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FACE-TRANSITIVE POLYHEDRA WITH RECTANGULAR FACES

H. S. M. Coxeter, F.R.S.C. and Branko Grünbaum[1]

Abstract. Isohedral polyhedra with non-square rectangles as faces are described.

Polyhedra endowed with symmetry properties have attracted attention since antiquity. In more modern times the investigations extended to polyhedra with self-intersections, and also to isohedral and to isogonal polyhedra, in which the symmetries are assumed only to act transitively on faces or on vertices. A culmination of one direction was the determination of all uniform polyhedra, that is, possibly self-intersecting isogonal polyhedra in which the faces are regular polygons. The enumeration itself was given by Coxeter, Longuet-Higgins and Miller [2], and its completeness was established by Sopov [5] and Skilling [4]. Wenninger [6] presents photos of cardboard models of these uniform polyhedra, and instructions for building them; Har'El [3] describes methods for calculating and drawing the uniform polyhedra and their isohedral reciprocals. Among the latter are several rhombic isohedra (see Coxeter [1]), that is, isohedral polyhedra in which the faces are congruent rhombi. However, although various other isohedra have been described as well, isohedra having nonsquare rectangles as faces seem not to have been considered. (The polyhedron shown in Figure 1 and mentioned in in the literature does not contradict this statement: each of its faces is a pentagon, the polyhedron is equivalent to the regular dodecahedron, and the apparent rectangular shape is accidental.) We shall describe here two isohedral polyhedra having as faces non-square rectangles. Several other polyhedra of this kind exist, but due to their complicated structure they will be described separately.

The polyhedron P_1 shown in Figure 2 (and, partially disassembled, in Figure 3) has 24 rectangles as faces, and its symmetry group is the full octahedral group [3, 4] in the Coxeter notation and $m3m$ or O_h in the crystallographic notation. The density at the center of P_1 is 5. P_1 has two orbits of vertices, one coinciding with the vertices of a cube, the other with those of a regular octahedron. The faces themselves are rectangles with ratio of sides $\sqrt{8/3} = 1.632993...$. The interpenetration of the faces of P_1 is quite complex; it is shown in Figure 4.

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Another aspect of the complexity of P_1 is indicated by its reciprocal P_1*, which is isogonal and is shown in Figure 5. Each of the 24 short edges of P_1 joins a vertex of the octahedron to a vertex of the cube. Of the 24 long edges, 12 are edges of the octahedron, while 12 are diagonals of the cube. In the sense that the 6 faces of a cube are arranged in 3 "zones" of 4, the faces of P_1 are arranged in 8 narrow zones of 3 and in 6 wide zones of 4. The map corresponding to P_1 has genus 6; it is shown in Figure 6.

