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HAMILTONIAN POLYGONS AND POLYHEDRA

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A *simple polygon* P in the plane is a collection of $n \geq 3$ distinct points $V_1, V_2, \ldots, V_n$ (called *vertices* of P) together with the n *edges* (closed straight line segments) V_1V_2, V_2V_3, ... , $V_{n-1}V_n$, V_nV_1, such that the edges are pairwise disjoint except for adjacent edges having a common vertex. The well known polygonal Jordan theorem asserts that the complement in the plane of each polygon P consists of two open regions, the bounded *interior* of P and the unbounded exterior of P. The points $V_1, V_2, \ldots, V_n$ constitute the *vertex set* of P. A segment I with endpoints in the vertex set of P is said to be a *diagonal* of P if the relative interior of I is contained in the interior of P; analogously, I is an *epigonal* of P if its relative interior is contained in the exterior of P.

It is easy to prove that:

(A) Given any set S of n distinct points in the plane, not all collinear, there is a simple polygon whose vertex set coincides with S.

We shall call such a polygon a *Hamiltonian polygon* of S. A simple proof of assertion (A), illustrated in Figure 1, is as follows. Given S, select a direction different from all the directions determined by pairs of points in S. Assume this direction to be horizontal. We shall find two disjoint paths from the highest point T of S to the lowest point L, with vertices in S and comprising all points of S. Since S is not contained in a line, at least one of the open halfplanes D_ℓ and D_r determined by the line D through T and L must contain points of S; without loss of generality we assume that S has points in D_r. Then the Hamiltonian polygon of S is the union of two descending paths from T to L; one with vertices $S \cap (D \cup D_\ell)$, and the other with intermediate vertices T, L and $S \cap D_r$. ◊

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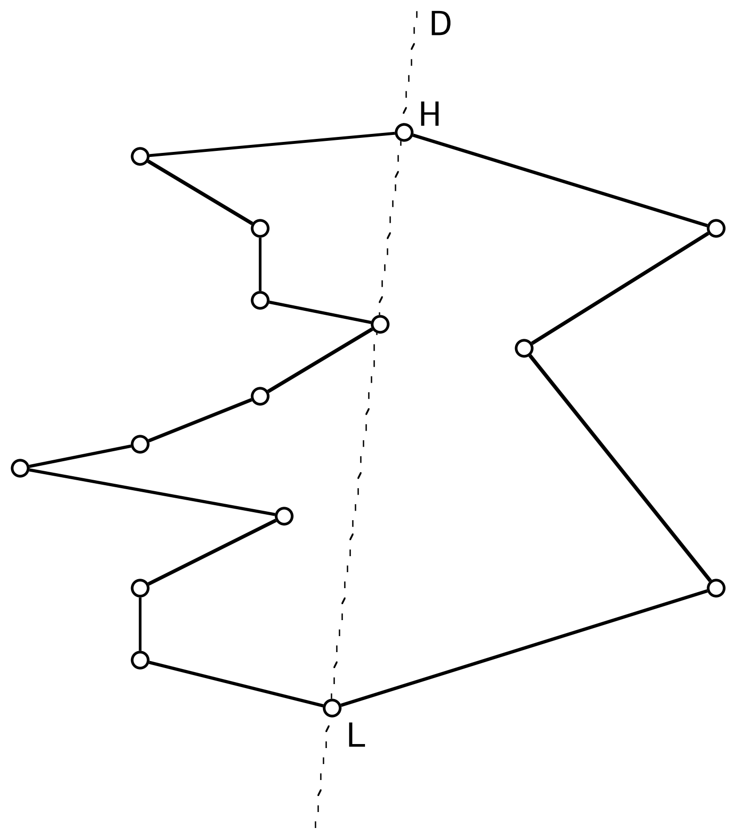

Figure 1.

Although this proof is quite trivial, assertion (A) is slightly tricky. In fact, it is sometimes incorrectly stated that *any* finite set of points admits a simple polygon whose vertex set coincides with S; see, for example, Mirzaian [3]. Such imprecision (leading to false statements) is not a rarity in the literature of computational geometry. But this should not detract from the great value of the questions and results arising from the new directions of inquiry that are motivated by the geometry of computers and computer graphics. Among earlier proofs of (A) I am aware of is a complicated one by Gemignani [1], as well as a simple one by the same author [2].

