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VENN DIAGRAMS II

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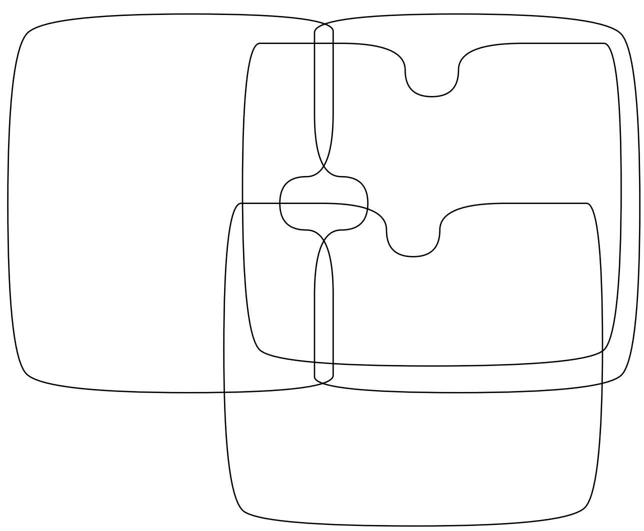
In this note we continue to use the terms introduced in Venn Diagrams I, to which we refer as VDI.[2] Here we shall concentrate on those particularly attractive Venn diagrams in which all curves are congruent; we shall call them *nice* diagrams.

All diagrams in Figures 1 and 7 of VDI are nice, and so is the diagram with four curves shown in Figure 1 below.



1 Supported in part by National Science Foundation grants DMS-8620181 and DMS-9008813.

2 *Geombinatorics* 1(1992), pp. 5 - 12. The reader will have no difficulties correctring several typos involving parentheses and brackets, in the middle of page 8, or deleting the last two lines on that page. However, in a more serious glitch, a crucial word was omitted from Conjecture 2 on page 9, which should read: "If a simple and exposed Venn diagram has the property that the intersection of the interiors of any subfamily of curves is connected, then it is isomorphic to a convex diagram."

Figure 1.

Some nice Venn diagrams (such as those in VDI Figures 1 and 7(a)) have the property that all curves are equivalent under isometric symmetries of the diagram; such diagrams are called *symmetric*. This definition is due to Henderson [2]. It is easy to see that every closed and piecewise smooth Jordan curve can be used to form a symmetric Venn diagram with two sets.

Conjecture 1. Every closed, piecewise smooth Jordan curve can be used to form a symmetric Venn diagram of three sets.

It is not clear whether the same should be expected of every closed Jordan curve. The conjecture is open even when weakened to require the formation of nice diagrams.

Various elementary sets can be used to form some symmetric Venn diagrams with five curves. We have already seen (Figure 7(a) of VDI) that ellipses can be used. Several non-simple symmetric Venn diagrams with ellipses are possible; two are shown in Figure 2 (the first is due to Schwenk [3]). Figure 3 (from Grünbaum [1]) shows that equilateral triangles can also serve that purpose.

