Are These Figures Oxymora?

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> Oxymoron [Greek *oxymoron*, a smart saying which at first view appears foolish, from *oxys*, sharp, and *morŏs*, dull, foolish] a figure of speech in which opposite or contradictory ideas or terms are combined.
>
> (After Noah Webster)

The century-old topic of point and line configurations straddles the fence between projective geometry and combinatorics. In this article we shall be concerned mainly with the geometric approach and will attempt to highlight certain aspects of such configurations that we find very interesting. We hope the reader will agree. These aspects seem not to be widely known, possibly because of the confusing nature of the usual terminology, and—even more—due to the general decline in familiarity with geometric facts. Also, although there is a great amount of known material, it is scattered in many papers, a large fraction of which appeared in rather inaccessible journals. Unfortunately, there is no book that would present a reasonable account of such material. It is remarkable that in an elementary topic such as configurations, there are still many unsolved questions, and that fruitful connections to other branches of mathematics and its applications are fueling a renewed interest. The reason for the following pages is the hope that they may help awaken in our students (and in other readers) an interest in geometry. The paper may also afford them a chance to "try their wings" in independently developing a nontrivial but easily accessible topic, and to experience the fact that "elementary" questions may be hard enough to have resisted solution even to this day. Although the material of this note traditionally would appear in the context of projective geometry, the reader may consider that all the points and lines are in the ordinary Euclidean plane. Many of the references are given for the sake of historical interest and understanding of the development, and we do not expect the reader to spend much time looking for them.

Cyclically inscribed multilaterals

By way of introduction to our topic, consider FIGURE 1. It is evident that A is cutting B's hair, B is cutting C's hair,..., and F is cutting A's hair. (Such groupings of individuals probably do not exist at present, but may soon be formed if the price of haircuts continues to increase!) Now consider FIGURE 2, representing a 9_3 *configuration*, that is, a family of nine points and nine lines, with three of the points on each of

*Research supported in part by NSF grants DMS-8620181 and DMS-9008813.

the lines, and three of the lines passing through each of the points. In this configuration, three *trilaterals* have been indicated by differing texture of the lines. Here we must say a few words regarding terminology. A "*multilateral*" or "*n*-lateral" is a cyclic sequence of n distinct points called, say, $A_1, A_2, \ldots, A_n$, and n lines $A_1A_2, A_2A_3, \ldots, A_{n-1}A_n, A_nA_1$ determined by the indicated pairs of points. "Trilaterals" and "quadrilaterals" are n-laterals with $n = 3$ or $n = 4$, respectively. In the sequel we shall simplify the notation by omitting the letter A, and labelling the points by natural numbers. To avoid confusion with other points of the plane, we shall call the points determining a multilateral its *vertices*, and we shall use this term also for the "points" of the various configurations. The difference between trilaterals and the more familiar triangles is that triangles are formed by segments determined by pairs of points, while trilaterals are formed by complete, unbounded lines; similarly for the difference between multilaterals and polygons—we shall have occasion to consider polygons later in this article. The early writers on configurations used the word "polygons" both when they meant "multilaterals" and when they meant "polygons."

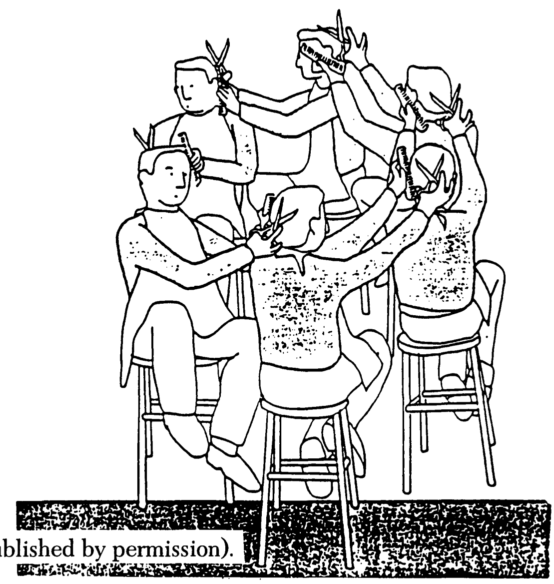


FIGURE 1

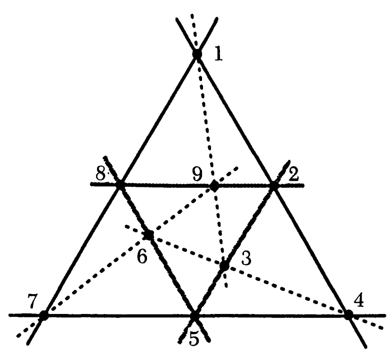


FIGURE 2
A 9_3 configuration, in which a cycle of three trilaterals, each inscribed in the preceding, has been indicated by different texture of the lines. Similar visual aids are used in the following diagrams.