The main aim of the present note is to discuss certain analogs of statement (A). Some deal with the 3-dimensional situation, while in others the point set S is replaced by a family of segments. In particular, one goal is to give wider circulation to an intriguing conjecture of

Mirzaian [3], which will be formulated below as Conjecture 2. We start by stating the following fact, which will be used later:

(B) The convex hull of S a finite set of distinct points of the plane, not all collinear, has a triangulation T in which all vertices are points of S. Moreover, given any family of segments with endpoints in S such that if any two of the segments intersect, the intersection is a common endpoint, the triangulation T can be chosen so as to include all these segments as edges.

A proof of (B) is immediate from the observation that if the given segments do not determine a triangulation, another segment with endpoints in S can be added which crosses none of the given ones.

Let now S be a family of n pairwise disjoint segments $S_i = A_iB_i$, for $i = 1,2,...,n$. The *vertex set* of S is the set $\{A_1, ... , A_n, B_1, ... , B_n\}$. A simple (2n)-gon P is a *circumscribing Hamiltonian* of S if the vertex set of P coincides with the vertex set of S, and every segment of S is either an edge or a diagonal of P. Analogously, P is a *noncrossing Hamiltonian* of S if their vertex sets coincide and each segment S_i is either an edge or a diagonal or an epigonal of P. A family S of segments is *full-dimensional* if there is no straight line that contains all the segments in S.

Mirzaian [3] formulated, among others, two conjectures that can be stated (in slightly corrected versions) as follows;

Conjecture 1. Every full-dimensional family of pairwise disjoint segments admits a circumscribing Hamiltonian.

Conjecture 2. Every full-dimensional family of pairwise disjoint segments admits a noncrossing Hamiltonian.

While Conjecture 1 is clearly stronger than Conjecture 2, a counterexample to it was soon found by Urabe and Watanabe [4]. This counterexample is reasonably complicated, consisting of 16 segments. A much simpler counterexample is shown in Figure 2; the easy verification that the segments in Figure 2 contradict Conjecture 1 is left to the reader.

On the other hand, Conjecture 2 seems to be well deserving of attention, since neither counterexamples nor a proof appear to be easy to find. In fact, even the special case in which the segments are restricted to be parallel to one of two directions is still undecided.

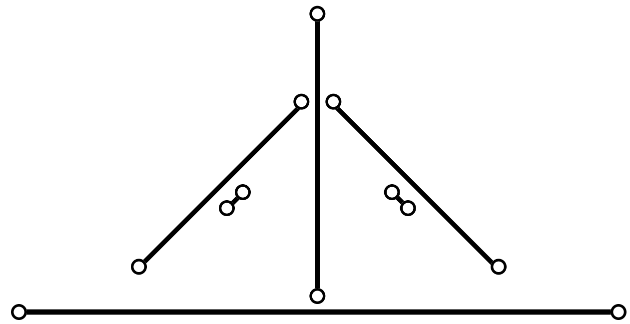

Figure 2.

However, even Conjecture 1 has an affirmative answer under some restrictions. Mirzaian [3] stated the following result, and gave a rather complicated proof under additional assumptions made in order "[t]o simplify the discussion"; however, it seems that removal of these assumptions from his proof is not straightforward.

(C) If S is a full-dimensional finite family of disjoint segments in the plane, such that each segment has at least one endpoint on the boundary of the convex hull of the union of all segments in S, then S admits a circumscribing Hamiltonian.

A simple proof of (C) is as follows. Let C denote the boundary of the convex hull of the union of all the segments S_j. In the illustrative example if Figure 3 C is shown by dotted lines, while thin solid lines mark the part of the circumscribing Hamiltonian H of S that is being constructed. Advancing along C in counterclockwise sense starting from an arbitrary point, we shall construct the H while completing one circuit around C. As we meet an endpoint of a segment S_j in S, there are two possibilities:

(i) Either the other endpoint of S_j also belongs to C, or to the part of H that has already been constructed; in this case we continue along C. The situation is illustrated by the segment marked 2 in Figure 3.