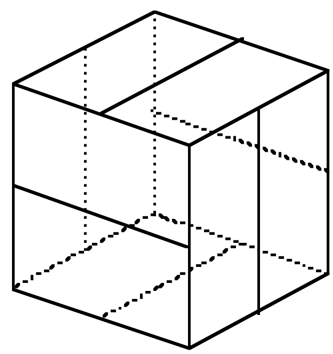
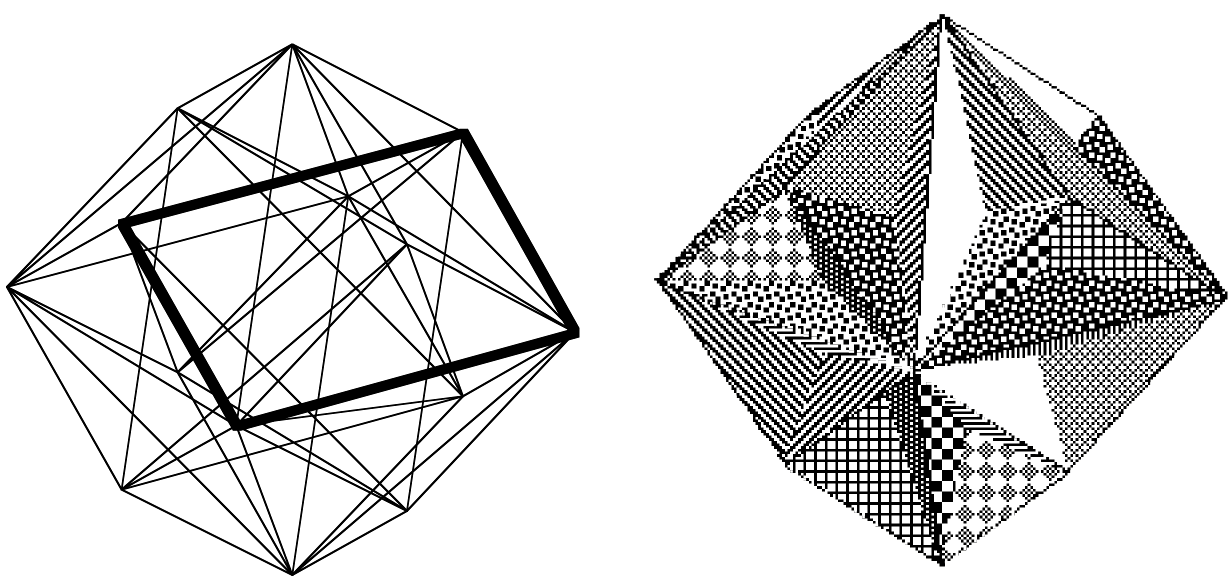


Figure 1. An isohedral polyhedron with faces that have rectangular shape but in fact are pentagons with pairs of collinear edges. Regarded as a map, it is equivalent to the regular dodecahedron {5, 3}.



(a) (b)

Figure 2. (a) A skeletal view of the isohedral polyhedron P_1 with 24 rectangular faces. One of the faces is emphasized by heavy lines. (b) A perspective view of a cardboard model of the same polyhedron P_1.

The stabilizer of each face of P_1 contains a reflection; by subdividing each face along this mirror we obtain an isohedral polyhedron with 48 rectangular faces.

The second isohedral polyhedron P_2 has 24 2-by-1 rectangles as faces; they lie by fours in the faces of a cube of edge 2. This orientable polyhedron of density 2 is illustrated in Figure 7. Its symmetry group is the full octahedral group, and its vertices form two orbits; eight coincide with the vertices of a cube, and twelve are at midpoints of the cube's edges. Each face of P_2 has one long edge coinciding with an edge of the cube, and the other long edge coincides with a midline of a face of the cube. Each of the two short edges of a face of P_2 connects a vertex of the cube with a vertex at the midpoint of an edge of the cube. There is a lot of overlap among the elements of P_2, but this leads to no problems.

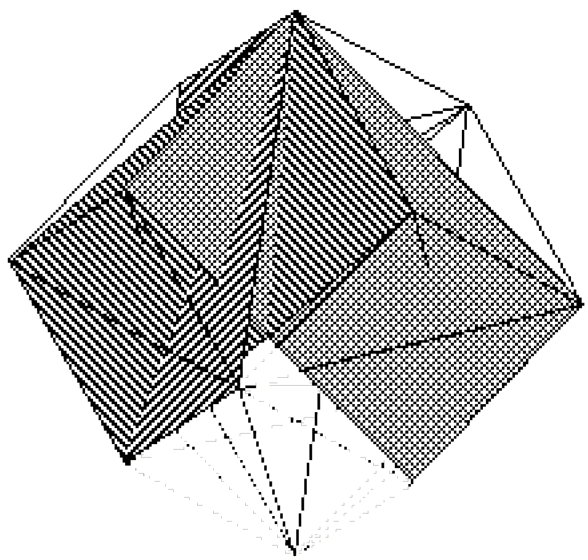


Figure 3. The skeleton and several faces of the polyhedron P_1, which illustrate their interpenetration.