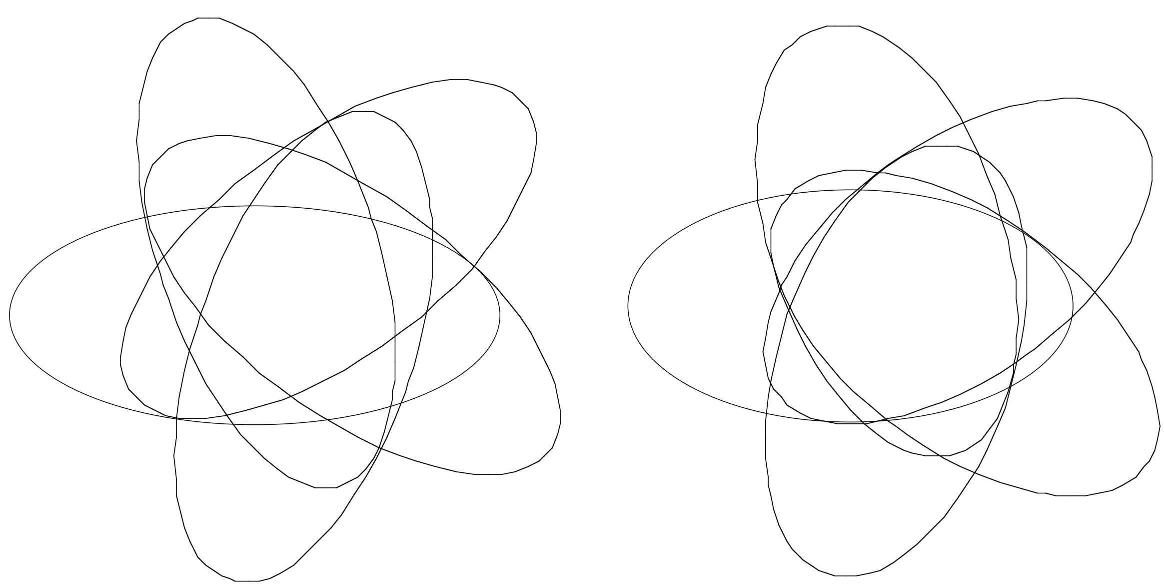


Figure 2.

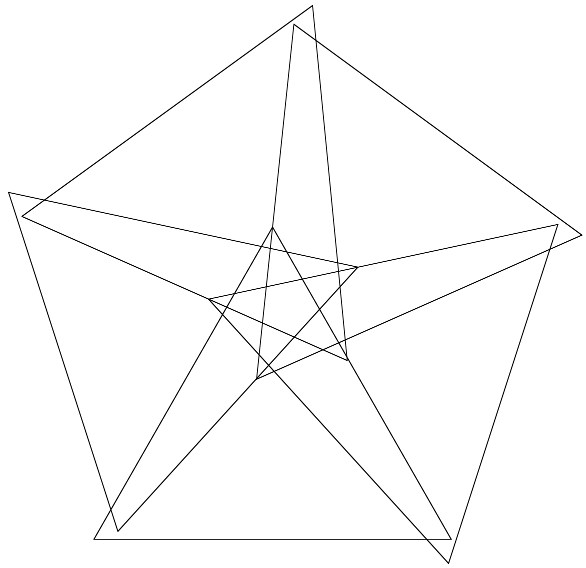


Figure 3.

The reader may be forgiven if Figure 4 inspires the conclusion that squares lead to symmetric Venn diagrams with 5 sets: the quadrangles used are not squares but rectangles, with sides in ratio about 1.02. Experimental evidence points to:

Conjecture 2. There is no symmetric Venn diagram with five squares.

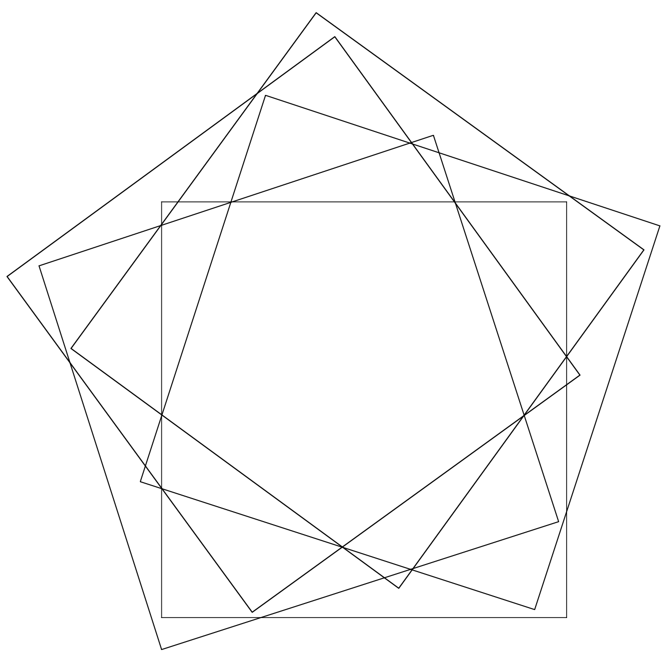


Figure 4.

It is not hard to verify that symmetric Venn diagrams with n sets can exist only if n is a prime number; hence the next-higher possible case is n = 7. In Figure 5 we show such a diagram; for clarity, one of the curves is emphasized. Henderson [2] found a symmetric Venn diagram with 7 hexagons, but his example was never published and appears to have been lost. In unpublished work, Carla Savage and Peter Winkler also found symmetric Venn diagrams with 7 sets. There exist several different types of such diagrams. In Figure 6 we show an example formed by pentagons, which is not isomorphic to the diagram in Figure 5. (Clearly, the fifth vertex of each pentagon in Figure 6 is far away, hence not shown.) Nothing more seems to be known about symmetric Venn diagrams. We venture:

Conjecture 3. For every positive prime n there exists symmetric Venn diagrams with n sets.

