ASTRAL (n_k) CONFIGURATIONS.

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(n_k) **configuration** is a family of n "points" and n "lines", "incident" with k lines and each line with k points. The ıt", "line" and "incident" can have a variety of meanings, but in - except when specifically stated otherwise — we consider only ıgurations formed by points and (straight) lines in the real ›lane, and we interpret "incidence" as meaning that a point lies We are interested in the **most symmetric** ones among the ›ns (n_k). An (n_k) configuration is called **astral** if its points, ıts lines, form $[(k+1)/2]$ transitivity classes under isometric of the configuration; it is clear that this is the minimal number ty classes in any (n_k) configuration, since no more than two line can be in the same transitivity class, and no more than two ıg through one point can be in the same transitivity class unless s through that point. It is not obvious, but it is true that astral ;urations exist if and only if n is a multiple of 12, and $n \geq 24$. · in preparation [1], we give a complete and constructive ı of the astral (n_4) configurations. (The first such ›ns, as well as some other very symmetric configurations, were in [3].) We also have detailed information about astral (n_3) ›ns, which will be published in [2]. For the present purposes it : to give just a few examples, in order to illustrate the concepts. we show four examples of astral (n_3) configurations, and in ur examples of astral (n_4) configurations. (Note that in order clarity, points are represented by solid dots, and unbounded ›een replaced by segments of the least possible length.) The easily confirm that **if** these families of points and lines are

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ons are astral. But this

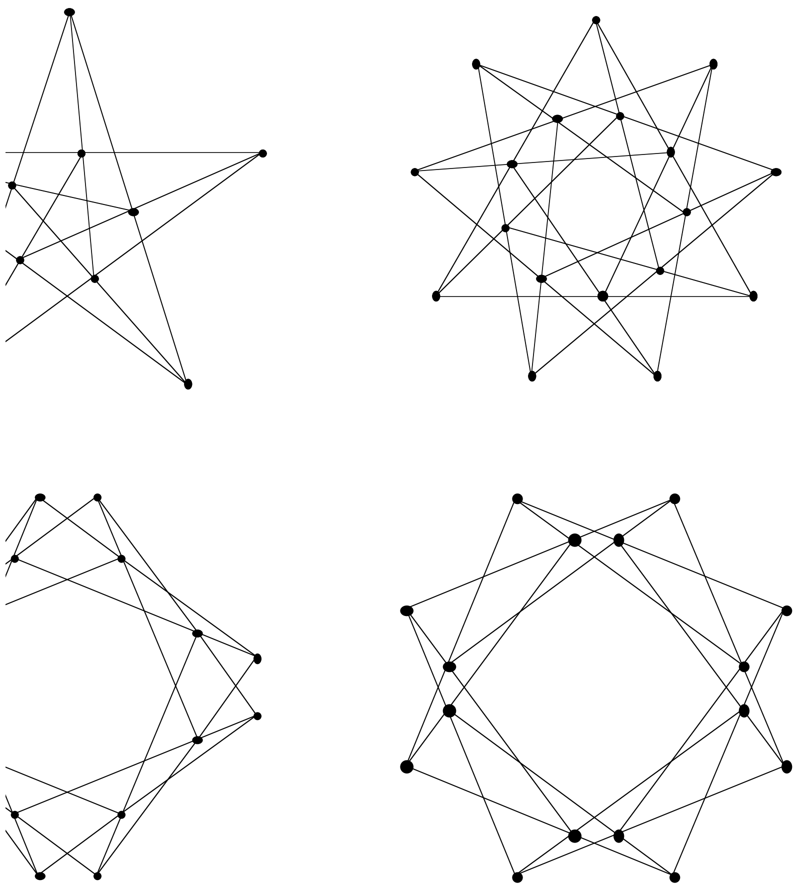

Figure 1.

y brings up two questions: first, do we really see configurations gram, and second, what about astral configurations (n_k) with

ore continuing reading, the reader may wish to scrutinize ne diagram in Figure 3, which shows an astral (n_6)

ıe as a surprize that we make the following.

