PARTITIONS OF MASS-DISTRIBUTIONS AND OF CONVEX BODIES BY HYPERPLANES

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1. Introduction. The following results are well-known (Neumann [7]; Eggleston [3], [4, p. 125–126], [5, p. 118]; Newman [8]:

(A) For any mass-distribution in the plane, such that the total mass contained in every half-plane is finite and depends continuously on the position of the half-plane, there exists a point P such that each half-plane which contains P, contains at least 1/3 of the total mass.

(B) For any convex body K in the plane there exists a point P such that for each half-plane H containing P the area of $H \cap K$ is at least 4/9 of the area of K.

The main object of the present note is to generalize (A) and (B) to higher-dimensional Euclidean spaces.

In the following m shall denote a fixed (non-negative) finite measure on the ring of subsets of E^n generated by the closed half-spaces in E^n. (For the terminology and results on measures see, e.g., Halmos [6].)

For a real λ, $0 \leqq \lambda \leqq 1/2$, we define $\mathscr{C}(m, \lambda)$ as the subset of E^n consisting of those points $P \in E^n$ which satisfy the condition: For any closed half-space $H \subset E^n$, with $P \in H$, the relation $m(H) \geqq \lambda \cdot m(E^n)$ holds.

Obviously, $\mathscr{C}(m, \lambda)$ is a compact, convex (possibly empty) set.

Using the notation of $\mathscr{C}(m, \lambda)$, Theorem (A) may be extended as follows:

THEOREM 1. *$\mathscr{C}(m, 1/(n+1)) \neq \phi$ for any measure m in E^n.*

Let $V(S)$ denote the volume (n-dimensional Lebesgue measure) of the set $S \subset E^n$. For any convex body $K \subset E^n$, we denote by m_K the measure (defined for all Lebesgue measurable subsets S of E^n) obtained by taking $m_K(S) = V(S \cap K)$. We denote $\mathscr{C}(m_K, \lambda)$ by $\mathscr{C}(K, \lambda)$.

Theorem (B) may now be generalized as follows:

THEOREM 2. *If K is any convex body in E^n then*

$$\mathscr{C}\left(K, \left(\frac{n}{n+1}\right)^n\right) \neq \phi .$$

We shall prove Theorems 1 and 2 in the following two sections.

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The last section contains remarks and comments.

2. Proof of Theorem 1.[1] If v is a unit vector (in E^n) and α is a real number, let $H(v, \alpha)$ be the closed half-space

$$H(v, \alpha) = \{x \in E^n; (x, v) \leqq \alpha\}.$$

Let $\alpha(v)$ be defined by

$$\alpha(v) = \min\left\{\alpha; m(H(v, \alpha)) \geqq \frac{n}{n+1} m(E^n)\right\},$$

(the minimum is attained since $m(H(v, \alpha))$ is continuous to the right as a function of α). Let $H(v) = H(v, \alpha(v))$ and

$$H^*(v) = \{x \in E^n; (x, v) \geqq \alpha(v)\}.$$

(Without loss of generality we shall in the sequel assume $m(E^n) = 1$.) Obviously,

$$\mathscr{C}\left(m \frac{1}{(n+1)}\right) \supset \bigcap_v H(v);$$

hence, if $\bigcap_v H(v) \neq \phi$ the proof is complete. On the other hand, if $\bigcap_v H(v) = \phi$, we shall show that

$$C\left(m \frac{1}{(n+1)}\right) \neq \phi$$

in the following way. The half-spaces $H(v)$ are closed convex sets, and it is easily seen that a finite number of them may be found such that their intersection is compact. By Helly's theorem on intersections of convex sets (see, e.g., Rademacher-Schoenberg [9]) the assumption $\bigcap_v H(v) = \phi$ implies the existence of an $n+1$ membered family of unit vectors v_i, $0 \leqq i \leqq n$, such that $\bigcap_{i=0}^n H(v_i) = \phi$. Using an inductive argument it is easily seen that we may assume that every n of the vectors v_i are linearly independent. Therefore (denoting $H_i = H(v_i)$ and $H_i^* = H_i^*(v_i)$) the set $S = \bigcap_{i=0}^n H_i^*$ is a non-degenerate simplex whose faces are contained in the hyperplanes $H_i \cap H_i^*$, $0 \leqq i \leqq n$. By the definition of $\alpha(v)$ we have $m(H_i^*) \geqq 1/(n+1)$ and $m(\operatorname{Int} H_i^*) \leqq 1/(n+1)$ for all i. Therefore $m(H_j \cap \operatorname{Int} H_i^*) \leqq 1/(n+1)$, and thus $m(H_j \cap H_i) \geqq (n-1)/(n+1)$ for all $i \neq j$. Now, since $\bigcap_{i=0}^n H_i = \phi$, we have