Although this seems not to have bothered them too much—see Hilbert [11], translator's footnote on p. 110 of the English version—we find it preferable and more logical to use different words for different concepts that often occur in the same context. We shall say that, in FIGURE 2, the trilateral determined by the vertices labeled 2, 5, 8 is *inscribed* in the trilateral 1, 4, 7, in analogy to the usual interpretation of the word. By "inscribed" we mean that the vertices of the former lie on the lines of the latter, one vertex on each line. It is obvious that the trilateral 1, 4, 7 is inscribed in the trilateral 3, 6, 9, which, in turn, is inscribed in the trilateral 2, 5, 8. Hence the situation is completely analogous to the one shown in FIGURE 1.

Although trilaterals may appear strange at first, the use of that concept leads to some neat mathematics. Some of it goes back to the beginnings of study of configurations (see, for example, [15]). To illustrate this, fix your attention on the cycle of trilaterals of FIGURE 2 when written in the following form, in which each row corresponds to a trilateral:

$$\begin{matrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9. \end{matrix}$$

This square array may be expanded to a similarly constructed rectangular array

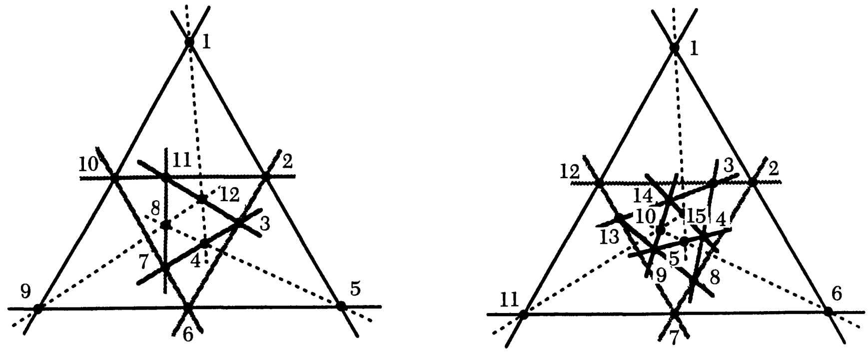
$$\begin{matrix} 1 & 5 & 9 \\ 2 & 6 & 10 \\ 3 & 7 & 11 \\ 4 & 8 & 12. \end{matrix}$$

The new array can be taken as a representation of FIGURE 3(a), which is a 12_3 configuration with a cycle of four trilaterals, each inscribed in the preceding one. Further expansion leads to the cycle of five trilaterals in FIGURE 3(b), and so on. For clarity, in these diagrams and in the following ones, only those parts of the lines involved have been shown that are necessary to show how one multilateral is inscribed in another; we used shadings of various kinds to distinguish the multilaterals involved.

When the original array is expanded in the right-hand direction we have

$$\begin{matrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12, \end{matrix}$$

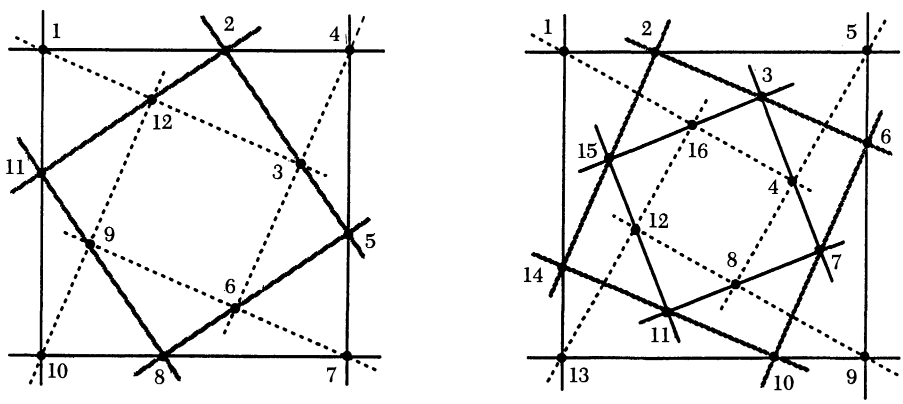
which represents FIGURE 4(a), a 12_3 configuration with a cycle of three inscribed quadrilaterals, then to FIGURES 4(b), 5, and so on. However, starting with $k = 5$, a somewhat surprising new possibility arises, namely a sequence of just two mutually inscribed "star-shaped" k-laterals, leading to configurations n_3 with $n = 2k \geq 10$; the cases $k = 5$ and 8 are illustrated in FIGURE 6. The existence of such configurations has been mentioned only relatively recently; the earliest publications we are aware of in which examples of such configurations are shown are van de Craats [22] and Zacharias [23].



