PORTUGALIAE MATHEMATICA

VOLUME 18

1 9 5 9

Publicação subsidiada por

Publication subventionnée par — Publication sponsored by

JUNTA DE INVESTIGAÇÃO MATEMÁTICA, SOCIEDADE PORTUGUESA DE MATEMÁTICA e FUNDAÇÃO CALOUSTE GULBENKIAN

Edição de

«GAZETA DE MATEMÁTICA, LDA.»

PORTUGALIAE MATHEMATICA
Rua Nova da Trindade, 1, 5.º-S
LISBOA-2 (PORTUGAL)

HERMANN & C.ie, Editeurs
6, Rue de la Sorbonne
PARIS (5eme)

PORTUGALIAE MATHEMATICA
Vol. 18 — Fasc. 3 — 1959

ON INTERSECTIONS OF SIMILAR SETS *

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1. Introduction. Among other variants of HELLY's theorem on intersections of convex sets, HADWIGER and DEBRUNNER [8] have established the following result:

Given any family of congruent and mutually intersecting circles (= circular discs) in the plane, there exist three points such that each circle of the family contains at least one of them. The number three cannot be reduced to two.

In the present paper we shall deal with similar problems.

To any compact convex set K we associate two numbers, $h(K)$ and $H(K)$, defined as follows:

h(K) *is the least cardinal* n *such that whenever a family* $\mathcal{K} = \{x_\alpha + K\}$ *of mutually intersecting translates of* K *is given, there exist* n *points such that each member of* $\mathcal{K}$ *contains at least one of them.*

H(K) *is the least cardinal* n *such that whenever a family* $\mathcal{K} = \{x_\alpha + \lambda_\alpha K\}$ *of mutually intersecting translates of sets homothetic to* K *is given, there exist* n *points such that each member of* $\mathcal{K}$ *contains at least one of them.*

Obviously, both $h(K)$ and $H(K)$ are invariant under non-singular affine transformations of K. Moreover, $h(K)$ and $H(K)$ remain unchanged if $K \subset E^h$ is imbedded in a space of higher dimension (E^k is the k-dimensional Euclidean space ; therefore, without loss of generality we shall in the sequel assume that Int $K \neq \emptyset$.

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* Received April 1959.

(1) This research was supported by the United States Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command, under contract No. A F 49 (638)-253.

Using the notion of $h(K)$ the above result of HADWIGER and DEBRUNNER may be stated as follows: If K is a circle then $h(K)=3$.

Another well-known result is (see Sz.—NAGY [14], HANNER [10], NACHBIN [12]): $h(K)=1$ for $K \subset E^n$ if and only if K is a parallelohedron; then also $H(K)=1$.

In § 2 of the present paper we shall establish the finiteness of $h_n = \max\{h(K);\ K \subset E^n\}$ and $H_n = \max\{H(K);\ K \subset E^n\}$. In §§ 3 and 4 we shall consider in some detail $h(K)$ and $H(K)$ for centrally symmetric $K \subset E^2$. In § 5 we discuss some unsolved problems and offer some conjectures.

