$$I_+(x) = \int_0^a e^{-xh_+}g\,dt, \qquad I_-(x) = \int_0^a e^{-xh_-}g\,dt.$$

Now h_+ and h_- both satisfy the conditions of Erdélyi's theorem. Thus

$$I_\pm(x) = \sum_0^{N-1} c_k x^{-(k+\lambda)/\nu} + c_N(\epsilon)x^{-(N+\lambda)/\nu} + o(x^{-(N+\lambda)/\nu}),$$

where it is easy to see from Erdélyi's proof that

(1) c_k is independent of ϵ for $k=0, 1, \cdots, N-1$,

(2) $c_N(\epsilon)$ is linear in ϵ: $c_N(\epsilon)=c_N \mp \epsilon c_N'$,

(3) the o-term depends on ϵ.

Thus, since $I_+ \leqq I_1 \leqq I_-$ we easily get

$$\left|\left(I - \sum_0^N c_k x^{-(k+\lambda)/\nu}\right) x^{(N+\lambda)/\nu}\right| \leqq \epsilon c_N' + o(1) + O(e^{-Ax}\cdot x^{(N+\lambda)/\nu}).$$

Letting $x\to\infty$ and then $\epsilon\to 0$ we see that $I=\sum_0^N c_k x^{-(k+\lambda)/\nu}+o(x^{-(N+\lambda)/\nu})$.

The precise evaluation of the c's in terms of the a's and b's is described by Erdélyi. In particular we might mention that if b_n is the first nonvanishing b then c_n is the first nonvanishing c and then $c_n=a_0^{(n+\lambda)/\nu}\Gamma((n+\lambda)/\nu)b_n/\nu$. If all the b's vanish then these same calculations show that $I(x)=o(x^{-(N+\lambda)/\nu})$.

ON A PROBLEM OF L. FEJES-TÓTH*

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L. Fejes-Tóth has shown† that a point in 3-space may be "hidden" by six congruent spheres; more precisely, six congruent, nonintersecting closed spheres in E^3 may be placed around a point P (not belonging to any of the spheres) in such a way that each ray issuing from P intersects at least one of the spheres§. He also raised the question as to the least number of spheres for which such an arrangement is possible.

We shall show that five congruent, disjoint spheres may not "hide" a point, and prove the slightly stronger

Theorem. *Let a point P and five congruent, open, disjoint spheres S_i, $1\leqq i\leqq 5$, be given in E^3. If P belongs to none of the spheres S_i then there exists a ray R, issuing from P, such that $R\cap(\bigcup_{i=1}^5 \bar{S}_i)$ is either the point P, or is empty.*

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† Unpublished to the best of our knowledge. We are indebted to Professor V. L. Klee for this information.

§ It is not difficult to see that such configurations may be obtained by arbitrarily small, suitable transformations of the six spheres of radius 1 whose centers are (in cylindrical coordinates (r, ϕ, z)): $(0, 0, 1)$, $(0, 0, -1)$, $(\sqrt{3}, \phi_0, 0)$, $(\sqrt{3}, -\phi_0, 0)$, $(\sqrt{3}, 3\phi_0, 0)$, $(\sqrt{3}, -3\phi_0, 0)$ where $\phi_0=\arccos\sqrt{\frac{2}{3}}$. The "hidden" point is $(0, 0, 0)$.

The proof of the theorem will follow from some simple lemmata.

Throughout this note we denote by S_i congruent, open, disjoint spheres, and by P a (fixed) point belonging to none of the spheres S_i. With any family of spheres S_i we associate the family of (open) spherical caps C_i on the surface of a fixed sphere S with center P, where C_i is obtained by projecting S_i onto Bd S by rays issuing at P. We denote by α_i the (angular) diameter of C_i, and by A_i the center of S_i.

LEMMA 1. *For any two spheres* S_1 *and* S_2 *we have* $\operatorname{diam}(C_1 \cap C_2) < \frac{1}{2}\pi$.