(a) (b)

FIGURE 3

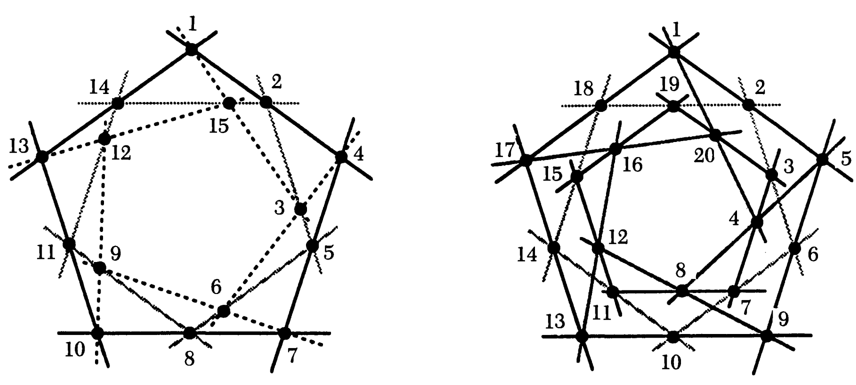
A 12_3 and a 15_3 configuration, with cycles of four or five inscribed trilaterals.



(a) (b)

FIGURE 4

A 12_3 and a 16_3 configuration, with cycles of three of four inscribed quadrilaterals.



(a) (b)

FIGURE 5

A 15_3 and a 20_3 configuration, with cycles of three or four inscribed 5-laterals.

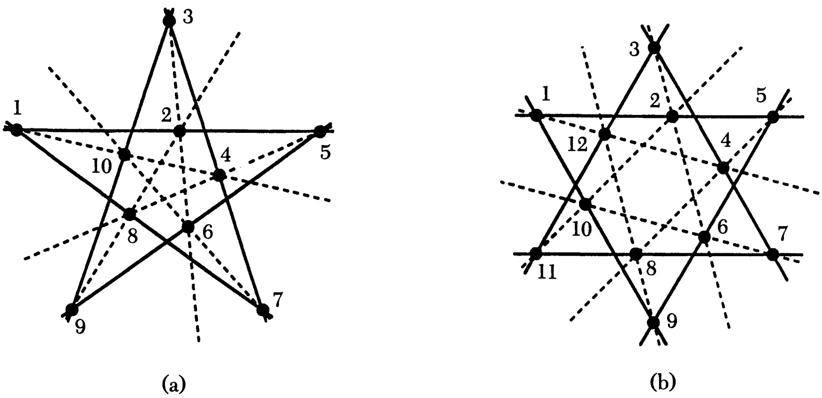


FIGURE 6
A 10_3 and a 16_3 configuration, with two mutually inscribed "star-shaped" 5-laterals or 8-laterals.

It may be noted that in constructing the configurations of these figures, the early multilaterals can be inscribed as desired, but for the final one the possibilities are severely limited by the requirement that its lines must pass through the vertices of the first multilateral. This shows that there is a nontrivial question hiding behind our drawings: Are the lines shown as straight really straight, and do they in fact pass through the vertices in the way the diagrams lead us to believe? You may enjoy experimenting a little on your own, by drawing examples of configurations in which the multilaterals are based on regular polygons (as in our diagrams) to see at what stage the freedom of choice stops. Once you reach such an experimental insight, you may try to prove its validity in general (most easily by using an argument based on the continuity of the way lines depend on the points determining them, and vice versa).