(ii) Or else, the other endpoint of the segment S_j just encountered does not belong to C or to the part of H already constructed. (For example, consider segments 3, 4, 5 or 6.) Thinking of the just traversed part of C (from the last endpoint of a segment S_{j-1} we encountered, to the newly met endpoint of S_j) as a rubber band, we stretch it so as to reach the far endpoint of S_j. In doing so we may have to stretch the rubber band around the endpoints of one or several other

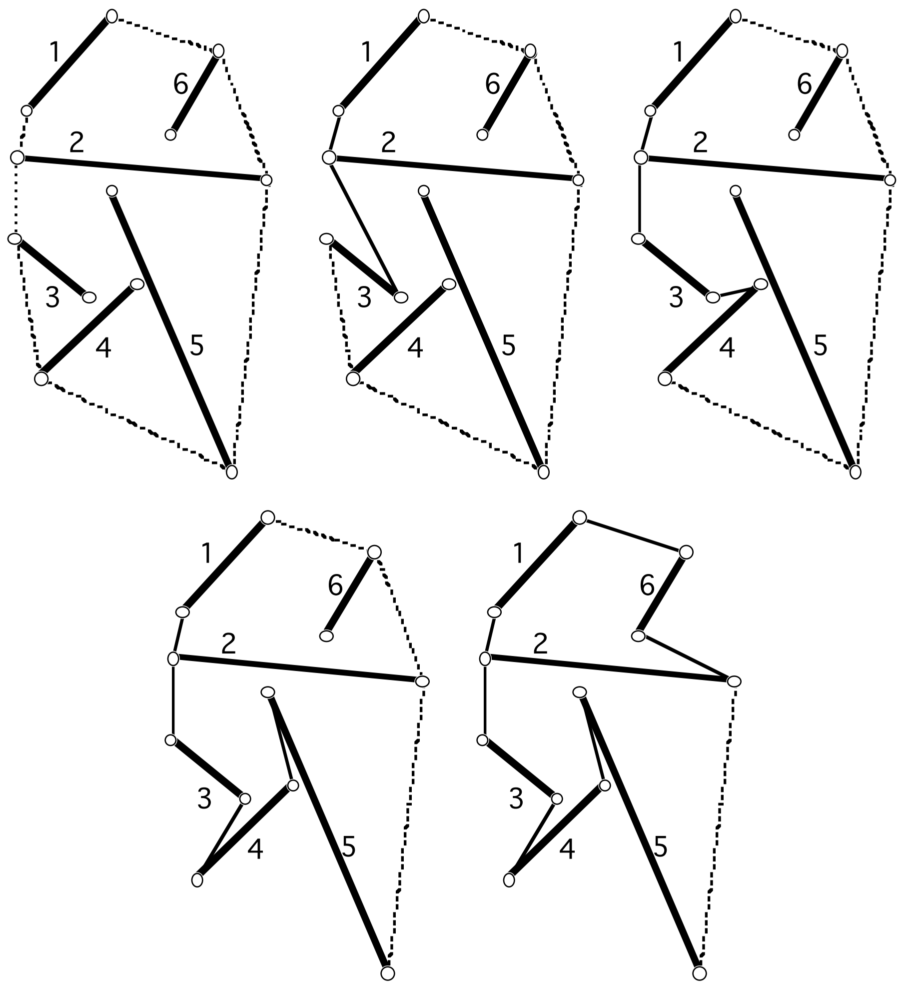

Figure 3.

segments. For each of the latter endpoints, if it has already been visited by the part of H previously constructed, we "detach" that earlier section of the rubber band, and let it contract as far as possible without getting detached from any other endpoint. (In Figure 4, this happens when reaching each of the segments labelled 4 or 5.) ◊

It should be noted that the circumscribing Hamiltonian constructed by this method has the property that each of its points is the endpoint of an open ray (halfline) that is contained completely in the exterior of H. In other words, H is completely visible from the outside.

There seems to have been no mention in the literature of 3-dimensional problems analogous to most of the questions discussed above. A few rather immediate generalizations come to mind.