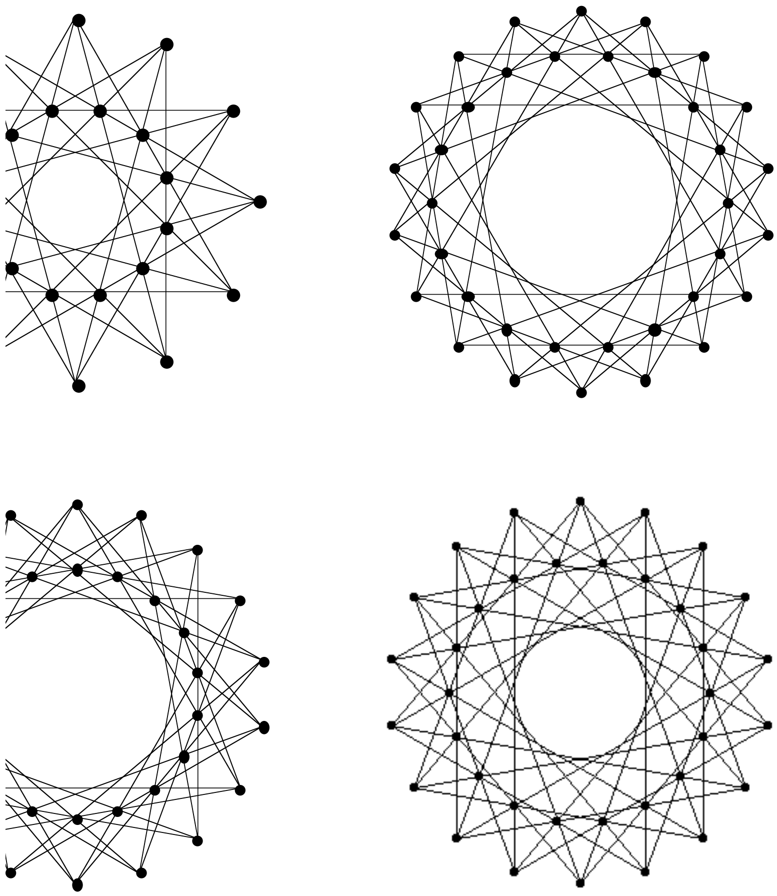

Figure 2.

ɛ **1.** If $k \geq 5$, there exist no (n_k) configurations.

apparent contradiction is resolved by observing that Figure 3 **ırs** to consist of 180 straight lines meeting the 180 marked ırees. In fact, if the incidences are to be satisfied the "lines" slightly bent — or if they are straight, triplets of lines form ;les instead of passing through the same point! With a little bit be at distance 1 from the center and if straight lines are drawn

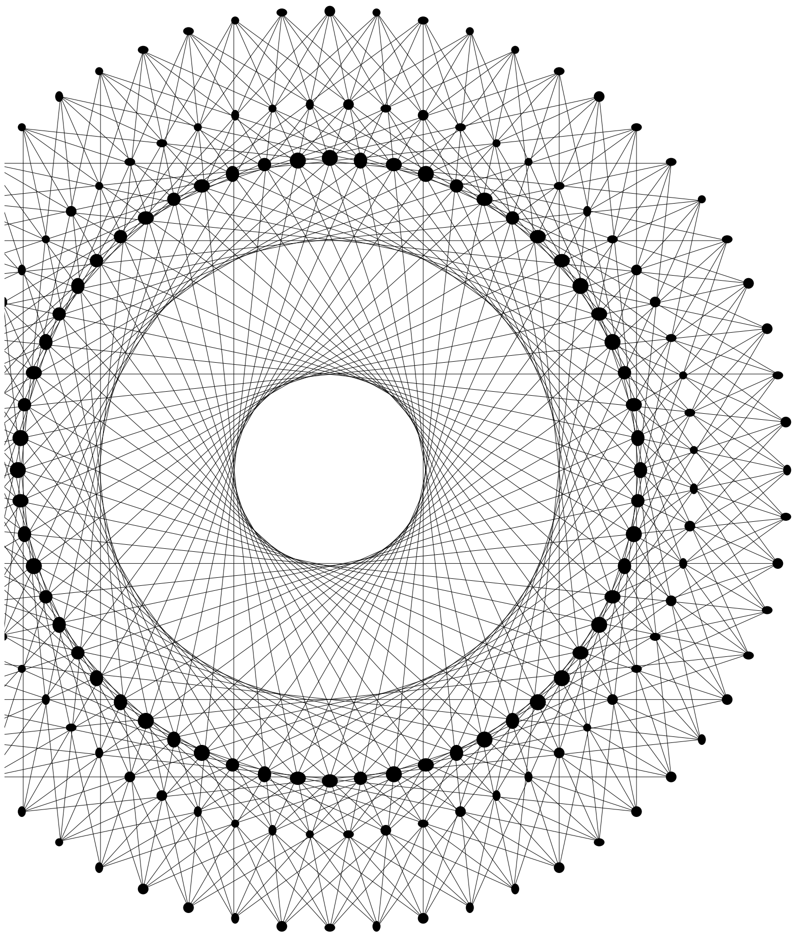

Figure 3.