Configuration tables

As a bonus for the work that has been done in constructing the sequences of configurations $9_3, 12_3, 15_3, \ldots; 12_3, 16_3, 20_3;\ 15_3, 20_3, 25_3, \ldots,$ and so on, a second aspect of line configurations appears. By taking pairs of vertices, determined by following the natural order of the integers used to label the vertices, and considering the segments they define, we obtain a closed circuit of segments in each figure. They define a *polygon* in the configuration. Since it contains all the vertices, in analogy with graph theory we shall call it a *Hamiltonian polygon* of the configuration. For the 9_3 configuration of FIGURE 2, the Hamiltonian polygon is formed by the segments $\overline{12}$, $\overline{23}$, $\overline{34}$, $\overline{45}$, $\overline{56}$, $\overline{67}$, $\overline{78}$, $\overline{89}$, $\overline{91}$, as shown in FIGURE 7(a). Additional examples are shown in FIGURES 7(b, c), 8, and 9. Each such Hamiltonian polygon determines a *Hamiltonian multilateral* (consisting of the lines that contain the sides of the polygon); this multilateral is *self-inscribed* in the sense of our definition. (In some publications —especially older ones—the expression "self-inscribed and circumscribed polygon" is used. Besides being unwieldy, it is also unreasonable if the word "polygon" is taken in its usual meaning.) The senior of the authors of this article, thinking that a former engineering colleague might be interested in the resemblance of FIGURES 7, 8, 9 to gearwheels, showed them to him and was surprised by his comment that his immediate reaction was to wonder if the series of drawings had any connection with the fashionable areas of fractals, chaos, etc.!

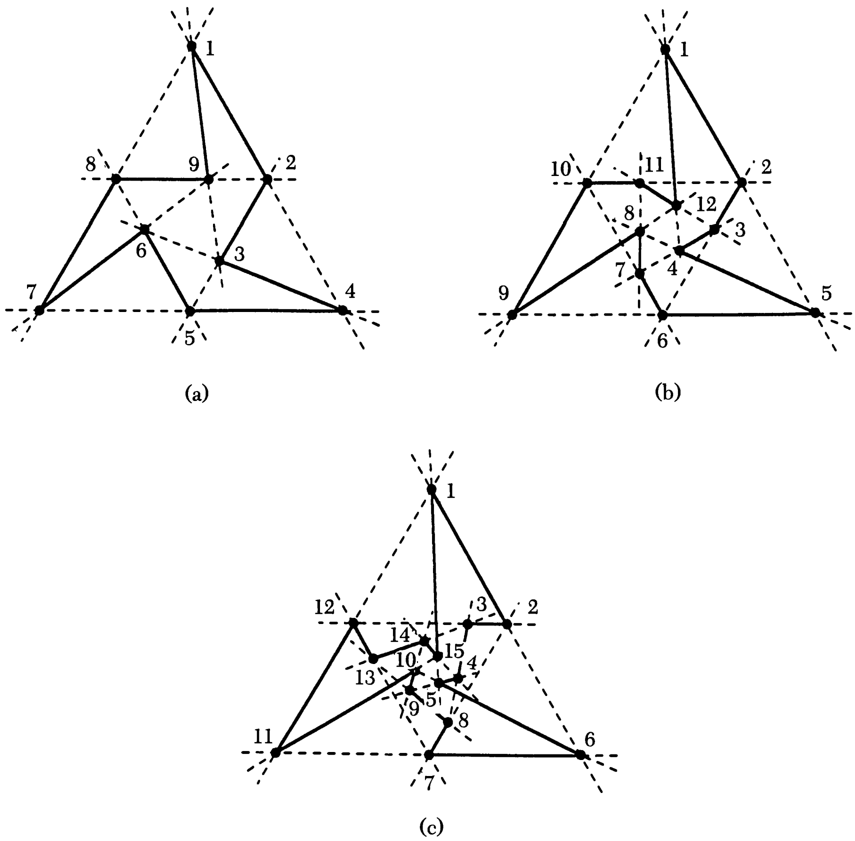


FIGURE 7
Hamiltonian polygons in the configurations of FIGURES 2 and 3.