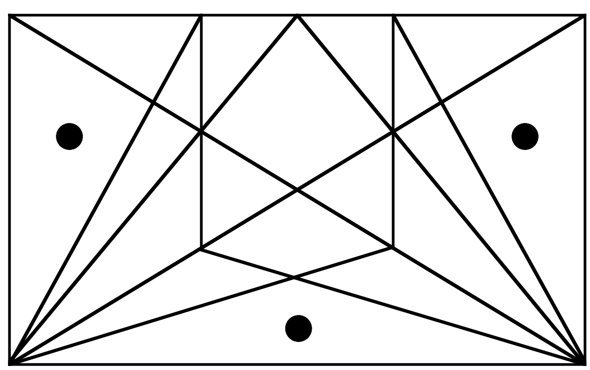
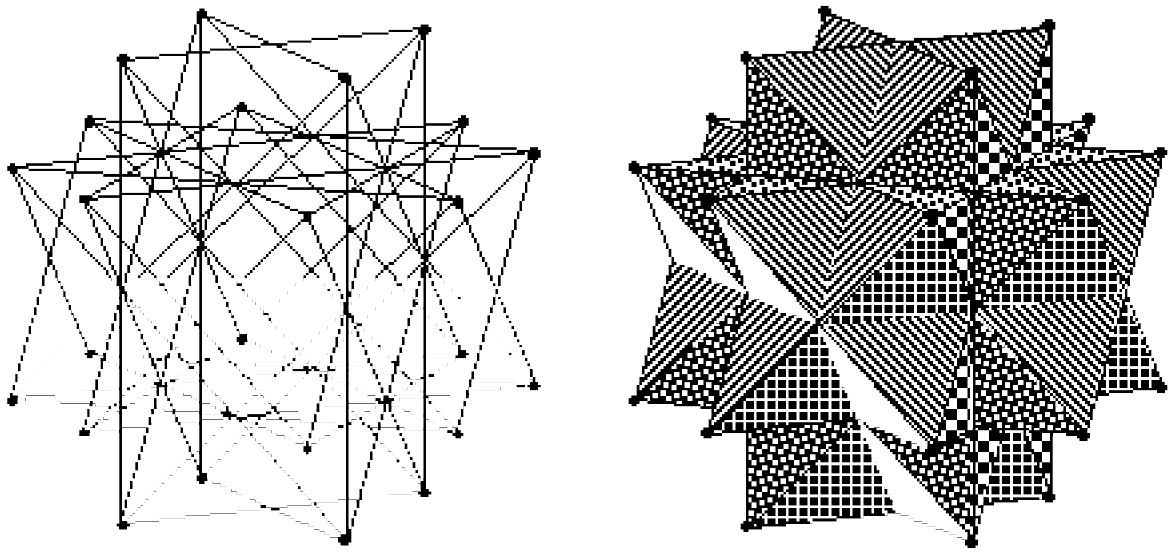


Figure 4. One face of the polyhedron P_1, with the traces of the faces that intersect it. The top edge is "hidden" in the cardboard model, in which only the three triangles marked by bullets are visible from the "outside".

A map equivalent to P_2 has genus 3 and is shown in Figure 8; it may as well be used as a net for the polyhedron, if the near-rectangles are replaced by rectangles. The faces of P_2 are arranged in 6 narrow zones of 4 and in 3 wide zones of 8.



(a) (b)

Figure 5. (a) A skeletal view of the isogonal polyhedron P_1* which is reciprocal to P_1. Two faces, one hexagonal and one octagonal, are emphasized. (b) A perspective view of a cardboard model of the same polyhedron P_1*.