§ 2. Bounds for $h(K)$ and $H(K)$. We start with

THEOREM 1. — $H_n = \max\{H(K);\ K \subset E^n\}$ *is finite.*

PROOF. Let $\mathcal{K} = \{x_\alpha + \lambda_\alpha K\}$ be a family of mutually intersecting translates of sets homothetic to $K \subset E^n$. If $\inf \lambda_\alpha = 0$, there exists a point common to all the members of $\mathcal{K}$. We may therefore assume that $\inf \lambda_\alpha > 0$ and (using the affine invariance of $H(K)$) $\inf \lambda_\alpha = 1$. Without loss of generality we may also assume that the set K is a member of $\mathcal{K}$. Let C denote an ellipsoid contained in K such that a translated of nC, concentric with C, contains K. The existence of such an ellipsoid C was established by JOHN [11]; another proof follows very easily from the remark that LÖWNER's ellipsoid of K (i. e. the ellipsoid of smallest volume containing K, see e. g. DANZER, LAUGWITZ and LENZ [2], ZAGUSKIN [15]) may be taken as nC. Since $H(K)$ is an affine invariant of K, we may assume that $C = S^n$ is the (solid) unit sphere of E^n; thus $S^n \subset K \subset nS^n$. Now we divide $\mathcal{K}$ into two subfamilies, $\mathcal{K}_1 = \{x_\alpha + \lambda_\alpha K \in \mathcal{K};\ |x_\alpha| \leq n+2\}$ (where $|x|$ denotes the distance from x to 0) and $\mathcal{K}_2 = \{x_\alpha + \lambda_\alpha K \in \mathcal{K};\ |x_\alpha| > n+2\}$. Each member $x_\alpha + \lambda_\alpha K$ of $\mathcal{K}_1$ contains the sphere $x_\alpha + S^n$ whose center x_α belongs to $(n+2)S^n$. Since $(n+2)S^n$ is compact there exist N_1 points of $(n+2)S^n$ (N_1 depending only on n) such that at least one of them is contained in any sphere $x + S^n$ with $x \in (n+2)S^n$. It follows that each member of $\mathcal{K}_1$ contains at least one of these N_1 points.

On the other hand, if $x_\alpha + \lambda_\alpha K \in \mathcal{K}_2$, it follows from $x_\alpha + \lambda_\alpha S_n \subset x_\alpha + \lambda_\alpha K \subset x_\alpha + n\lambda_\alpha S^n$ and $K \cap (x_\alpha + \lambda_\alpha K) \neq \emptyset$ that there exists a point $y_\alpha \in \text{Front}\, nS^n \cap (x_\alpha + \lambda_\alpha K)$ such that

$2 < |y_\alpha - x_\alpha| \leq n\lambda_\alpha$. The convex hull of λ_α and $x_\alpha + \lambda_\alpha S^n$ is contained in $x_\alpha + \lambda_\alpha K$ and therefore $(x_\alpha + \lambda_\alpha K) \cap \mathrm{Front}\,(n+1) S^n$ contains the intersection of Front $(n+1)S^n$ with a cone whose vertex belongs to Front nS^n and which has a vertex angle 2ϕ satisfying $\sin\phi \leq \frac{1}{n}$. This intersection, and thus also $x_\alpha + \lambda_\alpha K$, obviously contains a spherical cap of Front $(n+1)S^n$ with angular radius ρ satisfying $\sin\rho > 1/n(n+1)$. Now, there exists a finite number N_2 of points of Front $(n+1)S^n$ (N_2 depending only on n) such that at least one of them belongs to any spherical cap of Front $(n+1)S^n$ having angular radius exceeding arc sin $\frac{1}{n(n+1)}$. Therefore at least one of these N_2 points belongs to each member of $\mathcal{K}_2$; it follows that each member of $\mathcal{K}$ contains at least one of the $N_1 + N_2$ points chosen above, which proves Theorem 1.

Obviously $h(K) \leq H(K)$ for any set K. Theorem 1 implies therefore that $h_n = \max\{h(K); K \subset E^n\}$ is finite.

Remarks. (i) The proof of Theorem 1 may clearly be used to obtain some very rough bounds for H_n. The reasoning may be slightly simplified for h_n, and we can show that $h_n < (3n^{3/2})^n$, while for centrally symmetric $K \subset E^n$ the better estimate $h(K) < (\sqrt{2}\,n)^n$ holds. Since these estimates are probably very far from the best possible ones, we omit the profs.

(ii) If K is a compact, convex subset of linear topological space, it is possible that $h(K) \geq \aleph_0$. The unit cell S of the Hilbert space space l^2, which is compact in the weak topology of l^2, may be used to construct such an example. Il $\{e_n\}$ denotes an orthonormal basis for l^2, then any two of the sets $\{\sqrt{2}\,e_n + S\}$ intersect (in one point only); but the intersection of any three of them is void and therefore $h(S) \geq \aleph_0$. Similar examples may be found in other Banach spaces; it would be interesting to know something about the situation in Hilbert or Banach spaces for sets compact in the norm topology.

(iii) If $h^*(K)$ and $H^*(K)$ are defined in a way analogous to the definition of $h(K)$ and $H(K)$, but for families of sets congruent, resp. similar, to K, it is possible to show by a method very similar to that used in the proof of Theorem 1 that

$h^*(K) \leq H^*(K) < \infty$ for any set $K \subset E^n$ with non-void interior. But obviously $\sup \{h^*(K); K \subset E^n, \text{Int } K \neq \emptyset\}$ is not finite.