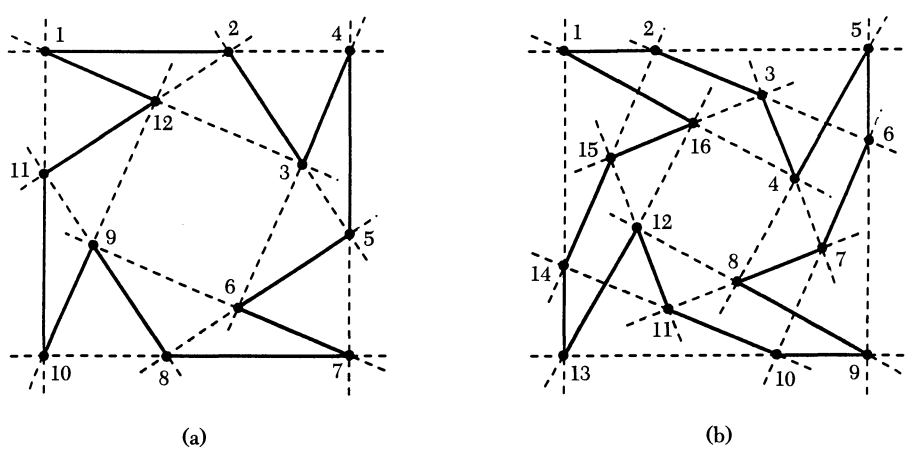


FIGURE 8
Hamiltonian polygons in the configurations of FIGURE 4.

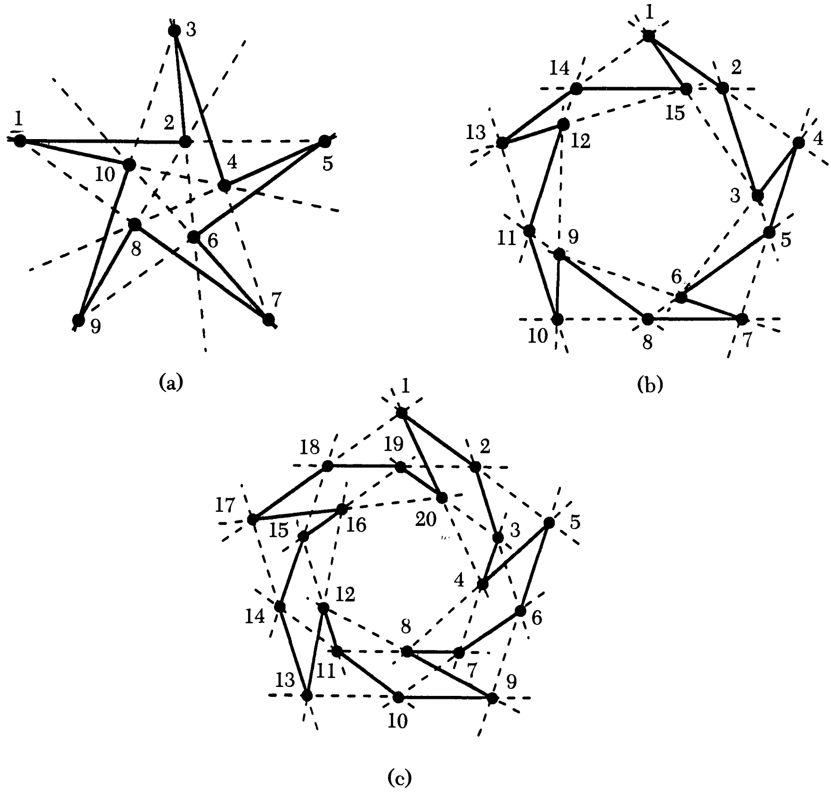


FIGURE 9
Hamiltonian polygons in the configurations of FIGURES 5 and 6(a).

Now, a few words concerning the numbering of the points in our drawings. The combinatorial aspect of configurations starts with the fact that every labeled geometric configuration—such as the one in FIGURE 2—leads to a rectangular table of integers where the labels of the three vertices on a line appear as the three entries of a column. Thus FIGURE 2 corresponds to the following table.

TABLE 1

1	2	3	4	5	6	7	8	9
2	3	4	5	6	7	8	9	1
4	5	6	7	8	9	1	2	3

We used the first two rows of this table to define the polygon and the multilateral described in the preceding paragraph. The third row, of course, lists in each column the third vertex of the configuration on each of the lines through the first two vertices of the same column. The table is an example of a *combinatorial* (or "strategic") 9_3 configuration. The interplay between combinatorial and geometric configurations leads to many results and to a variety of interesting unsolved problems.