$$\frac{n}{n+1} \geqq m(H_i) \geqq m\left[H_i \cap \left(\bigcup_{j \neq i} H_j\right)\right] \geqq \frac{1}{n-1} \sum_{\substack{0 \leqq j \leqq n \\ j \neq i}} m(H_i \cap H_j)$$
$$\geqq \frac{1}{n-1} \cdot n \cdot \frac{n-1}{n+1} = \frac{n}{n+1}.$$

[1] The author is indebted to Professor B. M. Stewart for the correction of an error in the original proof.

Thus, for all i, equality signs hold throughout. In particular,

$$m\Big(\bigcap_{\substack{0\leqq j\leqq n\\ j\neq i}} H_j\Big) = \frac{1}{n+1}$$

for all i (i.e., the support of m is contained in the "vertex-regions" of the simplex $S = \bigcap_i H_i^*$), and it is immediately verified that

$$\mathscr{C}\Big(m; \frac{1}{(n+1)}\Big) \supset S \neq \phi \ .$$

This ends the proof of Theorem 1.

3. **Proof of Theorem 2.** Let G_k denote the centroid of the convex body $K \subset E^n$. We shall prove Theorem 2 by establishing the stronger statement $G_K \in \mathscr{C}(K, \alpha_n)$, where $\alpha_n = (n/(n+1))^n$. Assuming, to the contrary, that $G_k \notin \mathscr{C}(K, \alpha_n)$, there exists a hyperplane L containing G_K such that the volume of the part of K contained in one of the half-spaces determined by L is less than $\alpha_n \cdot V(K)$. We shall obtain a contradiction from this assumption.

Let G_K be the origin of an orthogonal system of coordinates $(x_1, \cdots, x_n)$ of E^n, such that L is the hyperplane determined by $x_1 = 0$.

Let H^+ be the half-space $\{(x_1, \cdots, x_n); x_1 \geqq 0\}$ and H^- the other closed half-space determined by L. We may assume that $V(K \cap H^-) < \alpha_n \cdot V(K)$. For any set $S \subset E^n$ we shall use the notations $S^- = S \cap H^-$ and $S^+ = S \cap H^+$. Let $\hat{K}$ be the set obtained from K by spherical symmetrization ("Schwarzsche Abrundung", Bonnesen-Fenchel [1, p. 71]; "Schwarz rotation process", Eggleston [5, p. 100]) with respect to the x_1-axis (i.e., $\hat{K}$ is the union of the $(n-1)$-dimensional spheres obtained by taking in each hyperplane $L_t = \{(x_1, \cdots, x_n); x_1 = t\}$ an $(n-1)$-dimensional sphere with center $(t, 0, \cdots, 0)$ and $(n-1)$-dimensional volume equal to that of $K \cap L_t$). It is well known that $\hat{K}$ is a convex body, and obviously $V(\hat{K}^-) = V(K^-)$, $V(\hat{K}^+) = V(K^+)$ and $G_{\hat{K}} = G_K$. Therefore $V(\hat{K}^-) < \alpha_n \cdot V(\hat{K})$ and $G_{\hat{K}} \notin \mathscr{C}(\hat{K}, \alpha_n)$. Let C^- denote the (orthogonal) hypercone with base $\hat{K} \cap L$ and vertex $(c, 0, \cdots, 0) \in H^-$, where c is chosen in such a way that $V(C^-) = V(\hat{K}^-)$. Let C be the hypercone obtained by extending C^- (along its generators) into H^+ in such a way that $V(C^+) = V(\hat{K}^+)$. With C thus defined, it is easily verified that the x_1-coordinate of G_{C^-} (resp. G_{C^+}) is not greater than that of $G_{\hat{K}^-}$ (resp. $G_{\hat{K}^+}$). Therefore, $G_C \in H^-$, and thus the hyperplane L^*, parallel to L and passing through G_C, divides C into two parts in such a way that the part contained in H^- has a volume smaller than $\alpha_n \cdot V(C)$. But by a simple computation we find (since the centroid of a hypercone divides its height in the ratio $1 : n$) that the volume in question equals $\alpha_n \cdot V(C)$. The contradiction reached proves the theorem.

