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HOLEY ISOGONAL COLUMNS

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This is a sequel to the note [3] on infinite polyhedra. We shall use the terminology introduced there, giving here only brief descriptions of the main concepts. By *infinite polyhedron* P we mean an infinite collection of planar convex polygons (*faces* of P) with pairwise disjoint interiors; moreover, each side of each face of P is also a side of precisely one other face and is an *edge* of P. The endpoints of edges of P are *vertices* of P, and the faces, edges and vertices satisfy the conditions usually assumed for polyhedra and listed explicitly in [3]. We stress, however, that since we do not require that the polyhedron be convex, there is no point in assuming that its faces lie in distinct planes. In fact, plane tilings fall under our definition of infinite polyhedra.

We are interested in *isogonal* (or *vertex transitive*) polyhedra, that is, polyhedra for which the vertices are all mutually equivalent under symmetries of the polyhedron. In [3] we were mainly concerned with *uniform* polyhedra, by which is meant that the polyhedron is isogonal and all its faces are regular polygons; we also assumed that the polyhedron is periodic in three independent directions.

Isogonal (and, in particular, uniform) polyhedra that are periodic in two independent directions have been investigated in a number of publications. There is ample literature on planar tilings of this type (see, for example, [4, Chapters 2 and 6] and the references given there). Many uniform polyhedra periodic in two dimensions but not contained in a plane are shown in [7], and an exhaustive study of isogonal polyhedra of this kind is contained the forthcoming doctoral thesis of William T. Webber. Some of these "slabs" have "holes", that is, are topologically of infinite genus. The simplest examples are those obtained by starting with square tilings in two parallel planes, deleting a suitable subset of their tiles and connecting them by "tubes" formed by four squares.

Isogonal (or uniform) polyhedra that are periodic in one dimension have also been investigated; due to the appearance they have, such polyhedra may be called "columns".

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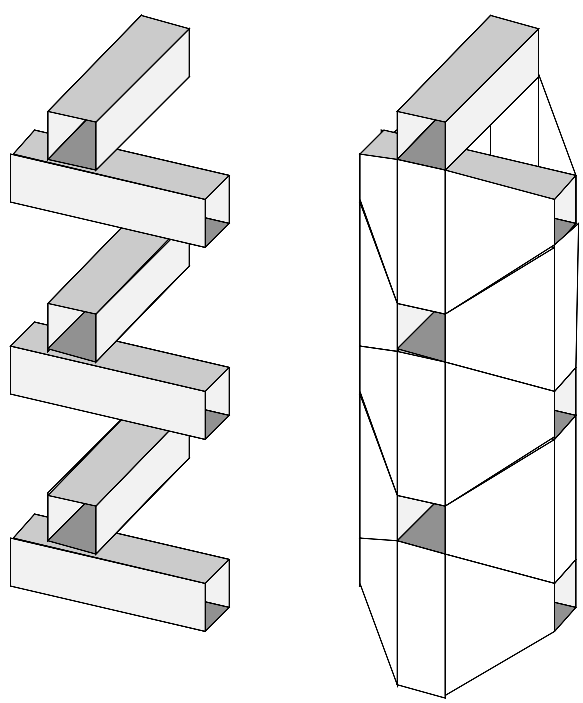
Uniform columns are easily constructed by stacking (in an appropriate way) prisms and/or antiprisms (and deleting the common bases of pairs of the components). Illustrations of these and some other columns are given in [7]. Also, a stack of regular tetrahedra forms an interesting column called a "tetrahelix" by Buckminster Fuller (see [2] and [1]; the latter deals with extensions to higher dimensions as well), and there is a variety of other possibilities of creating isogonal columns.

However, there is no mention in the literature of uniform or isogonal columns of higher genus. The purpose of the present note is to fill this gap.

We start by describing an infinite family of "holey" isogonal columns, illustrated by two examples in Figures 1 and 2. Since a diagram of the completed polyhedron does not convey the structure of the polyhedron very clearly, we describe and illustrate the construction itself. We start with the "holes" (or "interior passages"), as shown in parts (a) of our illustrations. In general, for an integer $n \geq 2$, each "hole" consists of a prism, the basis of which is a polygon with all angles equal and with $2n$ sides that

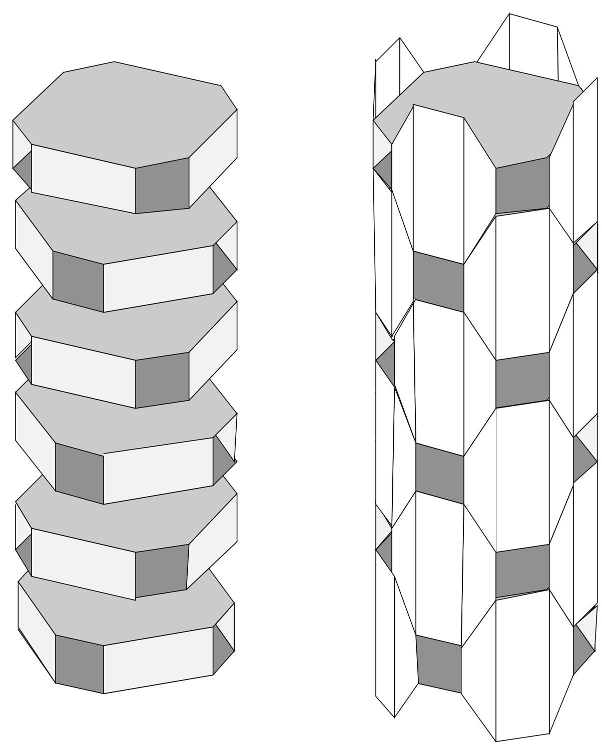


(a) (b)

Figure 1.

alternate in length (and the two lengths are distinct); for use in the infinite column, the mantle-faces of the prisms that contain the shorter sides of bases have been deleted. The prisms-with-openings are placed vertically one above the other, equally spaced and rotated through an angle of π/n (= 180°/n) with respect to each other. While the situation depicted in Figure 1(a) is the simplest one (n = 2), the case n = 4 shown in Figure 2(a) is probably a better illustration of the general construction. With the "holes" in place, the columns are completed by adding a suitable "mantle" on the outside, which connects the "holes" and creates a polyhedron.

It is obvious that the polyhedra just described are isogonal, with five quadrangles (two from the "hole" and three from the "mantle") meeting at each vertex. Besides similarities and the integer n, the polyhedra depend on three parameters that can have arbitrary positive values: the ratio (greater than 1) of the two lengths used for the sides of the prisms, the height of the prisms, and the spacing of the prisms in the column.



(a) (b)

Figure 2.

Part of the interest in the polyhedra just described is due to the following conjecture:

Conjecture 1. There are no isogonal columns of infinite genus other than the ones obtained by the above construction.

If true, this implies an explanation for the absence of mention in the literature of uniform columns of infinite genus. We formulate it as:

Conjecture 2. There exist no uniform columns of infinite genus.

References.

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