ese points, then the pairwise intersection points that **seem** to
niddle ring of configuration points will vary in their distance
nter by about 0.01 -- easy enough to hide under a "blob", but
assure that the six lines do not meet at one point.

ons in Figures 1 and 2 are also "fakes" ? The reader may wish
at these are, indeed, the "real thing" — but with a slight

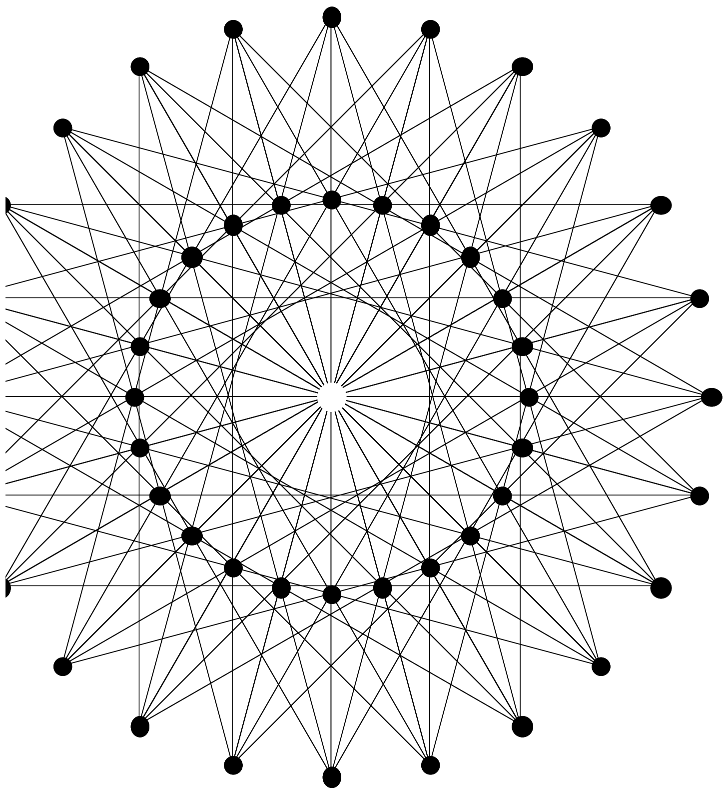

Figure 4.

n the two situations. Assuming, in all cases, that the points are
ated on concentric circles, for the configurations in Figure 2
rguments show that the question of incidences reduces to the
certain trigonometric formulas — which are, in fact, true. On
and, there is not enough symmetry in the configurations of
proceed with such a method. Instead, the position of the inner
ints is determined by solving some linear equations whose
are trigonometric functions depending on the particular
on. Moreover, in certain cases, the equations allow for a real-

ese calculations see [1] and [2].)

ijecture 1 depends heavily on the conditions we imposed so far. the conditions a little — for example, by using the "extended plane" model of the projective plane as the setting for our ons, then it is easy to find astral (n_5) configurations. Indeed, if uintuplet of parallel lines in Figure 4 we adjoin their common finity", we obtain an astral (60_5) configuration in the projective tead of the extended Euclidean plane model the reader may use, ame result, the model of the projective plane in which each point is represented by a pair of antipodal points of a fixed each projective line is represented by a great circle.) In the etting we venture not only the guess that there are no astral urations for $n \geq 6$, but more generally:

e **2.** In the real projective plane there exist no configurations $k \geq 6$ in which the points as well as the lines form $[(k+1)/2]$ classes under the group of projective transformations.

ious other related directions of investigation may be pursued; ention only one of these. If we place two copies of an astral iguration in two parallel planes, so that the centers of the ons are perpendicularly above each other, and the possible twist iguration with respect to the other is suitably chosen, we obtain $2n)_k)$ configuration which spans the Euclidean 3-dimensional lso, placing six copies of either one of the two astral ons (16_3) shown in the bottom row of Figure 1 in suitable on the face-planes of a cube, we obtain an astral (96_3) on that spans 3-space; similarly for copies of some other ons, appropriately placed on the face-planes of other regular We believe that these are the **only** methods if obtaining astral ons that span 3-space. More precisely:

e **3.** If an astral configuration spans the Euclidean 3-space then of congruent copies, without incidences among distinct copies, planar configuration.

