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STABLE COLORING

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In a recent issue of the American Mathematical Monthly, Raphael M. Robinson proposed an interesting problem [4] which he attributes to David Gale:

> The hyperbolic plane is tiled with equilateral triangles meeting seven at each vertex. Can the tiles be colored with seven colors in such a way that no two tiles of the same color meet, even at a vertex?

The problem is highly nontrivial, and Robinson's affirmative solution is very ingenious. Thus the following definition seems to lead to interesting unsolved problems.

We shall say that a planar map (or a tiling) M is *stably colored* by c colors if each country (tile) of M is assigned one of the c colors in such a way that any two countries whose closures meet are assigned different colors. The *stable coloring number* of a map M is the smallest c such that M can be stably colored by c colors. Clearly, if a map is 3-valent, its stable coloring number is the same as its chromatic number. It is equally obvious that the stable coloring number of a map of maximal valence v is greater than or equal to v. We shall say that the stable coloring number of M is *minimal* if it is equal to v.

One can check easily that of the five regular maps, the tetrahedron and dodecahedron admit no minimal stable coloring, while the cube, octahedron and icosahedron have such colorings. (A stable 5-coloring of the icosahedral map is shown in Figure 1.) Also, the three Euclidean regular tilings of the plane admit minimal stable colorings, and Robinson's result shows that the regular 7-valent tiling of the hyperbolic plane by triangles also admits a minimal stable coloring. This then leads to the following question, which is probably very hard:

Conjecture 1. If $v \geq 4$, every regular v-valent tiling of the hyperbolic plane admits a minimal stable coloring.

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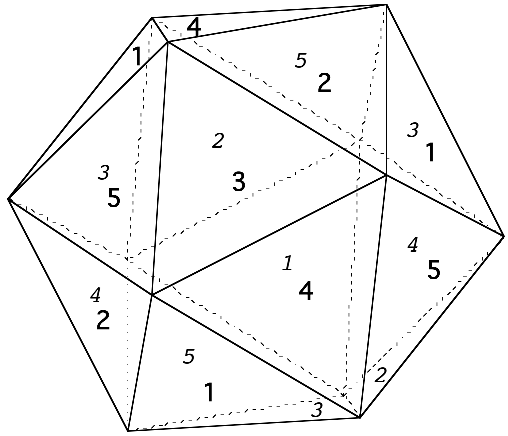

Figure 1.

We note in passing that it is well known (see, for example, [2, Section I.3.3] or [3, Section 4.7], and the references given there) that for every v and every p such that $1/v + 1/p < 1/2$ there exists a regular v-valent tiling of the hyperbolic plane in which each country (tile) is a regular p-gon. It can be checked easily that 3-valent tilings of the hyperbolic plane by regular p-gons with odd p do not admit minimal stable colorings; they do have stable 4-colorings (since every finite submap has a stable 4-coloring).

It would be of interest to find conditions under which finite maps of maximal valence v admit minimal stable colorings. Among v-valent maps in which all vertices form one transitivity class under automorphisms of the map, some admit minimal stable colorings while others need v+1 colors for a stable coloring. In Figures 2 and 3 we show one example of each possibility, for one of the more complicated maps of this kind.

This remark, and a rather large number of experiments on other maps, led to the following conjecture:

Conjecture 2. Any planar map of maximal valence v admits a stable coloring with at most v+1 colors.