4. Remarks. (i) It is very easy to find examples which show that the bounds in Theorems 1 and 2 are the best possible. From the proofs given, it is also easy to deduce that if $\mathscr{C}(K, \alpha_n + \varepsilon) = \phi$ for all $\varepsilon > 0$ then K is a simplex, and that $\mathscr{C}(m, 1/(n+1) + \varepsilon) = \phi$ for all $\varepsilon > 0$ only if the support of m is contained in the "vertex-regions" of some (possibly degenerate) simplex, and all the "vertex-regions" have the same measure.

(ii) The proof of Theorem 1 may be somewhat simplified if the measure m is assumed to be continuous (as in Theorem (A)). The advantage of the more general form is that it includes, e.g., measures generated by finite point-sets, surface-area etc.

(iii) The following obvious corollary of Theorem 2 is interesting because of its independence on the dimension:

For any convex body $K \subset E^n$ we have

$$G_K \in \mathscr{C}(K, e^{-1}) = C(K, 0.3678\cdots) .$$

(iv) It would be interesting to find the analogue of Theorem 2 obtained by substituting the $(n-1)$-dimensional surface area $A(K)$ for the volume $V(K)$ of $K \subset E^n$. The problem seems to be unsolved even for $n = 2$.

(v) It is easily proved that for any continuous mass-distribution in the plane there exists a pair of orthogonal lines such that each "quadrant" determined by them contains 1/4 of the total mass. The analogous statement is not true for n mutually orthogonal hyperplanes in E^n; does it become true if the condition of orthogonality is omitted?

(vi) It is well known (Buck and Buck [2]) that for any continuous mass-distribution in the plane there exist three concurrent straight lines such that each of the six "wedges" determined by them contains 1/6 of the total mass. Does this fact generalize to E^n when the three lines are replaced by $n+1$ hyperplanes with a common $(n-2)$-dimensional intersection?

Added in proof. After submitting the present note for publication, the following facts came to our attention:

(i) Theorems (A) and B are proved, and Theorem 1 suggested, in I. M. Jaglom—W. G. Boltjanski, Konvexe Figuren, Berlin, 1956, pp. 16, 18, 27, 104–106, 116, 135–136 (this is a translation of the Russian original, which appeared in 1951); Theorem (b) is there attributed (without references) to A. Winternitz.

(ii) A proof of Theorem 1 (using Brouwer's fixed-point theorem), together with some related results, was given in B. J. Birch, On 3N points in a plane, Proc. Cambridge Philos. Soc., 55 (1959), 289–293.

(iii) A proof of Theorem 2, very similar to the one given in the present paper, was found independently by P. C. Hammer; it is contained in a paper "Volumes cut from convex bodies by planes", submitted to "Mathematika".

(iv) The relation $\mathscr{C}\left(m, \frac{1}{2}\right) \neq \phi$ (resp. $\mathscr{C}\left(\mathrm{K}, \frac{1}{2}\right) \neq \phi$ holds for any distribution of masses (resp. convex body) with a center of symmetry. Obviously, $\mathscr{C}\left(m, \frac{1}{2}\right) \neq \phi$ is possible also for mass-distributions without a center. The conjecture (trivial for the plane) that $\mathscr{C}\left(K, \frac{1}{2}\right) \neq \phi$ characterizes centrally symmetric convex bodies was first established Professor F. J. Dyson; it is hoped that a proof will be published soon.

(v) Results generalizing Theorem 1 were established by R. Rado in the paper, "A theorem on general measure", J. London Math. Soc., 21 (1946), 291–300. Rado's proof also uses Helley's theorem, but in a fashion different from the one used in the present paper.

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