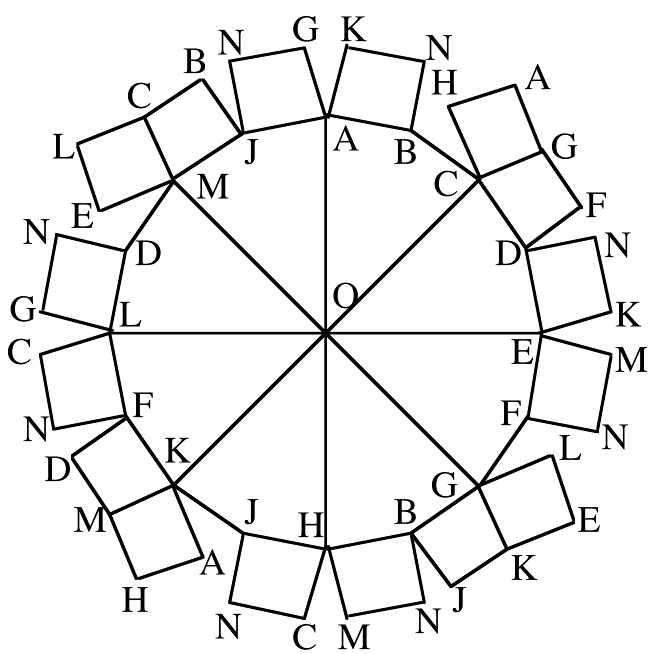


Figure 6. The map corresponding to the polyhedron P_1. The lettering defines the identification of vertices.

If the faces of P_2 are subdivided into two squares each, there results an orientable isohedral polyhedron of genus 3 and density 2 with 48 faces. Its appearance is just like Figure 7, but all elements except the vertices of the cube are doubled up. By a slight modification of the construction one may obtain orientable isohedral polyhedra with $24k$ square faces, of density k and genus $3k$–3, for every $k \geq 1$.

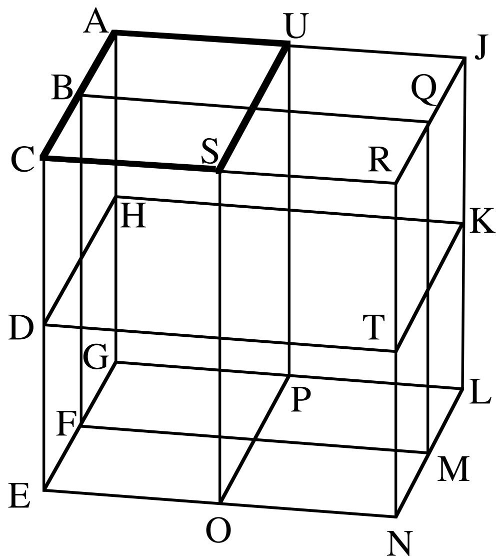


Figure 7. The polyhedron P_2 which has as faces 24 rectangles of size 2-by-1. One face is emphasized by heavy lines.

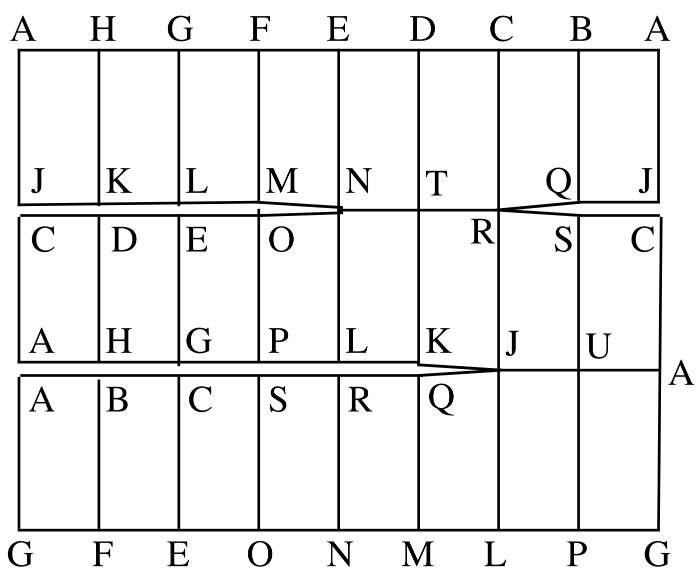


Figure 8. A map equivalent to the polyhedron P_2.

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H. S. M. Coxeter
Department of Mathematics
University of Toronto
Toronto, Ontario
Canada M5S 1A1

Branko Grünbaum
Department of Mathematics
University of Washington, Box 354350,
Seattle, WA 98195-4350, USA
e-mail: grunbaum@math.washington.edu