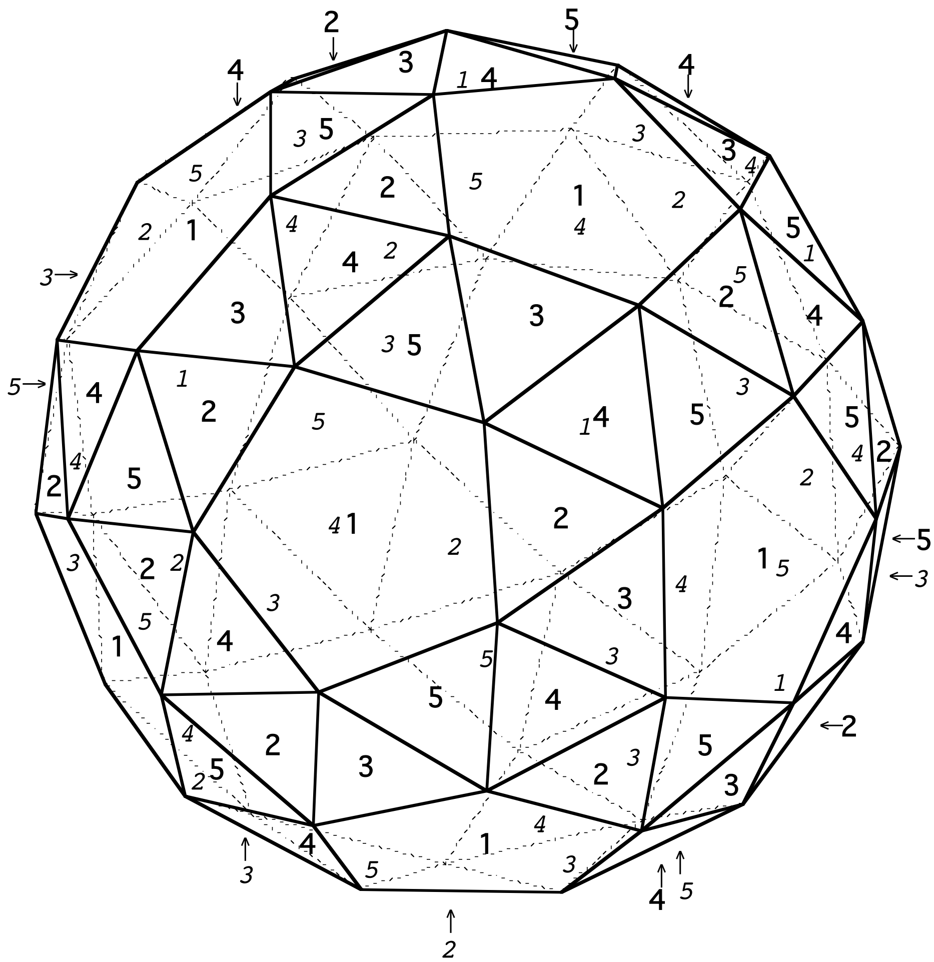

Figure 2.

Stable coloring can be investigated for maps on manifolds of any genus g, or on nonorientable surfaces. Examples show that the stable coloring numbers of such maps may grow rapidly with the genus. For instance, the regular toroidal map denoted $\{4,4\}_{3,0}$ by Coxeter and Moser [1, Section 8.3] is a 4-valent map with 9 quadrangular countries all of which must receive distinct colors in any stable coloring. At present there is too little numerical or other evidence to justify making any guesses as to the relation between genus, maximal valence and stable coloring number of maps on such manifolds.

References

[1] H. S. M. Coxeter and W. O. J. Moser, *Generators and Relations for Discrete Groups*. Springer, Berlin 1980.

[2] L. Fejes Tóth, *Regular Figures*. Pergamon, New York 1964.

[3] B. Grünbaum and G. C. Shephard, *Tilings and Patterns*. Freeman, New York 1987.

[4] R. M. Robinson, Problem 10349. Amer. Math. Monthly 100(1993), page 952.

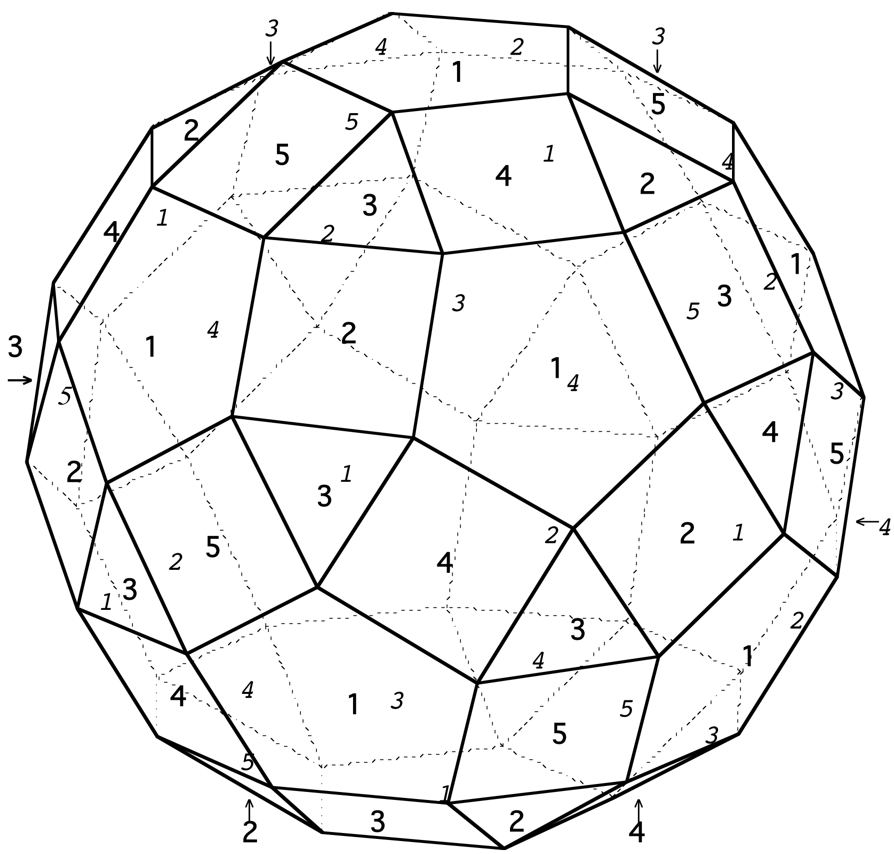

Figure 3.

Added in proof (18 March 1994). It is easy to verify that the following graph, found by Benjamin Schoenberg, is a counterexample to Conjecture 2 for $v = 4$. At present, I have no reasonable substitute for this conjecture; efforts to find examples with $v = 4$ and needing more than six colors for any stable coloring have not been successful so far. On the other hand, it is not certain that any finite number of colors will be sufficient in all cases. It would also be very interesting to investigate whether there exist counterexamples to Conjecture 2 for values of v greater than 4.