Grünbaum, An enumeration of astral (n_4) configurations. (In
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Grünbaum and J. F. Rigby, "The real configuration (21_4)."
Math. Soc .(2) 41(1990), 336 - 346.

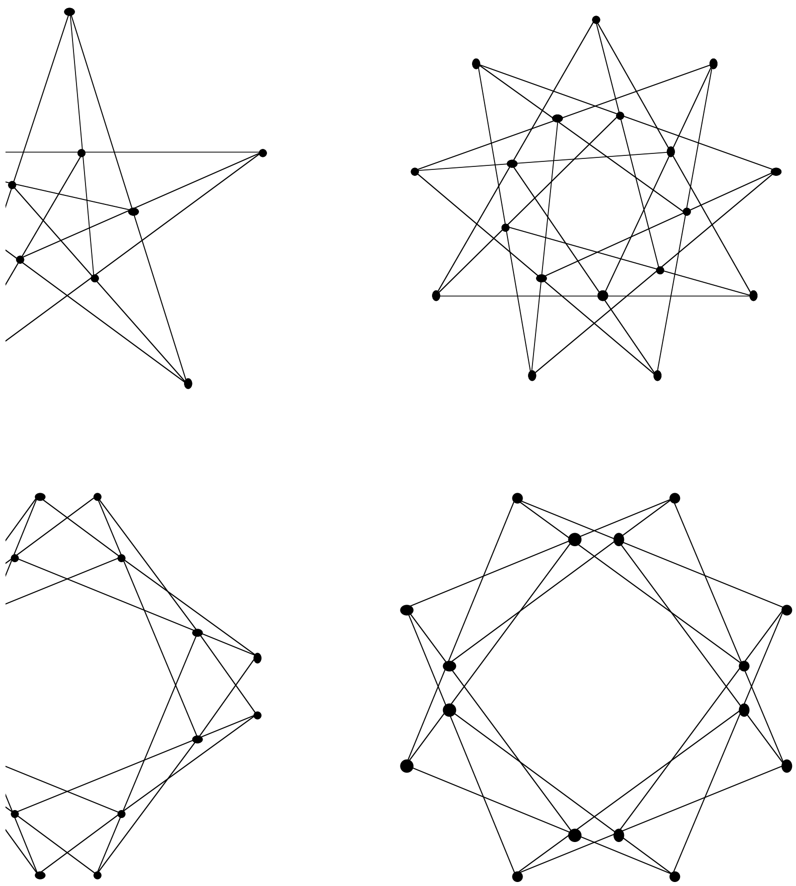

Figure 1.

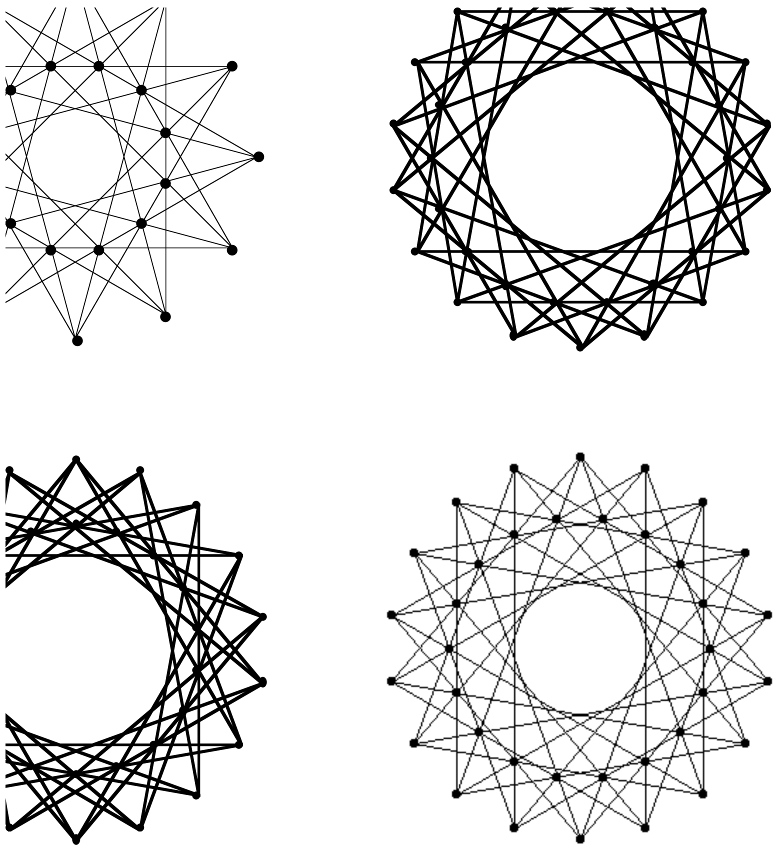

Figure 2.