Rules for the formation of various combinatorial n_3 configurations can be given easily (see, for example, [14]). In most enumerations of configurations, the first step consists in the construction of combinatorial configurations, followed by elimination of repetitions, that is, configurations that can be made to coincide by renaming the vertices and the lines. But such combinatorial configurations may or may not be realizable as geometric configurations in the Euclidean (or projective) plane, thus leading to many questions. Briefly, the known facts are as follows. To form a combinatorial n_3 configuration, n must be greater than 6. There is a unique (up to labeling) 7_3 and a unique 8_3, both of considerable importance in many parts of modern mathematics; neither of these two configurations is geometrically realizable (see, for example, [5]). There exist three distinct 9_3 combinatorial configurations that are all realizable [11], 10 distinct 10_3 combinatorial configurations all but one of which are realizable (see [4]), as well as 31 11_3 and 229 12_3 combinatorial configurations, all of which are realizable. For the 11_3 configurations, which have been independently enumerated by several authors (see [13], [2], [7]), diagrams for all cases have been provided by Daublebski [3]. However, diagrams can be misleading (see FIGURE 10 for a "fake" configuration that pretends to realize the one 10_3 configuration that is not geometrically realizable). The question of the realizability of the 11_3 configurations was finally settled only recently by Sturmfels and White [20], [21]. Daublebski [3] enumerated the combinatorial configurations 12_3 and found 228 different ones. This was long considered to be the correct number, but recently Gropp [9] found an additional configuration 12_3, which is specified in Table 2. (We are indebted to Dr. Gropp for making available to us his still unpublished results and for permission to refer to them.) The 228 configurations 12_3 found by Daublebski were shown by Sturmfels and White [20], [21] to be geometrically realizable. Gropp's "new" configuration is also geometrically realizable; a diagram for it is shown in FIGURE 11, and coordinates for the vertices (which show that the realization is no "fake") can be computed quite easily. (The coordinates underlying the diagram in FIGURE 11 were found by assuming arbitrary but convenient positions, compatible with the collinearities, for all points except 1, 5, and 7, and computing the location of point 1 for which all required collinearities take place.) Gropp [8] reports that there are 2,036 combinatorial configurations 13_3, and 21,399 combinatorial configurations 14_3. One of the 14_3 configurations is not connected; it consists of two copies of the 7_3 configuration, and is therefore not geometrically realizable. It is not known whether all 13_3 configurations and the 21,398 connected 14_3 configurations are geometrically realizable. For $n > 14$ no enumeration of the possible configurations (combinatorial or geometric) has been carried out. The statement by Steinitz [18], [19] that for $n \geq 11$ all n_3 configurations are "probably realizable" is contradicted by the example in FIGURE 12.

TABLE 2

1	2	3	4	5	6	7	8	9	10	11	12
2	4	5	8	1	7	12	10	3	9	6	11
3	6	7	12	4	1	9	2	11	5	8	10

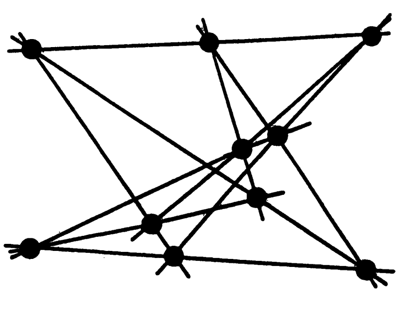


FIGURE 10

A geometric "realization" of one of the 10 combinatorial configurations 10_3—the one that, in fact, is not realizable. In this diagram lines that are only approximately straight have been used, thus leading to the illusion of realizability.

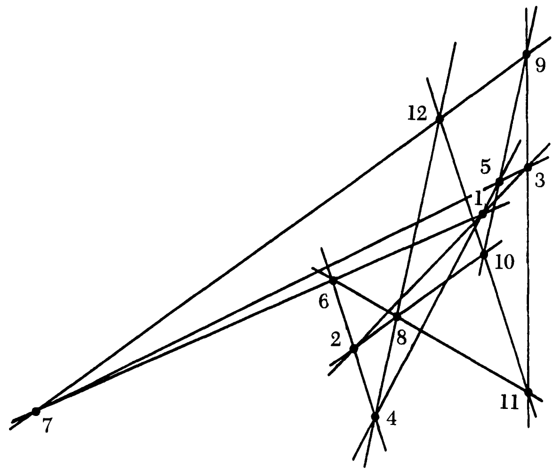


FIGURE 11

A geometric realization of the 12_3 configuration discovered by Gropp [9] and specified in Table 2.