Proof. Since $\operatorname{diam}(C_1 \cap C_2)$ does not decrease if the centers A_i of the spheres S_i are moved toward P we may, without loss of generality, assume that P belongs to Bd S_1 and that the closed spheres $\overline{S}_1$ and $\overline{S}_2$ touch in a point. Let T be the plane passing through P and tangent to $\overline{S}_1$. If A_2 belongs to T or to that open half-space determined by T which contains A_1, an easy computation yields $\operatorname{diam}(C_1 \cap C_2) = \alpha_2 \leqq \arcsin \frac{2}{3}\sqrt{2} < \frac{1}{2}\pi$. On the other hand, if T separates A_1 from A_2 it follows by elementary trigonometry that the angle ϕ, spanned at P by $T \cap \overline{S}_2$, is given by

$$\phi = 2 \arcsin \frac{[(1 - \cos \vartheta) \cos \vartheta]^{1/2}}{\sin \vartheta},$$

where $\vartheta = \measuredangle PA_1A_2$. Since in this case $\operatorname{diam}(C_1 \cap C_2) \leqq \phi$, and $\phi < \frac{1}{2}\pi$ for $\vartheta > 0$, we have to settle only the case $\vartheta = 0$. But then $C_1 \cap C_2 = \varnothing$, *i.e.*, $\operatorname{diam}(C_1 \cap C_2) = 0$. This completes the proof of the lemma.

LEMMA 2. *If* $\overline{C}_1 \cap \overline{C}_2 \cap \overline{C}_3 \neq \varnothing$ *then* $\min(\alpha_1, \alpha_2, \alpha_3) \leqq \arcsin \frac{2}{3}\sqrt{2} < \frac{5}{12}\pi$.

Proof. From the assumption of the lemma it follows that there exists a ray R, issuing from P, such that $R \cap \overline{S}_i \neq \varnothing$ for $i = 1, 2, 3$. The centers A_1, A_2, A_3, of the spheres are contained in one of the closed half-spaces determined by the plane T passing through P and orthogonal to R. Let A_3 be not nearer to P than A_1 and A_2 (*i.e.*, $\alpha_3 \leqq \alpha_1, \alpha_2$). Since any motions of S_1 and S_2 (subject to the limitations $S_i \cap S_j = \varnothing$ for $i \neq j$, $\overline{S}_i \cap R \neq \varnothing$ for $i = 1, 2, 3$, and $\alpha_1, \alpha_2 \geqq \alpha_3$) do not change $\alpha_3 = \min \alpha_i$, it is easily seen that S_1 and S_2 may be moved (keeping S_3 fixed) in such a way that $A_1, A_2 \in T$ (and therefore R is their common tangent). But in this case a simple computation shows that $\alpha_3 \leqq \arcsin \frac{2}{3}\sqrt{2}$, which ends the proof of Lemma 2.

LEMMA 3. *For any three spheres* S_i, $i = 1, 2, 3$, *we have* $\min(\alpha_1, \alpha_2, \alpha_3) \leqq \frac{2}{3}\pi$.

Proof. Let T be a plane containing the centers A_i of the spheres S_i. (If the A_i are not collinear, T is unique.) If P is orthogonally projected onto T, the angles α_i increase. But if $P \in T$, then α_i obviously equals the angle spanned at P by the circle $T \cap \overline{S}_i$, and the assertion of the lemma is clearly true. Indeed, we have $\min \alpha_i < \frac{2}{3}\pi$ unless $\overline{S}_i \cap \overline{S}_j \neq \varnothing$ and P is the centroid of the (equilateral) triangle A_1, A_2, A_3, in which case $\alpha_i = \frac{2}{3}\pi$ for $i = 1, 2, 3$.