(IV) The definition of $h^*(K)$ is meaningful also if K is a subset of C^n, the boundary of S^{n+1}. Again, if the n-dimensional interior of K is not void, $h^*(K)$ is finite. HADWIGER [7] has proved a theorem which implies $h^*(K) \leq n+2$ for sets $K \subset C^n$ wich contain a spherical cap of angular radius ρ, where $\cos \rho \leq 1/(n+1)$. On the other hand, he raised at the end of the same paper a related problem, which may be formulated as follows: Given a family $\mathcal{K}$ of compact, convex subsets of C^n, such that for any two members K_1, K_2 of $\mathcal{K}$ there exists a point $x \in K_1$ with $-x \in K_2$, do there exist $n+2$ points such that each member of $\mathcal{K}$ contains at least one of them? As HADWIGER remarks, it is easy to show that the answer is affirmative for $n=1$.

The following example shows that already for $n=2$ the answer is negative, even if the members of $\mathcal{K}$ are supposed to be congruent convex bodies contained in an open hemisphere, and if any number is substituted for $n+2=4$.

EXAMPLE 1. For each natural $k>1$ there exists a family of k pairwise disjoint subsets of C^2, all congruent to a sperical convex body which does not intersect a great circle, such that any pair of sets contains a pair of antipodal points.

The construction of the family may be described as follows: We take $k \geq 3$ equidistant points on the «equator» of C^2 and, on great circles through these points (equi-inclined to the equator in a, say, «north-east» to «south-west» direction), we consider arcs which contain the points and reach in both directions up to the intersections with the neighboring great circles. If we diminish slightly the size of the k slanted «orange peels» determined by the k arcs, the resulting family clearly has all the required properties.

§ 3. $h(K)$ for centrally symmetric $K \subset E^2$

Given a centrally symmetric convex body $K \subset E^2$, the plane becomes a MINKOWSKI plane with norm $\|x\|$, according to which K is the unit cell; $K = \{x; \|x\| \leq 1\}$. All the metric concepts used in §§ 3 and 4 are with respect to this norm.

We prove first

THEOREM 2. *If* $K \subset E^2$ *has a center of symmetry then* $h(K) \leq 3$.

PROOF. JUNG's constant J_K of a MINKOWSKI plane with unit cell K is defined as the minimum of positive numbers λ having the proporty: Given any set A of diameter not greater than 2, there exists a translate of λK containing A. We shall need the following proposition established in [5]: $J_K \cdot K$ is contained in the union of three suitable translates of K.

Let now $\mathcal{K} = \{x_\alpha + K\}$ be a family of mutually intersecting translates of K. Then $\| x_\alpha - x_\beta \| \leq 2$, *i.e.* the set $X = \{x_\alpha\}$ has diameter less than or equal to 2. Therefore X is contained in a translate of $J_K . K$, and therefore also in the union of appropriate translates $y_i + K, i = 1, 2, 3$, of K. It follows from the symmetry of K that each member of $\mathcal{K}$ contains at least one of the points y_i. This ends the proof of Theorem 2.

REMARK. HADWIGER and DEBRUNNER [8] use in their proof of $h(S^2) = 3$ a very similar argument, based on PÁL's [13] theorem (according to which any subset of the plane can be covered by a regular hexagon of diameter $2/\sqrt{3}$ times the diameter of the set). They note also that the three points may be chosen as the vertices of an equilateral triangle. The same is true also for the three points whose existence is established in Theorem 2, if the sides of the triangle are measured in the norm determined by K.

The following result shows that the estimate of $h(K)$ given by Theorem 2 is the best possible for a large classe of sets K.

THEOREM 3. *If* $K \subset E^2$ *has a center of symmetry and is strictly convex then* $h(K) = 3$.

