UNSOLVED PROBLEMS

Edited by **Richard Nowakowski**

In this department the MONTHLY presents easily stated unsolved problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial or related results. Typescripts should be sent to Richard Nowakowski, Department of Mathematics & Statistics & Computing Science, Dalhousie University, Halifax NS, Canada B3H 3J5, rjn@cs.dal.ca

Which Coronas Are Simple?

Branko Grünbaum

If $\mathcal{T}$ is an isohedral tiling of the Euclidean d-dimensional space by copies of a convex tile T, is the corona of each tile a topological disk?

A *tiling* $\mathcal{T}$ is a collection of closed topological disks with disjoint interiors (the *tiles* of $\mathcal{T}$), such that the union of all tiles is the whole space; if all tiles are congruent, then $\mathcal{T}$ is *monohedral*, and their common shape is said to be a *prototile* of $\mathcal{T}$. A tiling $\mathcal{T}$ is *isohedral* provided the symmetry group of $\mathcal{T}$ acts transitively on the tiles. For tilings of the plane ($d = 2$) these concepts are illustrated in Figures 1 and 2. The *corona* of a tile T of $\mathcal{T}$ is the union of all tiles that meet T, including T; for example, in the usual lattice tiling of the plane by unit squares the corona of

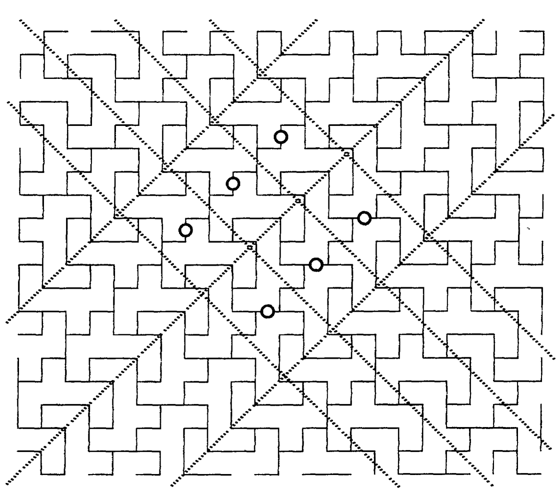

Figure 1. An example of an isohedral tiling of the plane, with a heptomino as prototile. The symmetries of the tiling are: translations in a lattice of directions (not marked), half-turns (several centers are marked by hollow dots), and glides (glide reflections) along axes in two perpendicular directions (several glide axes are marked by dotted lines).

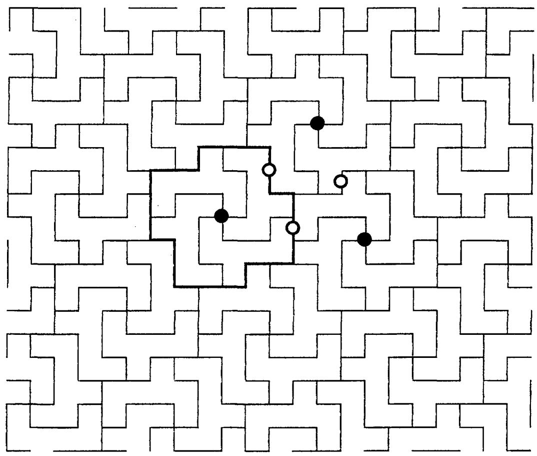

Figure 2. A monohedral but not isohedral tiling of the plane, which has as prototile the same heptomino as in the tiling in Figure 1. Besides translations, this tiling has only halfturns as symmetries (several centers are marked by solid or hollow dots). Quadruplets of tiles meeting at points such as those marked by solid dots give the impression that the tiling has 90° rotations as symmetries; however, a closer inspection reveals that such rotations are not symmetries of the tiling. This tiling was shown in Figure 9.4.3(c) of [**1**], with the assertion that it is isohedral; the error was pointed out in [**2**], where also the isohedral tiling of Figure 1 was presented.

each tile is the set of nine squares, which form the 3 by 3 square consisting of the given tile, the four tiles sharing an edge with it, and the four tiles meeting it at its vertices.

Since all types of isohedral tilings of the plane are known [**1**, Chapter 6], the affirmative answer to our question for $d = 2$ can be established by inspection, even without the assumption that the prototile is convex. However, already for $d = 3$ the answer seems not to be known. We can show by examples that in 3-dimensional space the answer is negative if either (i) nonconvex prototiles are admitted, or (ii) monohedral tilings with convex prototiles are considered.