Proof of the theorem. Let us assume that there exists a configuration of five spheres S_i contradicting the theorem. If $P \in \bar{S}_1 \cap \bar{S}_2$ then $\bar{C}_3 \cap \bar{C}_4 \cap \bar{C}_5$ must cover a great circle on Bd S. But this is impossible since, by Lemma 2, $\alpha_i \leqq \arcsin \frac{2}{3}\sqrt{2} < \frac{1}{2}\pi$ for $i=3, 4, 5$. Therefore P belongs at most to one of the spheres $\bar{S}_i$. By the assumption made on S_i we have $\bigcup_{i=1}^{5} \bar{C}_i \supset$ Bd S. We shall prove this to be impossible. Let $\alpha_1 \geqq \alpha_2 \geqq \alpha_i$ for $i=3, 4, 5$. Since in this case $\operatorname{diam}(\bar{C}_1 \cap \bar{C}_2) = \operatorname{diam}(C_1 \cap C_2) < \frac{1}{2}\pi$ by Lemma 1, the complement of $C_1 \cup C_2$, and thus also $\bar{C}_3 \cup \bar{C}_4 \cup \bar{C}_5$, contains an arc $> \frac{3}{2}\pi$ of a great circle on Bd S. Now, at least two of the three numbers $\alpha_3, \alpha_4, \alpha_5$ are smaller than $\arcsin \frac{2}{3}\sqrt{2}$ (by Lemma 2) since obviously there exist points of Bd S covered by three of the sets $\bar{C}_i$ and not all such points are contained in one of the sets $\bar{C}_i$. Assume $\alpha_4, \alpha_5 \leqq \arcsin \frac{2}{3}\sqrt{2}$; then $\alpha_3 > \frac{3}{2}\pi - 2 \arcsin \frac{2}{3}\sqrt{2} > \frac{2}{3}\pi$. But since $\alpha_1 \geqq \alpha_3$, $\alpha_2 \geqq \alpha_3$, we have reached a contradiction to Lemma 3. This completes the proof of the theorem.

Remark. It is easily seen that 4 spheres of different sizes may "hide" a point. On the other hand it seems probable that 4 spheres of two (possibly also three) different sizes may not "hide" a point.

THE FALSITY OF A CERTAIN ASYMPTOTIC RELATION

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On page 56 of the booklet* by Ernst Trost there is the statement that $p_1 \cdot p_2 \cdot \cdots \cdot p_{\pi(N)} \sim e^N$, which for $N = p_n$ implies

$$M_n = \prod_{p \leqq p_n} p \sim e^{p_n}. \tag{1}$$

This statement was suspect because its acceptance together with some newly derived results of F. B. Correia implied the truth of Cramér's conjecture which is $d_n = p_{n+1} - p_n = O(\log^2 p_n)$. The following theorem is therefore offered to show the falsity of Trost's statement.

THEOREM. *The relation* (1) *is false.*

Proof. If we suppose the contrary, then $M_n = e^{p_n}(1+o(1))$ and $\log M_n = p_n + \log(1+o(1)) = p_n + o(1)$. Thus

$$\log p_{n+1} = \log M_{n+1} - \log M_n = p_{n+1} - p_n + o(1) = d_n + o(1),$$

$$\frac{d_n}{\log p_n} = \frac{\log p_{n+1}}{\log p_n} + o\left(\frac{1}{\log p_n}\right) = \frac{\log p_{n+1}}{\log p_n} + o(1).$$

Since $\log p_{n+1} \sim \log p_n$ we have $(\log p_{n+1})/(\log p_n) = 1 + o(1)$ and $d_n/\log p_n = 1 + o(1) + o(1) = 1 + o(1)$, so that $\lim_{n\to\infty} d_n/\log p_n = 1$. This is a contradiction since it is known† that $\limsup_{n\to\infty} d_n/\log p_n = \infty$.

* Primzahlen, Bd. II, Basel-Stuttgart, 1953.

† Karl Prachar, Primzahlverteilung, Berlin-Gottingen-Heidelberg, 1957, p. 157.