PROOF. Let x_1, x_2, x_3, be points such that $\| x_1 - x_2 \| = \| x_2 - x_3 \| = \| x_3 - x_1 \| = 2$ but no translate of K contains all three points x_1, x_2, x_3. (The existence of such points follows at once from the reasoning used in the proof of Lemma 1 of [5].) We define the «Reuleaux triangle» T, with vertices x_i, by $T = \bigcap_{i-1}^{3} (x_i + 2K)$. It is easily proved that the diameter of T is 2. From the choice of x_1, x_2, x_3 it follows at once that there exist points $x_{ij} \epsilon$ Front T, $i, j = 1, 2, 3, i \neq j$, such that:

(a) x_{ij} belongs to the interior of that arc of Front T determined by x_i and x_j which does not contain the third vertex x_k;

(b) no translate of K containing x_i contains both x_{ij} and x_{ik}.

Then the family $\mathcal{H} - \{x_i + K, i = 1, 2, 3\} \cup \{x_{ij} + K, i, j = 1, 2, 3, i \neq j\}$ proves that $h(K) \geq 3$. Indeed, assuming that there exist two points, p and q, such that each member of $\mathcal{H}$ contains at least one of them, it follows that one of the points is contained in $x_i + K$ for two different values of i; with appropriate notations this implies $p = \frac{1}{2}(x_1 + x_2)$. Since $p \notin (x_3 + K) \cup (x_{31} + K) \cup (x_{32} + K)$, this implies $q \in (x_3 + K) \cap (x_{31} + K) \cap (x_{32} + K)$. But the last relation is equivalent to $x_3, x_{31}, x_{32} \in q + K$, and thus contradicts the condition (b). This ends the proof of Theorem 3.

The following example is of interest since it shows that in Theorem 3 it is impossible to replace the assumption «K is strictly convex» by «K not a parallelogram.» (Parallelograms are the only «exceptional sets» in certain combinatorial problems of a related nature; see, e. g., [6].)

Example 2. If H is a regular hexagon then $h(H) = 2$.

Indeed, let $\mathcal{H} = \{x_\chi + H\}$ be any family of mutually intersecting translates of H. Using a suitable affine transformation, we may assume that the set $X = \{x_\chi\}$ has diameter 2, and arrange the notation in such a way that $\|x_1 - x_2\| = 2$. Obviously $X \subset (x_1 + 2H) \cap (x_2 + 2H)$, and the complement C of $\frac{1}{2}(x_1 + x_2) + H$ in $(x_1 + 2H) \cap (x_2 + 2H)$ has two connected components. Let us consider $R = X \cap C$. If $R = \emptyset$, $\frac{1}{2}(x_1 + x_2)$ belongs to each member of $\mathcal{H}$, so we may assume $R \neq \emptyset$. Then, since diam X $= 2$, only the following two cases are possible:

(I) R is contained in one of the components of C. Since each of the components of C may be covered by a suitable translate $y + H$ of H, it follows that in this case each member of $\mathcal{H}$ contains either $\frac{1}{2}(x_1 + x_2)$ or y (or both).

(ii) R is contained in one of the closed half-planes determined by a straight line passing through $\frac{1}{2}(x_1+x_2)$ and parallel to a pair of sides of $\frac{1}{2}(x_1+x_2)+H$ containing x_1 and x_2. Let x_2 belong to that half-plane which contains R. Then X is contained in the union of y_1+H and y_2+H, where y_1 and y_2 are such that: $y_i+H \subset x_1+2H$ for $i=1,2$, and $(y_1+H)\cup(y_2+H)$ covers that side of x_1+2H which contains x_2. Obviously each member of $\mathcal{H}$ then contains at least one of the points y; and thus $h(H)=2$ is established.

§ 4. H(K) for centrally symmetric $K \subset E^2$. Turning now to H(K) we prove first:

THEOREM 4. *If* $K \subset E^2$ *has a center of symmetry then* $H(K) \leq 7$.

PROOF. Let $\mathcal{H}=\{x_\alpha+\lambda_\alpha K\}$ be a family of mutualy intersecting sets. As in the proof of Theorem 1, we may without loss of generality assume that $K \in \mathcal{H}$ and that $\inf \lambda_\alpha = 1$. It is well-known (see [6] for references) that there exist points $y_i, i=1,2,3,4,5,6$, such that (taking $y_7=y_1$):

(i) $\|y_i\|=2$ and

(ii) $\|y_i-y_{i+1}\|=2$ for all i.

