1, V_2, V_3, V_4]$ is circumscribed to the pentagon $Q = [W_0, W_3, W_1, W_4, W_2]$ in such a way that $V_i V_{i+1}$ passes through $W_i$ and is parallel to $W_{i+4} W_{i+1}$. Then

$$\prod \left[ \frac{V_i W_i}{W_i V_{i+1}} \right] = 1$$

as is easily proved using APH. Analogous properties hold for all $n$-gons with $n$ odd and $n \geqslant 5$.

Finally we remark that by choosing a suitable line as the 'line at infinity' and using cross-ratios instead of ratios of lengths, all the theorems of this paper can be converted into theorems in projective geometry. In many ways, however, the presentation in affine geometry which we have chosen seems simpler and intuitively more attractive.

Table I. The Second Transversality Property for $n$-gons with $4 \leqslant n \leqslant 16$.

The number after each symbol indicates the relevant theorem; thus 2(iii) indicates that the property is established in part (iii) of Theorem 2.

| | | | |
|---|---|---|---|
| $n = 4$ | | | |
| 4(1; 0, 1; 0, 2) | 2(i) | | |
| $n = 5$ | | | |
| 5(1; 0, −2; 0, −4)[a] | 2(i) | | |
| $n = 6$ | | | |
| 6(3; 0, 1; 0, 2) | 2(i) | 6(3; 0, 1; 1, −1) | 2(ii) |
| 6(3; 0, 2; 2, 4) | 2(iii) | 6(3; 0, 1; 0, −1) | 3(i) |
| 6(3; 0, 2; 0, −2) | 3(i) | 6(3; 1, −1; −1, 2) | 3(iii) |
| $n = 7$ | | | |
| 7(1; 0, 2; 0, 4)[a] | 2(i) | 7(1; 0, 2; 2, −2) | 2(ii) |
| 7(1; 0, 4; 4, 1) | 2(iii) | 7(1; 0, 1; 4, 5) | 2(iv) |
| 7(1; 0, 1; 1, 4) | 2(vi) | 7(1; 0, −2; −2, 2) | 2(vi) |
| $n = 8$ | | | |
| 8(1; 0, −3; 0, −6) | 2(i) | 8(1; 0, −3; −3, 3) | 2(ii) |
| 8(1; 0, 2; 2, 4) | 2(iii) | 8(1; 0, 1; 2, 3)[a] | 2(iv) |
| 8(1; 0, −2; 3, 1) | 2(v) | 8(1; 0, −2; −2, 3) | 2(vi) |
| 8(4; 0, 1; 0, −1) | 3(i) | 8(4; 0, 3; 0,−3) | 3(i) |
| 8(4; 1, −1; −1, 3) | 3(iii) | | |
| $n = 9$ | | | |
| 9(3; 0, −1; 0, −2) | 2(i) | 9(3; 0, 2; 0, 4) | 2(i) |
| 9(3; 0, −4; 0, −8) | 2(i) | 9(3; 0, −1;−1, 1)[a] | 2(ii) |
| 9(3; 0, 2; 2, −2) | 2(ii) | 9(3; 0, −4; −4, 4) | 2(ii) |
| 9(3; 0, 1; 1, 2) | 2(iii) | 9(3; 0, 4; 4, 8) | 2(iii) |
| 9(3; 0, −2; −2, −4) | 2(iii) | | |
| $n = 10$ | | | |
| 10(1; 0, 3; 0, 6) | 2(i) | 10(1; 0, 3; 3, −3) | 2(ii) |
| 10(1; 0, −4; −4, −8) | 2(iii) | 10(1; 0, 1; 2, 3) | 2(iv) |
| 10(1; 0, −2; 7, 5) | 2(v) | 10(1; 0, −2; −2, 4) | 2(vi) |
| 10(5; 0, 1; 0, −1) | 3(i) | 10(5; 0, 2; 0, −2) | 3(i) |
| 10(5; 0, 3; 0, −3) | 3(i) | 10(5; 0, 4; 0, −4) | 3(i) |
| 10(5; 1, 2; −1, −2) | 3(i) | 10(5; 1, −2; −1, 2) | 3(i) |
| 10(5; 1, 3; −1, −3) | 3(i) | 10(5; 1, −3; −1, 3) | 3(i) |
| 10(5; 1, −1; 2, −2) | 3(ii) | 10(5; 1, −1; −1, 4) | 3(iii) |
| 10(5; 1, −1; −2, 3) | 3(iii) | 10(5; 2, −2; −1, 4) | 3(iii) |
| 10(5; 2, −2; −2, 3) | 3(iii) | 10(5; −1, 4; −2, 3) | 3(iv) |
| $n = 11$ | | | |
| 11(1; 0, −4; 0, −8) | 2(i) | 11(1; 0, −4; −4, 4) | 2(ii) |
| 11(1; 0, 3; 3, 6) | 2(iii) | 11(1; 0, 1; 3, 4)[a] | 2(iv) |
| 11(1; 0, −2; 4, 2) | 2(v) | 11(1; 0, 1; 1, 6) | 2(vi) |