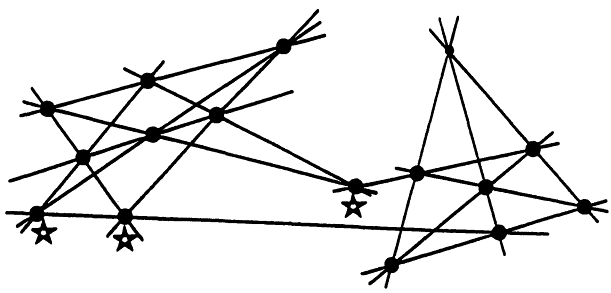


FIGURE 12

Another "fake" configuration. Although it appears to be a 16_3 configuration of points and lines, the combinatorial configuration 16_3 that seems to underlie it cannot be realized by straight lines meeting by threes as shown: The nine points and eight lines in the left half of the diagram satisfy the conditions of the famous theorem of Pappus from projective geometry, which implies that the three points marked by asterisks are necessarily collinear.

Regarding questions of realizability, it may be mentioned that some years ago the junior author of this article conjectured that every n_3 configuration that is geometrically realizable in the Euclidean plane is also geometrically realizable in the *rational plane*, that is, in the subplane of the Euclidean plane consisting of those points both of whose coordinates are rational numbers. This conjecture has been verified by Sturmfels and White [20], [21] for all the configurations 11_3, and for the 228 configurations 12_3 found by Daublebski [3]. That Gropp's configuration 12_3 is realizable in the rational plane recently was communicated to us by Bernd Sturmfels; the conjecture is still undecided for all values of n larger than 13.

Hamiltonian multilaterals

The reader has doubtlessly observed that the third row of TABLE 1 is surprisingly regular. Tables of this nature—in which each row is just a cyclic permutation of the first—are called *cyclic*; the tables for all figures shown in this note (except FIGURES 10 to 14) are either cyclic, or differ from cyclic ones by cyclic displacements of some subsets of the integers. For example, the third row, shown below, of the table for FIGURE 3(b) is of the latter kind, involving a cyclic permutation of the boxed integers.

$$6 \quad \boxed{12} \quad 8 \quad 9 \quad 10 \quad 11 \quad \boxed{2} \quad 13 \quad 14 \quad 15 \quad 1 \quad \boxed{7} \quad 3 \quad 4 \quad 5$$

Different labelings of the vertices of our configurations lead to different self-inscribed (that is, Hamiltonian) multilaterals, though not every labeling yields such a multilateral. In [6] the 36 different 9-laterals of the Pappus configuration 9_3 (that is essentially different from the 9_3 configuration in FIGURE 2) are shown, together with the 24 10-laterals for the Desargues 10_3 configuration; see also [1] and [6] for six different cycles of three mutually inscribed trilaterals in the Pappus 9_3, and for six pairs of mutually inscribed 5-laterals in the Desargues 10_3. The geometrically not realizable 7_3 and 8_3 combinatorial configurations also admit cyclic tables (see, for example, [11]). On the other hand, it should be stressed that not every geometric n_3 configuration admits a Hamiltonian multilateral. Clearly, an 18_3 consisting of two unrelated copies of 9_3 cannot have a Hamiltonian multilateral. However, even connected configurations (that is, configurations in which it is possible to reach from any vertex to any other vertex by steps along lines of the configuration) may lack Hamiltonian multilaterals. This was first established by Steinitz [17], refuting a claim made by Kantor [12] that every connected n_3 has a Hamiltonian multilateral. It is known (see Gropp [10]) that all connected configurations n_3 with $n \leq 14$ have Hamiltonian multilaterals. The smallest known example of a non-Hamiltonian connected geometric configuration is the 22_3 shown in FIGURE 13, but it may well be that there exist smaller ones.