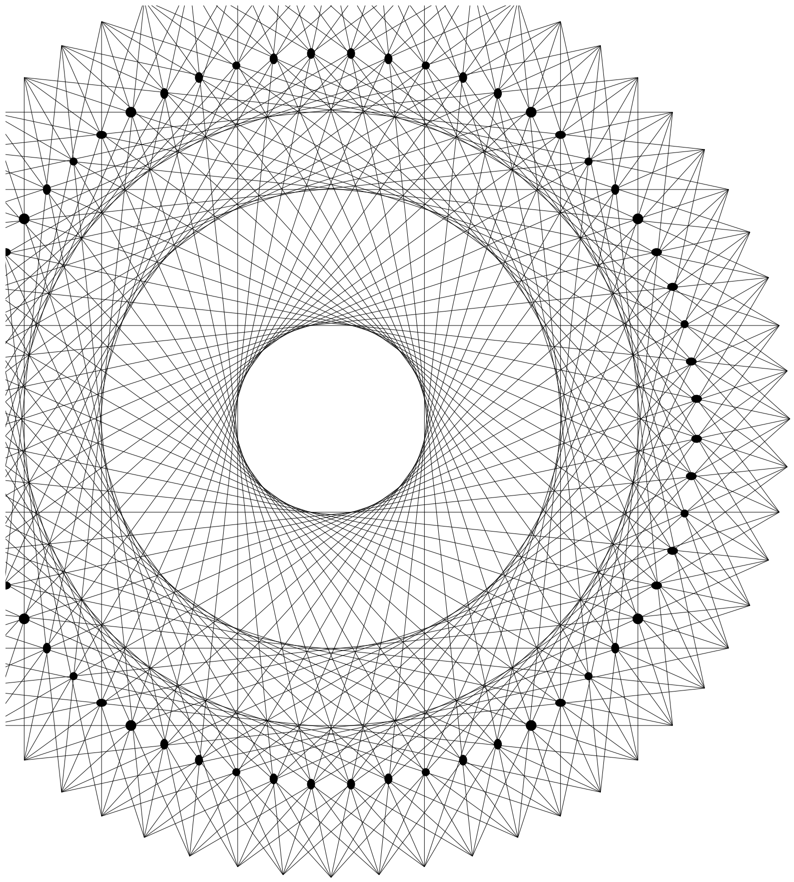

Figure 3.

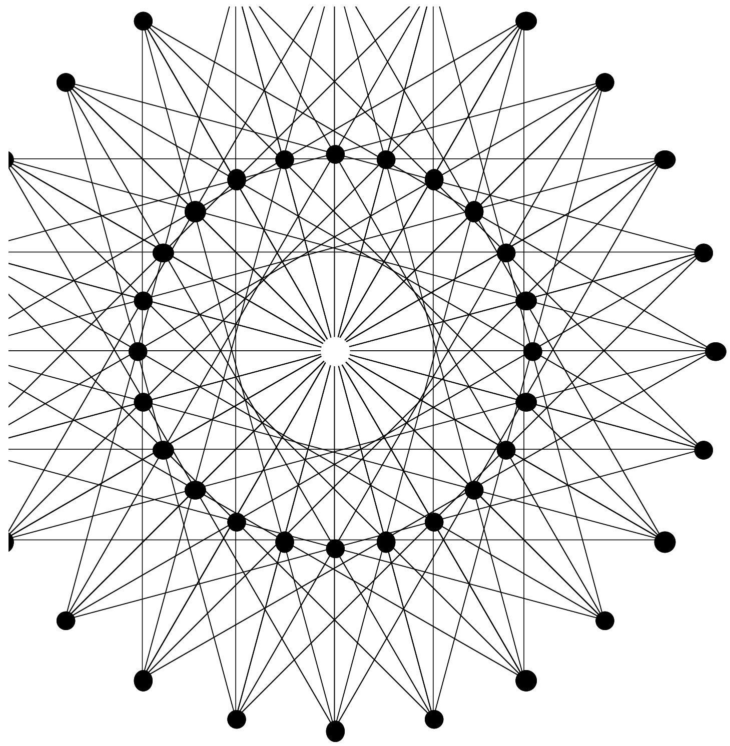

Figure 4.