Table I *contd.*

| | | | |
|---|---|---|---|
| $n = 12$ | | | |
| 12(3; 0, −1; 0, −2) | 2(i) | 12(3; 0, −5; 0, −10) | 2(i) |
| 12(3; 0, −1; −1, 1) | 2(ii) | 12(3; 0, −5; −5, 5) | 2(ii) |
| 12(3; 0, 2; 2, 4) | 2(iii) | 12(3; 0, −2; −2, −4) | 2(iii) |
| 12(3; 0, 3; 2, 5) | 2(iv) | 12(3; 0, 3; −2, 1) | 2(iv) |
| 12(3; 0, −6; 1, −5) | 2(v) | 12(3; 0, −6; 5, −1) | 2(v) |
| 12(6; 0, 1; 0, −1) | 3(i) | 12(6; 0, 5; 0, −5) | 3(i) |
| 12(6; 1, 2; −1, −2) | 3(i) | 12(6; 1, −2; −1, 2) | 3(i) |
| 12(6; 1, 4; −1, −4) | 3(i) | 12(6; −1, 4; 1, −4) | 3(i) |
| 12(6; 1, −1; 2, −2) | 3(ii) | 12(6; 1, −1; −1, 5) | 3(iii) |
| 12(6; 1, −1; −2, 4) | 3(iii) | 12(6; 2, −2; −1, 5) | 3(iii) |
| 12(6; −1, 5; −2, 4) | 3(iv) | 12(3; 1, 7; −1, 5) | 4(i) |
| 12(3; 0, 3; −1, 5) | 4(ii) | 12(3; 0, 3; 1, 7) | 4(ii) |
| 12(3; 0, 3; 1, 2) | 4(iii) | 12(3; 0, 3; −1, 4) | 4(iii) |
| $n = 13$ | | | |
| 13(1; 0, 4; 0, 8) | 2(i) | 13(1; 0, 4; 4,−4) | 2(ii) |
| 13(1; 0, −5; −5, −10) | 2(iii) | 13(1; 0, 1; 8, 9) | 2(iv) |
| 13(1; 0, −2; 9, 7) | 2(v) | | |
| $n = 14$ | | | |
| 14(1; 0, −5; 0, −10) | 2(i) | 14(1; 0, −5; −5, 5) | 2(ii) |
| 14(1; 0, 4; 4, 8) | 2(iii) | 14(1; 0, 1; 4, 5) | 2(iv) |
| 14(1; 0, −2; 5, 3) | 2(v) | 14(7; 0, 1; 0, −1) | 3(i) |
| 14(7; 0, 2; 0, −2) | 3(i) | 14(7; 0, 3; 0, −3) | 3(i) |
| 14(7; 0, 4; 0, −4) | 3(i) | 14(7; 0, 5; 0, −5) | 3(i) |
| 14(7; 0, 6; 0, −6) | 3(i) | 14(7; 1, 2; −1, −2) | 3(i) |
| 14(7; 1, −2; −1, 2) | 3(i) | 14(7; 1, 3; −1, −3) | 3(i) |
| 14(7; 1, −3; −1; 3) | 3(i) | 14(7; 1, 4; −1, −4) | 3(i) |
| 14(7; 1, −4; −1, 4) | 3(i) | 14(7; 1, 5; −1, −5) | 3(i) |
| 14(7; 1, −5; −1, 5) | 3(i) | 14(7; 2, 3; −2, −3) | 3(i) |
| 14(7; 2, −3; −2, 3) | 3(i) | 14(7; 2, 4; −2, −4) | 3(i) |
| 14(7; 2, −4; −2, 4) | 3(i) | 14(7; 1, −1; 2, −2) | 3(ii) |
| 14(7; 1, −1; 3, −3) | 3(ii) | 14(7; 2, −2; 3, −3) | 3(ii) |
| 14(7; 1, −1; −1, 6) | 3(iii) | 14(7; 1, −1; −2, 5) | 3(iii) |
| 14(7; 1, −1; −3, 4) | 3(iii) | 14(7; 2, −2; −1, 6) | 3(iii) |
| 14(7; 2, −2; −2, 5) | 3(iii) | 14(7; 2, −2; −3, 4) | 3(iii) |
| 14(7; 3, −3; −1, 6) | 3(iii) | 14(7; 3, −3; −2, 5) | 3(iii) |
| 14(7; 3, −3; −3, 4) | 3(iii) | 14(7; −1, 6; −2, 5) | 3(iv) |
| 14(7; −1, 6; −3, 4) | 3(iv) | 14(7; −2, 5; −3, 4) | 3(iv) |
| $n = 15$ | | | |
| 15(3; 0, −1; 0, −2) | 2(i) | 15(3; 0, 4; 0, 8) | 2(i) |
| 15(3; 0, −1; −1, 1) | 2(ii) | 15(3; 0, 4; 0, −4) | 2(ii) |
| 15(3; 0, −2; −2, −4) | 2(iii) | 15(3; 0, −7; −7, −14) | 2(iii) |
| 15(3; 0, 3; −2, 1) | 2(iv) | 15(3; 0, 3; 8, 11) | 2(iv) |
| 15(3; 0, −6; 1, −5) | 2(v) | 15(3; 0, −6; 11, 5) | 2(v) |

