ON SOME COVERING AND INTERSECTION PROPERTIES IN MINKOWSKI SPACES

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1. **Introduction** Let X denote a Minkowski space (i.e. a finite dimensional normed linear space over the reals) and let $S = \{x \in X; \|x\| \leq 1\}$ denote the unit cell of X. In this note we shall be concerned with two numbers, E_X and J_X, determined by the geometric properties of X.

DEFINITION 1. The *expansion constant* E_X of X is the greatest lower bound of real numbers $\mu \geq 0$ which possess the following property:

Given any family $\{x_i + \alpha_i S; i \in I\}$ of mutually intersecting cells (in other words, given any family $\{x_i\}$ of points and any family $\{\alpha_i\}$ of non-negative numbers such that $\|x_i - x_j\| \leq \alpha_i - \alpha_j$ for all $i, j \in I$); then

$$\bigcap_{i \in I} (x_i + \mu \alpha_i S) \neq \phi .$$

DEFINITION 2. *Jung's constant* J_X of X is the greatest lower bound of real numbers μ which possess the following property:

Given any family $\{x_i + S; i \in I\}$ of mutually intersecting cells (i.e. given any family $\{x_i\}$ such that $\|x_i - x_j\| \leq 2$ for all $i, j \in I\}$; then

$$\bigcap_{i \in I} (x_i + \mu S) \neq \phi .$$

We note the following immediate consequences of the above definitions:

(i) By Helly's theorem on intersections of convex sets, the index set I may be assumed to consist of not more than $n + 1$ elements, where n is the dimension of X.

(ii) Standard compactness arguments show that E_X and J_X are not only the greatest lower bounds, but even the minima of the numbers μ defining them.

(iii) $1 \leq J_X \leq E_X \leq 2$.

(iv) J_X may equivalently be defined as the smallest number such that a cell of that diameter may cover, after a suitable translation, any set of diameter ≤ 1 .

The definitions of E_x and J_x apply obviously to more general spaces. We will show elsewhere [4] that E_x plays an important role in the problem of extensions of transformations. In this context Nachbin [9] and Aronszajn-Panitchpakdi [1] have studied concepts closely related to the property $E_x = 1$.

Szökefalvi-Nagy [12] and Hanner [6] proved that Minkowski spaces whose Jung constant equals unity are precisely those having a parallelohedron as unit cell; the corresponding result for the expansion constant had been established by Nachbin [9].

For the Euclidean space E^n the value $J_{E^n} = [2n/(n+1)]^{\frac{1}{2}}$ was determined by Jung [7]. Bohnenblust [2] proved that $J_x \leq 2n/(n+1)$ if X is an n-dimensional Minkowski space. Leichtweiss [8] gave a simpler proof of Bohnenblust's result and characterized those n-dimensional Minkowski spaces X for which $J_x = 2n/(n+1)$.

The main object of the present note is the determination of the exact upper bound of E_x for n-dimensional spaces X and the characterization of those spaces for which the bound is attained.

2. Bound for the expansion constants. We show first of all that Bohnenblust's bounds for Jung's constants apply also to expansion constants.

THEOREM 1. *If X is an n-dimensional Minkowski space then $E_x \leq 2n/(n+1)$.*

Proof. Let $\mathscr{S} = \{x_i + \alpha_i S ; i \in I\}$ be any family of mutually intersecting cells in X. We shall show that

$$(1) \qquad \bigcap_{i \in I} \left(x_i + \frac{2n}{n+1} \alpha_i S \right) \neq \phi .$$

As remarked above in (i), it is sufficient to prove (1) in case I has $n+1$ elements; we shall assume $I = \{0, 1, \cdots, n\}$. We shall also assume that $\alpha_i > 0$ for each $i \in I$; indeed if $\alpha_i = 0$ then x_i belongs to all members of $\mathscr{S}$ and (1) obviously holds.

Let us denote

$$(2) \qquad \alpha = \sum_{i=0}^{n} \alpha_i .$$

It is convenient to deal separately with two possible cases:

$$(3) \qquad \text{(i) } \alpha \geq (n-1)\alpha_k \text{ for all } k \in I$$

(ii) there exists a $k \in I$ such that $\alpha < (n-1)\alpha_k$.

In both cases we shall establish (1) by exhibiting a point belonging to

$$(4) \qquad \bigcap_{i=0}^{n} (x_i + \mu \alpha_i S)$$

for a suitable $\mu \leq 2n/(n + 1)$.