Already the first open case, n = 11, appears to pose considerable problems for experimental solution.

Concerning composite n we can expect, at best, to have nice (but not symmetric) diagrams. Figure 1 shows such a diagram for n = 4. In case n = 6 we can use the following technique. Starting from the symmetric diagram with five ellipses shown in Figure 7(a) of VDI, by threading a curve through all 32 regions we obtain the Venn diagram of 6 curves shown in Figure 7. By taking a curve that is a combination of the two shapes in Figure 7, we can form the nice Venn diagram of six curves shown in Figure 8. In an analogous way we may obtain a nice Venn diagram with 8 curves starting from the symmetric diagrams in Figure 6. There seems to be no straightforward way of creating nice Venn diagrams with 9 or more curves.

Conjecture 4. For every positive integer n there exist nice Venn diagrams with n curves.

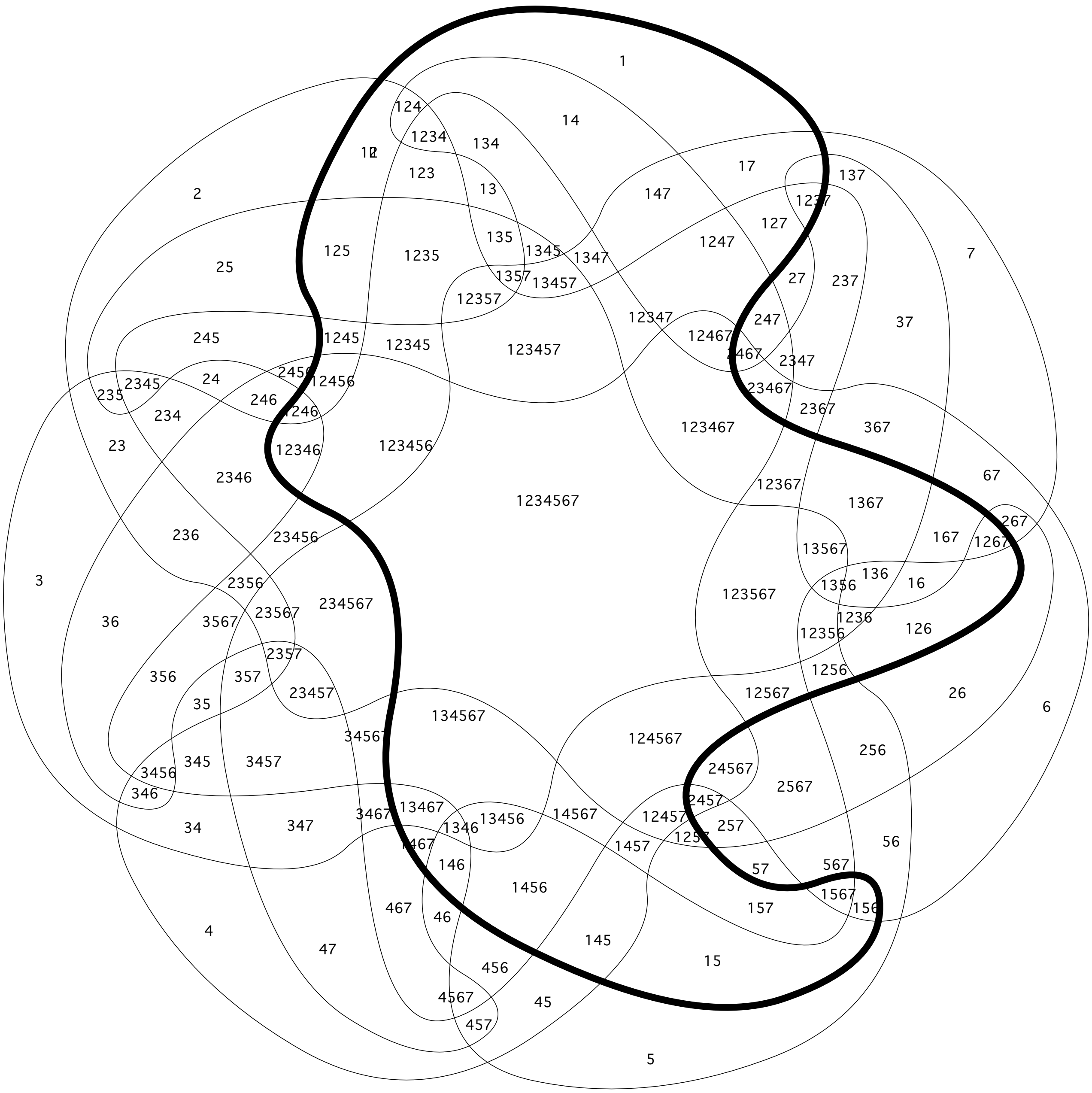

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Figure 5.