Table I *contd*.

| | | | |
|---|---|---|---|
| $n = 16$ | | | |
| 16(1; 0, 5; 0, 10) | 2(i) | 16(1; 0 −11; −11, 11) | 2(ii) |
| 16(1; 0, −6; −6, −12) | 2(iii) | 16(1; 0, 1; 10, 11) | 2(iv) |
| 16(1; 0, −2; 6, 4) | 2(v) | 16(8; 0, 1; 0, −1) | 3(i) |
| 16(8; 0, 3; 0, −3) | 3(i) | 16(8; 0, 5; 0, −5) | 3(i) |
| 16(8; 0, 7; 0, −7) | 3(i) | 16(8; −1, 6; 1, −6) | 3(i) |
| 16(8; 1, 6; −1,−6) | 3(i) | 16(8; 1, 2; −1, −2) | 3(i) |
| 16(8; 1, −2; −1, 2) | 3(i) | 16(8; 1, 3; −1, −3) | 3(i) |
| 16(8; 1, −3;−1, 3) | 3(i) | 16(8; 1, 5; −1, −5) | 3(i) |
| 16(8; 1, −5;−1, 5) | 3(i) | 16(8; 2, 3; −2, −3) | 3(i) |
| 16(8; 2, −3; −2, 3) | 3(i) | 16(8; 2, 5; −2, −5) | 3(i) |
| 16(8; 2, −5; −2, 5)[a] | 3(i) | 16(8; 1, −1; 2, −2) | 3(ii) |
| 16(8; 1, −1; 3, −3) | 3(ii) | 16(8; 2, −2; 3, −3) | 3(ii) |
| 16(8; 1, −1; −1, 7) | 3(iii) | 16(8; 1, −1; −2, 6) | 3(iii) |
| 16(8; 1, −1; −3, 5) | 3(iii) | 16(8; 2, −2; −1, 7) | 3(iii) |
| 16(8; 2, −2; −3, 5) | 3(iii) | 16(8; 3, −3; −2, 6) | 3(iii) |
| 16(8; 3, −3; −1, 7) | 3(iii) | 16(8; 3, −3; −3, 5) | 3(iii) |
| 16(8; −1, 7; −2, 6) | 3(iv) | 16(8; −1, 7; −3, 5) | 3(iv) |
| 16(8; −2, 6; −3, 5) | 3(iv) | 16(4; 1, 9; −1, 7) | 4(i) |
| 16(4; 0, 4; −1, 7) | 4(ii) | 16(4; 0, 4; 1, 9) | 4(ii) |
| 16(4; 0, 4; 1, 3) | 4(iii) | 16(4; 0, 4; −1, 5) | 4(iii) |
| 16(2; 0, 2; 1, 9) | 5 | | |

[a]Diagrams illustrate these cases.

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