Geombinatorics 1(1991), No. 1, pp. 4 - 5

ODD POLYHEDRA

by Branko Grünbaum
University of Washington
GN-50, Seattle, WA 98195

Let P be a monohedral polyhedron in 3-dimensional Euclidean space, that is, a polyhedron having all faces congruent to each other. Here "polyhedron" is restricted to mean a surface of which one could build a faithful cardboard model: each face is a simple polygon (without self-intersections), and the faces meet only along common edges or vertices (so the surface has no self-intersections). On the other hand, neither the faces of a polyhedron nor the polyhedron itself need be convex.

Hugo Steinhaus asked (see Problem 193 in [1]) whether there exist any monohedral convex polyhedra with an odd number of faces. A negative answer was established thirty years ago (see [2]). It is easy to see (using parity arguments and Euler's theorem) that any monohedral convex polyhedron with an odd number of faces would have to have quadrangular faces; the elimination of that eventuality requires elementary but non-trivial arguments. Some geometric arguments are clearly necessary, since there exist convex polyhedra with an odd number of faces, all of which are quadrangles. The smallest example I am aware of has nine quadrangular faces (see the sketch below).

However, if convexity is eliminated from Steinhaus's question, we have an unsolved problem. In fact, no argument is known which would eliminate the possibility of a monohedral polyhedron with convex quadrangles as faces, the polyhedron itself being of genus 0 (that is, a distortion of the sphere) but not convex. Unsolved as well are the analogous questions for toroidal polyhedra, or polyhedra of higher genus. I conjecture that in all these cases the answer is also negative.

[1] H. Fast and S. Swierczkowski (Eds.), *The New Scottish Book*. Wroclaw, 1957,

[2] B. Grünbaum, On polyhedra in E^3 having all faces congruent. *Bull. Res. Counc. of Israel*, Vol. 8F (1960), pp. 215-218.