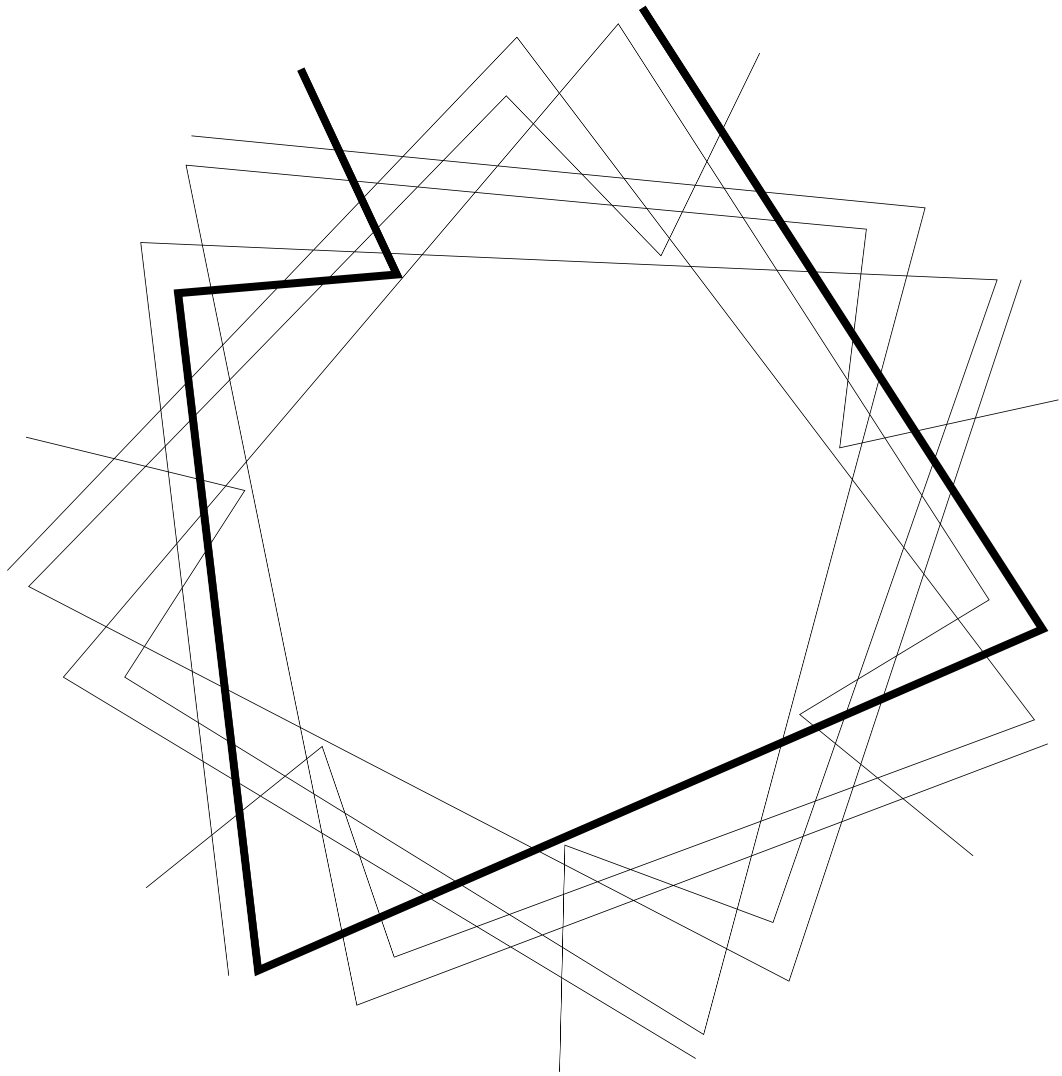

Figure 6.