We consider first the case (i) and we introduce the following notations:

$$\alpha'_i = 1/\alpha_i \quad \text{for } i \in I$$
$$(5) \qquad \alpha' = \sum_{i=0}^{n} \alpha'_i$$

$$(6) \qquad \sigma = \sum_{0 \leq i \leq j \leq n} (\alpha_i \alpha'_j + \alpha'_i \alpha_j) = \alpha\alpha' - (n + 1) \ .$$

Since $\alpha_i \alpha'_j + \alpha'_i \alpha_j \geq 2$, and $\alpha_i \alpha'_j + \alpha'_i \alpha_j = 2$ if and only if $\alpha_i = \alpha_j$, we have

$$(7) \qquad \sigma = n(n + 1) + \delta \text{ with } \delta \geq 0$$

and

$$(8) \qquad \delta = 0 \text{ if and only if } \alpha_0 = \alpha_1 = \cdots \alpha_n \ .$$

We also put

$$(9) \qquad \beta_i = \frac{\alpha'_i \alpha - n + 1}{\sigma - (n + 1)(n - 2)} = \frac{\alpha'_i \alpha - n + 1}{2(n + 1) + \delta} \text{ for } 0 < i < n$$

and we note that (3) and (7) imply $\beta_i \geq 0$ for $0 \leq i \leq n$ and $\sum_{i=0}^{n} \beta_i = 1$.
We shall show that the point

$$(10) \qquad z = \sum_{i=0}^{n} \beta_i x_i$$

belongs to (4), with

$$(11) \qquad \mu = \frac{\sigma - n(n - 2)}{\sigma - (n + 1)(n - 2)} = \frac{4n + \delta}{2n + 2 + \delta} \leq \frac{2n}{n + 1} \ ,$$

and thus prove the theorem in case (i).

Since all the points x_i enter symmetrically in our reasoning, it is sufficient to show that

$$(12) \qquad z \in x_0 + \mu \alpha_0 S \ ;$$

the relation (12) will be established by exhibiting a point $y \in x_0 + \alpha_0 S$ such that

$$(13) \qquad z - x_0 = \mu(y - x_0) \ .$$

Let $y_i = (\alpha_i x_0 + \alpha_0 x_i)/(\alpha_0 + \alpha_i)$ for $1 \leq i \leq n$ and

$$\gamma_i = \frac{(1 + \alpha_i \alpha')(\alpha'_i \alpha - n + 1)}{\sigma - n(n - 3)} = \frac{(1 + \alpha_i \alpha')(\alpha'_i \alpha - n + 1)}{4n + \delta}$$

for $1 \leq i \leq n$. Then $y_i \in x_0 + \alpha_0 S$ since $(x_0 + \alpha_0 S) \cap (x_i + \alpha_i S) \neq \phi$; on the other hand, $\gamma_i \geq 0$ because of (3) and (7), and an elementary computation shows that $\sum_{i=1}^{n} \gamma_i = 1$.

These relations imply that the point $y = \sum_{i=1}^{n} \gamma_i y_i$ satisfies $y \in x_0 + \alpha_0 S$, and equation (13) can be established by a straight-forward computation:

$$[\sigma - n(n-3)](y - x_0)$$

$$= -[\sigma - n(n-3)]x_0 + \sum_{i=1}^{n} (1 + \alpha_i \alpha_0')(\alpha_i' \alpha - n + 1)\frac{\alpha_i x_0 + \alpha_0 x_i}{\alpha_i + \alpha_0}$$

$$= -[\sigma - n(n-3)]x_0 + 2(n-1)x_0 + \sum_{i=0}^{n} (\alpha \alpha_i' - n + 1)x_i$$

$$[\sigma - (n+1)(n-2)]\left(\sum_{i=0}^{n} \beta_i x_i - x_0\right)$$

$$= [\sigma - (n+1)(n-2)](z - x_0) .$$

We turn now to the remaining case (ii). We change, if necessary, the notation in such a way that $0 < \alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n$. Then there exists one and only one k, with $2 \leq k < n$, such that

$$(k-1)\alpha_k \leq \sum_{i=0}^{k} \alpha_i < (k-1)\alpha_{k+1} .$$

Denoting by $\alpha^*, \alpha'^*, \sigma^*, \delta^*, \beta^*, z^*, \mu^*$ quantities analogous to those designated by the same letter without asterisk and defined by substituting in (2), (5), (6), (7), (9), (10), and (11) the letter k instead of n and adding asterisks, we may prove as above that

$$z^* = \bigcap_{i=0}^{k} (x_i + \mu^* \alpha_i S)$$

with $\mu^* \leq \dfrac{2k}{k+1} \leq \dfrac{2n}{n+1}$.