Let $z_i=\frac{1}{2}(y_i+y_{i+1}), i=1,2,3,4,5,6$. (Then $\|z_i-y_i\|=\|z_i-y_{i+1}\|=\|z_i-\frac{1}{2}y_i\|=\|z_i-\frac{1}{2}y_{i+1}\|=1$.) We shall establish Theorem 4 by proving that each member of $\mathcal{H}$ contains at least one of the points $0, z_1, z_2, z_3, z_4, z_5, z_6$. Indeed, let $0 \notin x_\alpha+\lambda_\alpha K \in \mathcal{H}$. Then there exists a point $x_\alpha{}^*$ such that $x_\alpha{}^*+K \subset x_\alpha+\lambda_\alpha K$ and $1 \leq \|x_\alpha{}^*\| \leq 2$. Now $x_\alpha{}^*$ is contained in one of the six sections of the «ring» $\{x; 1 \leq \|x\| \leq 2\}$ formed by the rays issuing from 0 to the points y_i. If $x_\alpha{}^*$ belongs to the sector determined by y_i and y_{i+1}, then $z_i \in x_\alpha{}^*+K$ and thus also $z_i \in x_\alpha+\lambda_\alpha K$. This ends the proof of Theorem 4.

Next we show that $h(K) < H(K)$ is possible.

EXAMPLE 3. $H(S^2) \geq 4 > h(S^2)$.

To prove this statement, we shall describe a family consisting of 21 circles and show that no set of three points has non-void intersection with each member of the family.

Let $L_k, 1 \leq k \leq 5$, denote the ray issuing from the origin O of a polar system of coordinates in the plane, defined by $\phi = \phi_k = \frac{2\pi}{5} k$, respectively. Taking each of the points $(1;\phi_k)$ as center, we construct a circle A_k with radius $\cos \frac{\pi}{10}$. (A_k and A_{k+2} intersect in one point.) Next, we construct the circles B_k having $(1.001;\phi_k)$ as centers and such a radius as to touch A_{k+2} (and A_{k-2}). To complete the family we take a circle E with center at O, circles C_k with centers at the points of $B_k \cap L_k$ nearer to the origin, and circles D_k with centers at $(0.002;\phi_k)$. The radius of each of the circles C_k, D_k, E is taken to be minimal subject to the condition of intersecting each of the circles B_n.

It is clear that any two of the 21 circles described have a non-void intersection. In order to prove that no three points a, b, c are such that each of the circles A_k, B_k, C_k, D_k, E contains at least one of them, we proceed in the following way:

Assuming that such points a, b, c exist, we have to consider only the following cases (permuting the indices k, and a, b, c, if necessary):

(i)	$a \in A_1 \cap A_2 \cap A_3$,	$b \in A_4 \cap A_5$,	$c \in E$
(ii)	$a \in A_1 \cap A_2 \cap A_3$,	$b \in A_4$,	$c \in A_5 \cap E$
(iii)	$a \in A_1 \cap A_2$,	$b \in A_3 \cap A_4$,	$c \in A_5 \cap E$

In case (i), a belongs to B_2; since b cannot belong to more than three of the circles $B_k, c \in B_j$ for some j. But this is impossible since a and b do not belong to any of the circles D_k, while $c \in B_j$ belongs only to three of them.

In case (ii), if c belongs to some B_j, the impossibility follows as in case (i); but if c does not belong to any of the circles B_k, b has to belong to four of them, which is impossible.

In case (iii), if $c \in B_5$ then D_2 and D_3 do not contain any of the points a, b, c; on the other hand, if $c \notin B_5$, then $a \in B_1 \cap B_2 \cap B_5$ and $b \in B_3 \cap B_4$. Therefore a does not belong to any C_k, while b belongs at most to C_3 and C_4; this implies that $c \in C_1 \cap C_2 \cap C_5$ which is impossible, since $E \cap A_5 \cap C_2 = \phi$.

Thus $H(S^2) \geq 4$, as claimed.