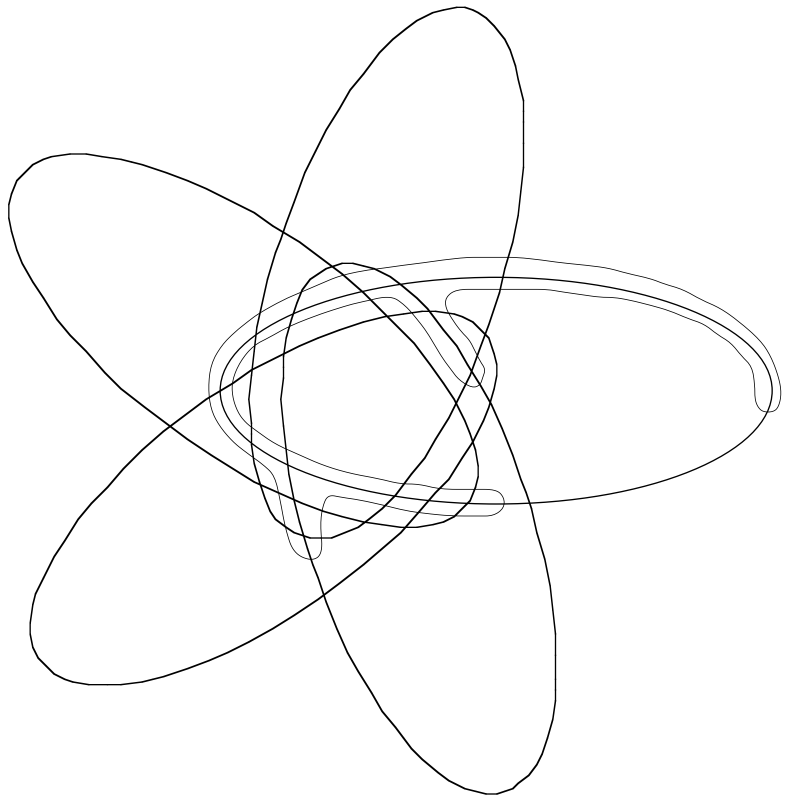

Figure 7.

References

[1] B. Grünbaum, On Venn diagrams and the counting of regions. *College Mathematics Journal* 15(1984), 433 - 435.

[2] D. W. Henderson, Venn diagrams for more than four classes. *American Mathematical Monthly* 70(1963), 424 - 426.

[3] A. J. Schwenk, Venn diagram for five sets. *Mathematics Magazine* 57(1984), 297.