Two copies of the prototile used in the first example are shown in Figure 3. Parts (a) and (b) give perspective views, and parts (c) and (d) schematic views from

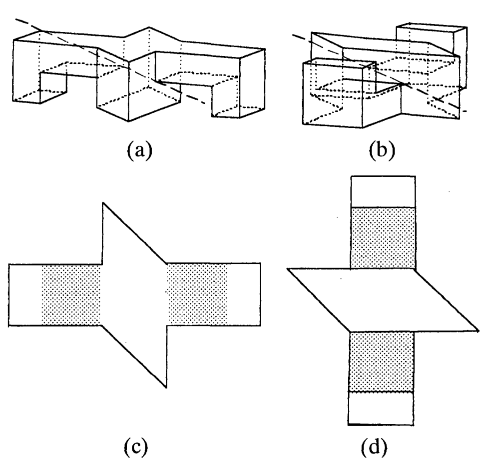

Figure 3. Perspective views (a) and (b), and schematic presentations of the prototile used in the first example.

above; the tile in (b) was obtained by turning the tile in (a) through 180° about the axis indicated by the long dashes. The shaded areas in (c) indicate the "vaults" on the bottom of the tile in (a). The shaded areas in (d) are "canals" on top of the tile in (b).

To construct the tiling, we first tile a horizontal slab by using translates of the tiles in (a) and (b). The construction of the slab is indicated schematically in Figure 4, in which the translates of the original tile are shown as in Figure 3(c), while each translate of the tile in Figure 3(b) is represented by its three parts that are visible from the top. Several examples of such tiles are indicated by three equal symbols (hollow or full squares, etc.) on these three parts.

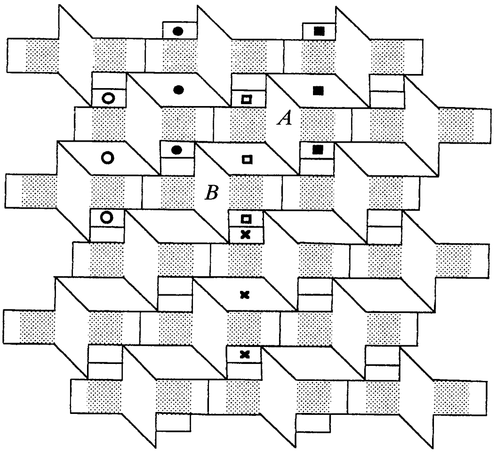

Figure 4. A slab of tiles used in the construction of the first example.

The construction of the tiling is completed by repeatedly reflecting the slab and its images in their bounding planes. Thus each tile has a partner-tile in the tiling, with which it forms a 2-hole toroidal set. The isohedrality of the tiling is obvious, and so is the fact that the corona of the tile denoted A contains the tile indicated by the hollow squares and its partner-tile; the tile B, which is not in the corona of A, shows that the corona of A is not a topological disk.

The prototile of the second example is a parallelepiped (skew box) shown in Figure 5. The tiling is even simpler to describe than the tiling just considered; it consists of horizontal layers, each layer being the familiar "herringbone" pattern. Let one layer be as shown in Figure 6 by the up-and-down zigzags of tiles (shown as transparent), and the adjacent one below it as shown by the left-and-right zigzags; repeating this part of the tiling by translation in the vertical direction gives the desired tiling. The corona of the dark tile is clearly not a topological disk.

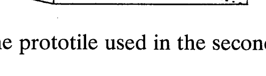

Figure 5. The prototile used in the second example.

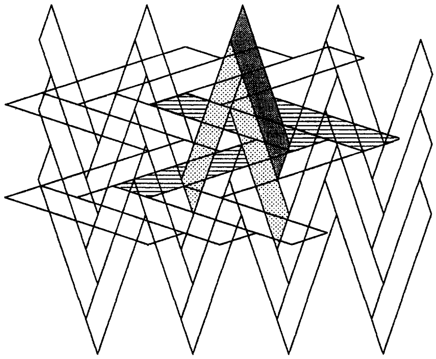

Figure 6. A schematic of the arrangement of tiles in two adjacent layers of the second example.

While it may be thought that one could find a negative answer to the original question by adjusting the shape of the prototile in Figure 5, this does not seem to be possible. In fact, we conjecture that the answer to the problem posed in the beginning is affirmative. Since a solution appears elusive, it may be worthwhile to try to settle at least the simpler variant in which the isohedral tiling by convex tiles is assumed to be *face-to-face*, which means that the intersection of any family of tiles in the tiling is a face, edge or vertex of each of the participating tiles. This case is still open in three-dimensional space. It is possible that for face-to-face tilings the requirement of isohedrality can be weakened to monohedrality.

We conjecture that in the plane every monohedral tiling has simple coronas. The Voderberg tile [**1**, page 123] shows that the proof of this conjecture may not be completely straightforward.

Another family of open questions arises if we insist on a particular symmetry group for the tiling. For example, could we require that the tiling have the symmetry group of the cubic lattice tiling, while allowing either arbitrary topological disks or only convex tiles as prototiles? What happens if no symmetry or convexity is required, but pairwise intersections of tiles are assumed to be simply connected?

REFERENCES

1. B. Grünbaum and G. C. Shephard, *Tilings and Patterns*. Freeman, New York, 1987.
2. K. Keating and A. Vince, Isohedral polyomino tiling of the plane. Preprint, June, 1997.

University of Washington, Box 354350, Seattle, WA 98195-4350
grunbaum@math.washington.edu