REMARK. The problem of determining $H(S^2)$ has been considered by various authors but, to the best of our knowledge, no final solution has been published. FEJES TÓTH [3, p. 97] mentions that P. UNGÁR and G. SZEKERES have established $H(S^2) \leq 7$ and that GALLAI has conjectured that $H(S^2) \leq 5$. According to HADWIGER and DEBRUNNER [9], A. HEPPES has shown that $H(S^2) \leq 6$, L. SZACHTO has proved that $H(S^2) \leq 5$, and FEJES TÓTH has conjectured that $H(S^2) \leq 4$; the latter conjecture is said to have been confirmed by (unpublished) investigations of L. DANZER.

§ 5. Remarks and problems. (i) For $n > 2$ it appears to be very hard to obtain reasonable estimates of $h(K)$ and $H(K)$, even for very simple n-dimensional sets K.

For $h(S^3)$, GALE'S [4] result that any subset A of E^3 may be covered by a regular octagon whose diagonals are $\sqrt{3} \cdot \mathrm{diam}\, A$ implies that $h(S^3) \leq 6$; the six points may, moreover, be taken as the vertices of a regular octagon of diameter $2(\sqrt{3}-1)$.

(ii) Although the methods used in the proof of Theorem 2 do not apply to case of asymmetric K, it seems very probable that $h_2 = 3$. More precisely, it seems that if K is a convex subset of E^2, not having a center of symmetry, then $h(K) = 3$.

(iii) No example being known of a set $K \subset E^n$ with $h(K) > n+1$, it is natural to conjecture that $h(K) \leq n+1$ for any n-dimensional set K. If true, even for centrally symmetric sets K only, this conjecture would imply the following statement (related to the famous, and for $n > 3$ stil undecided, conjecture of BORSUK [1] that any bounded subset of E^n may be partitioned into $n+1$ parts having smaller diameters): Any subset A of an n-dimensional MINKOWSKI space may be covered by $n+1$ cells having diameters equal to that of A.

For MINKOWSKI planes this conjecture (equivalent in this case to Teorem 2) was established in [5].

REFERENCES

[1] BORSUK, K., *Drei Sätze über die n-dimensional euklidische Sphäre*. Fund. Math., **20** (1933), 177-190.

[2] Danzer, L., Laugwitz, D., and Lenz, H., *Über das Löwnersche Ellipsoid und sein Analogon unter den einem Eikörper einbeschriebenen Ellipsoiden.* Archiv der Math., **8** (1957), 214-219.

[3] Fejes Tóth, L., *Lagerungen in der Ebene, auf der Kugel und im Raum.* Berlin-Göttingen-Heidelberg, 1953.

[4] Gale, D., *On inscribing n-dimensional sets in a regular n-simplex.* Proc. Amer. Math. Soc., **4** (1953), 222-225.

[5] Grünbaum, B., *Borsuk's partition conjecture in Minkowski planes.* Bull. Research Council of Israel, **7 F** (1957), 25-30.

[6] ———, *On a conjecture of* H. Hadwiger. Pacif. Journ. of Math. (to appear).

[7] Hadwiger, H., *Kleine Studie zur kombinatorischen Geometrie der Sphäre.* Nagoya Math. Journal, **8** (1955), 45-48.

[8] Hadwiger, H., and Debrunner, H., *Ausgewählte Einzelprobleme der kombinatorischen Geometrie in der Ebene.* Enseignement mathématique, (2) **1** (1955), 56-89.

[9] ———, *Kombinatorische Geometrie in der Ebene.* (in preparation).

[10] Hanner, O., *Intersections of translates of convex bodies.* Math. Scandinavica **4** (1956), 65-87.

[11] John, F., *Extremum problems with inequalities as subsidiary conditions.* Courant Anniv. Vol., 1948, pp. 187-204.

[12] Nachbin, L., *A theorem of the Hahn-Banach type for linear transformations.* Trans. Amer. Math. Soc., **68** (1950), 28-46.

[13] Pál, J., *Über ein elementares Variations-problem.* Math. fys. Medd., Danske Vid. Selsk., **3** (1920), No. 2.

[14] Sz. - Nagy, B., *Ein Satz über Parallelverschiebungen konvexer Körper.* Acta Sci. Math. Szeged, **15** (1954), 169-177.

[15] Zaguskin, V. L., *On circumscribed and inscribed ellipoids of extremal volume.* Uspehi Mat. Nauk, **13**, No. 6 (84) (1958), 89-93. (In Russian).