By *sphere-like polyhedron* P we mean a collection of planar polygonal regions (each a union of a simple convex polygon and its interior) called *faces* of P, such that any two of the faces meet, if at all, either along an edge of each, or at a vertex of each; moreover, the union of all the faces is homeomorphic to the 2-dimensional sphere (the boundary of a solid ball). The vertices and edges of the faces of P are also said to be vertices and edges of P. The set of all vertices of P is its *vertex set*. A sphere-like polyhedron P is *simplicial* if all its faces are triangles.

If S is a set of distinct points such that the vertex set of a sphere-like polyhedron P coincides with S, we say that P is a Hamiltonian polyhedron for S. The following is an analog of the planar result (A):

(D) If S is a set of distinct points, not all of which are coplanar, then there exists a simplicial Hamiltonian polyhedron P for S.

A proof of this assertion is more complicated than in the planar case, although may be seen to follow similar lines. To begin with, given S we select a plane Q such that no two points of S belong to a plane parallel to Q and no two points of S are on a line parallel to the direction L perpendicular to Q; for ease of expression we assume Q to be horizontal. Let B be the shadow boundary of the convex hull conv S of S in direction L, and let C = conv B. Although the reasoning is similar in both cases, it is convenient to distinguish the case in which C is 2-dimensional from the one in which C is 3-dimensional.

In the former case, let H be the plane that contains C; then at least one of the open halfspaces H_ℓ and H_u determined by H contains points of S. Without loss of generality assume that H_u contains some points of S. Let $S_\ell = S \cap (H \cup H_\ell)$, and $S_u = S \cap (B \cup D_u)$, and let P_ℓ and P_u be the projections of S_ℓ and S_u parallel to L onto the plane H. By statement (B), the convex hulls of P_ℓ and P_u can be triangulated without introducing any new vertices, and respecting the projections of the relevant edges of C; a slight strengthening of (B), which we shall not stop to prove, shows that the triangulation of the convex hull of P_u can be chosen so that no triangle has all three vertices in B. "Lifting" these triangulations from P_ℓ and P_u to S_ℓ and S_u, and taking the union of these two polyhedral surfaces, yields the desired simplicial Hamiltonian polyhedron for S.

In case C is 3-dimensional, we proceed similarly. Let $C_{\|}$ and C_u be the lower and upper parts of the surface of C, with common boundary B; if one of $C_{\|}$ and C_u is contained in a plane, we assume that this is $C_{\|}$. We denote by $S_{\|}$ the set of points of S that are on $C_{\|}$ or below $C_{\|}$, and by S_u the set of points of S that are on B or above $C_{\|}$. Let $P_{\|}$ and P_u be the orthogonal projections of $S_{\|}$ and S_u onto the plane P. By statement (B), the convex hulls of $P_{\|}$ and P_u can be triangulated without introducing any new vertices and without crossing the projection of any segment that is an edge of $C_{\|}$ or C_u, respectively. As before, we may assume that the triangulation of the convex hull of P_u is chosen so that no triangle has all three vertices in B. "Lifting" these triangulations from $P_{\|}$ and P_u to $S_{\|}$ and S_u, and taking the union of

the resulting two polyhedral surfaces, yields the desired simplicial Hamiltonian polyhedron for S. The polyhedron P is sphere-like, since its lower part is not above $C_{\|}$, and its upper part is strictly above $C_{\|}$ everywhere but along the common boundary B. ◊

We conclude with several open problems.

Conjecture 3. Given any finite family S of disjoint segments in the 3-dimensional space, not contained in any plane, there exists a simplicial Hamiltonian polyhedron P for the vertex set of S such that each segment in S is either an edge, or a diagonal or an epigonal of P.

Conjecture 4. Given any finite family S of distinct points in the d-dimensional Euclidean space, not contained in any hyperplane, there exists a simplicial polyhedron P homeomorphic to the (d-1)-dimensional sphere, such that the vertex set of P coincides with S.

It may be noted that a difficulty of extending the proof of (D) to a solution of Conjecture 4 arises from the fact that it is not clear in what form Statement (B) generalizes to higher dimensions.

References.

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[4] M. Urabe and M. Watanabe, On a counterexample to a conjecture of Mirzaian. *Computational Geometry: Theory and Applications* 2(1992), 51 - 53.