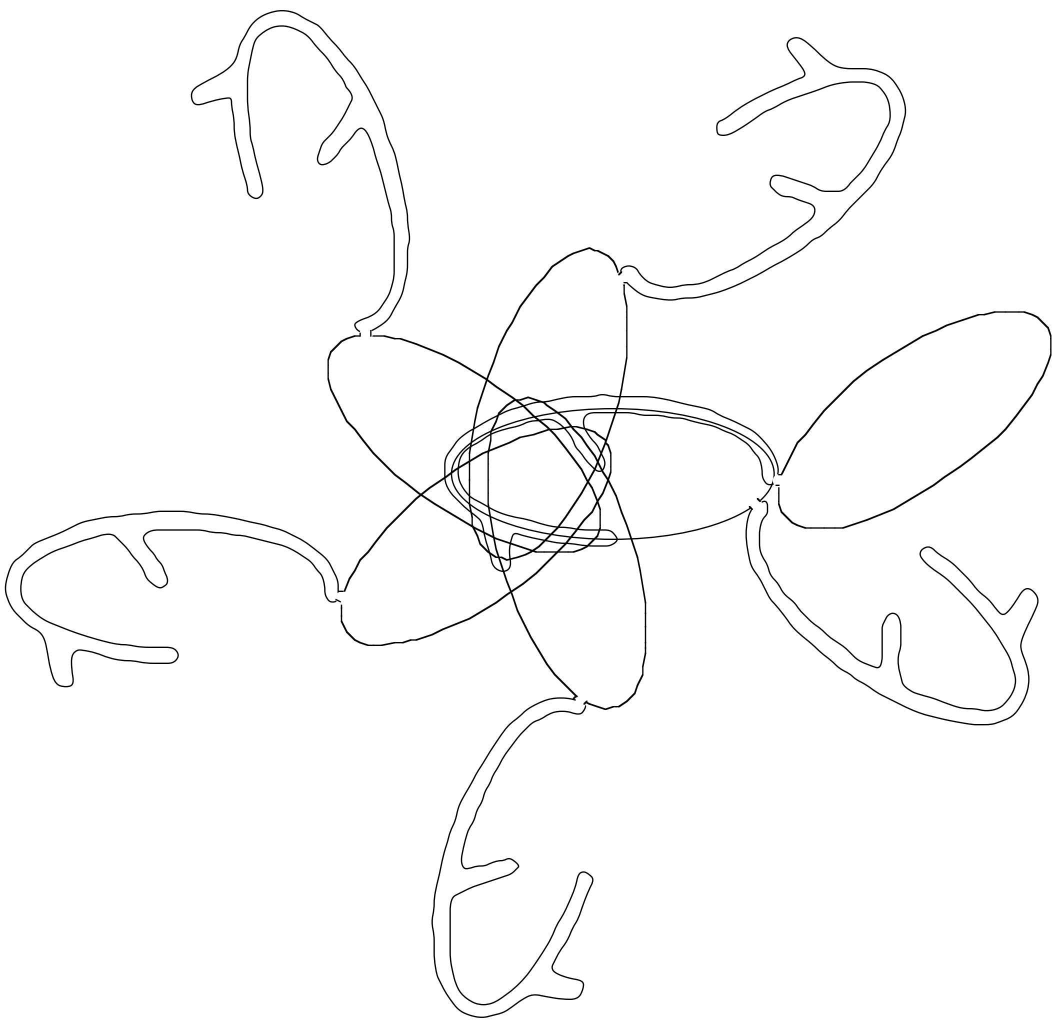

Figure 8.

Symmetric n-diagram: Venn diagram with n curves, such that rotation through $2\pi/n$ about a suitable point O is a symmetry of the diagram, which cyclically interchanges the n curves. Hence, in particular, all curves are congruent to each other; any curve of that shape is said to be a *prototype* of the symmetric diagram. Symmetric Venn diagrams were first defined and considered by Henderson [1963].

Figure 1 from [part 1] shows symmetric 1-, 2-, and 3-diagrams formed by circles. Among the other diagrams in [part I] only that in Figure 7(a) is symmetric.

Figure 1 shows three symmetric 5-diagrams. [ellipse, equilateral triangle, near-square 1∞1.02 rectangle] The first two are isomorphic, and are also isomorphic to the diagram in Figure 7(a) of [part I].

From the definition it follows easily that a symmetric n-diagram can exist only if n is a prime number. All symmetric 3-diagrams are simple. It is readily shown that every convex curve, as well as every polygon (convex or not), is the prototype of a symmetric 3-diagram. It seems not to be known whether every Jordan curve a prototype of a symmetric 3-diagram.

Note that a Venn diagram isomorphic to a symmetric diagram need not be symmetric, even if its curves are all congruent. On the other hand, continuing the numbering from Part I, we have

Conjecture 7. If a Venn diagram consists of congruent curves, and is *combinatorially symmetric* (that is, adjacency relations among regions and their neighbors are preserved under a suitable cyclic permutation of all the curves) then it can be deformed to a symmetric diagram.

Figure 2 shows three non-simple symmetric 5-diagrams with an ellipse as prototype. The diagram in part (a) was first found by Schwenk [1985]. It may be noted that in all diagrams of Figure 2, at all points in which three curves meet, two of them are tangent. In contrast, in the non-simple diagram in Figure 1(c), at any point common to two curves, the curves cross each other. It follows that this diagram is not isomorphic to any of the diagrams in Figure 2.

Problem 8. Find all isomorphism types of 5-symmetric Venn diagrams with strictly convex curves. Are there any others, besides the four diagrams with ellipses, shown in Figures 1(a) and 2?

It should be noted that Problem 8 is open even for diagrams with ellipses!

Conjecture 2. The 5-symmetric Venn diagrams with ellipses do not depend on the shape of the ellipse used; in other words, if a 5-symmetric Venn diagram of ellipses can be made using one ellipse, an isomorphic 5-symmetric diagram can be made using any other ellipse.

Conjecture 3. If C is any convex curve, every 5-symmetric Venn diagram with copies of C is isomorphic to one of the 5-symmetric Venn diagrams with ellipses.

Conjecture 4. For $n > 5$ there are no n-symmetric Venn diagrams with copies of any convex curve.
It may well be that Conjecture 4 is valid for all simple Jordan curves.