In this context we would like to clarify and correct a statement made in an earlier paper. Given a configuration n_3 (combinatorial or geometric), it is always possible to choose for its presentation a table in which all n labels appear in each of the rows (hence each appears precisely once in each row). This is a nontrivial fact, which was established by Steinitz [16]. It is equivalent to the assertion that for any given configuration, one can find a finite family of polygons such that each vertex of the configuration is also a vertex of exactly one of the polygons. (In other words, although —as mentioned above—not every configuration admits a Hamiltonian multilateral, every configuration does admit a family of multilaterals that together act in a "Hamiltonian" way.) Due to a misinterpretation of the sources, it was stated in Page and Dorwart [14], p. 83, that a representation of this kind is not always possible for configurations n_3 with $n = 12$ or larger. By Steinitz's theorem, this statement is incorrect.

Are these figures oxymora? The answer is **both** yes and no. A "cycle of inscribed multilaterals" and "self-inscribed multilaterals" are certainly *figures* (of speech) in which terms with meanings contradictory in the colloquial sense are combined; hence each is—by definition—an oxymoron. However, it is hoped that the reader has found the FIGURES (or drawings) shown in this article to be interesting as well as clear and unambiguous; they are not oxymora.

Questions for readers

For those who have found the above material stimulating, a few final challenges:

Can you find three cyclically inscribed quadrilaterals in the 12_3 configuration of FIGURE 3(a)?

Can you decide whether the 12_3 configurations in FIGURES 3(a) and 4(a) are essentially different, or can their vertices be relabeled so that the same triplets of labels correspond to collinear sets of vertices in both configurations?

Can you find a pair of mutually inscribed 6-laterals in the 12_3 configuration of FIGURE 4(a)?

Can you give a short argument to show that the 22_3 configuration in FIGURE 13 admits no Hamiltonian multilateral?

Can you find a series of three cyclically inscribed trilaterals in the Pappus configuration 9_3 shown in FIGURE 14? Can you find a Hamiltonian 9-lateral in it?

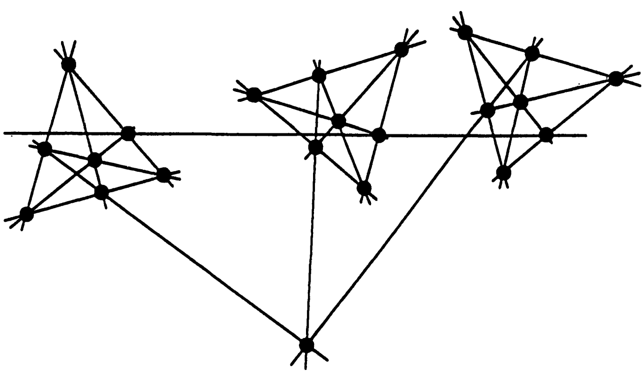


FIGURE 13
A 22_3 configuration that is connected but does not admit a Hamiltonian multilateral.

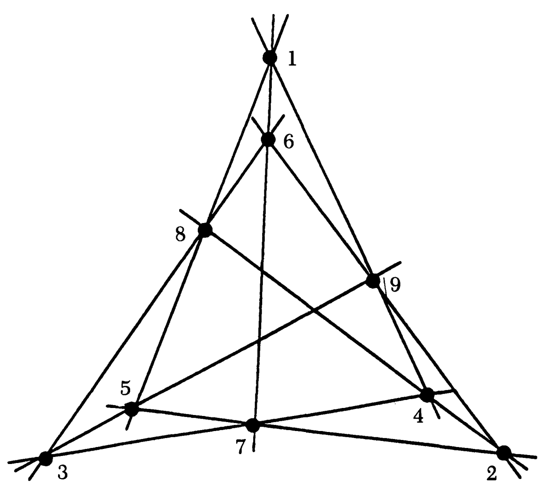


FIGURE 14
One geometric realization of the Pappus configuration 9_3.

Can you find a Hamiltonian multilateral in the 12_3 configuration of FIGURE 11 and TABLE 2?

Can you continue the series of "star-shaped" configurations shown in FIGURE 6, and find analogous $14_3, 18_3, \ldots$ configurations, consisting of pairs of mutually inscribed k-laterals? Can you find more than one way of continuing the series?

Note. The experimentation suggested at the end of Section 1, as well as many other investigations of configurations, can be carried out very conveniently using the "Geometer's Sketchpad" software for Macintosh computers. This software has just been released by Key Curriculum Press.

The authors are indebted to the anonymous referees for helpful comments.

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