We shall show that for any m satisfying $k + 1 \leq m \leq n$ we have

(14) $z^* \in x_m + \mu_m \alpha_m S$,

where

$$\mu_m = 1 + \frac{2\alpha^* \alpha_m'}{\sigma^* - (k+1)(k-2)} < 1 + \frac{2k-2}{2k+2+\delta^*} \leq \frac{2k}{k+1} < \frac{2n}{n+1} .$$

Indeed, with the notation

$$y_i^{(m)} = (\alpha_m x_i + \alpha_i x_m)/(\alpha_m + \alpha_i) \text{ for } 0 \leq i \leq k$$

$$\gamma_i^{(m)} = \frac{(\alpha_m + \alpha_i)(\alpha_i' \alpha^* - k + 1)}{2\alpha^* + \alpha_m[\sigma^* - (k+1)(k-2)]} \text{ for } 0 \leq i \leq k$$

it follows as above that

$$y_i^{(m)} \in x_m + \alpha_m S, \gamma_i^{(m)} \geq 0, \text{ and } \sum_{i=0}^{k} \gamma_i^{(m)} = 1.$$

Then, defining $y^{(m)} = \sum_{i=0}^{k} \gamma_i^{(m)} y_i^{(m)}$ we have $y^{(m)} \in x_m + \alpha_m S$ and

$$z^* - x_m = \mu_m(y^{(m)} - x_m)$$

which establishes (14) and thus ends the proof of Theorem 1.

REMARKS. 1. From the relation $J_X \leq E_X$ it follows that Theorem 1 implies Bohnenblust's result $J_X \leq \dfrac{2n}{n+1}$ for n-dimensional X. Using appropriate definitions, Leichtweiss [8] showed that the same inequality holds also for n-dimensional spaces whose unit cell is not assumed to be symmetric. But it follows easily from these definitions that the unsymmetric case is implied by the symmetric one.

2. Another way to generalize Bohnenblust's result on Jung's constant to spaces with unsymmetric "unit cell" is as follows:

Let K be any convex body in the n-dimensional Euclidean space E^n. A set $M \subset E^n$ has K-diameter ≤ 1 if any segment with end-points in M may be covered by a suitable translate of K. Any such M may be covered by a translate of nK, and the constant n is the best possible.

Indeed, the example of an n-dimensional simplex as K, and the centrally symmetric simplex as M shows that no number smaller than n is sufficient. (A closer analysis of the following reasoning shows that this is the only case in which αK, for a suitable $\alpha < n$, is not sufficient.) In order to prove that any set M of K-diameter ≤ 1 may be covered by a suitable translate of nK we remark that K is contained in a centrally symmetric convex body K^* which is contained in a translate of $\dfrac{1}{2}(n+1)K$.

(This result or statements equivalent to it have been proved by many authors; see, e.g., Leichtweiss [8, Lemma 2], Hammer [5], Süss [11], and the references given in these papers.) Then, denoting by x and y suitable points (translations) and using the inequality $J_{K^*} \leq \dfrac{2n}{n+1}$ we have

$$M \subset x + \frac{2n}{n+1} K^* \subset y + \frac{2n}{n+1} \cdot \frac{1}{2}(n+1)K = y + nK ,$$

which proves our assertion.

3. **Characterization of the extremal cases.** Let P be an n-dimensional simplex, and let $S_P^* = P - P = \{x = y - z; y, z \in P\}$. Among the $(n-1)$-dimensional faces of S_P^* there are exactly $2(n+1)$ which are $(n-1)$-dimensional simplices; we denote them by $\pm P_i, 0 \leq i \leq n$. Of them, $n+1$ are translates of the $(n-1)$-dimensional faces of P, the remaining $n+1$ being centrally symmetric to them. Let $\pm H_i$ denote

the closed half-space bounding S_P^* and containing $\pm P_i$ in its boundary. We define S_P^{**} as the intersection of all the half-spaces $\pm H_i$. Obviously, $S_P^* \subset S_P^{**}$.

DEFINITION 3.   A centrally symmetric convex set $K \subset E^n$ is called a *Leichtweiss body* if and only if for a suitable n-dimensional simplex P we have

$$S_P^* \subset K \subset S_P^{**} .$$

Now we have the following.

THEOREM 2.   *The following statements on the n-dimensional Minkowski space X are equivalent*:
   (a) $J_X = 2n/(n+1)$.
   (b) $E_X = 2n/(n+1)$.
   (c)   *The unit cell S of X is a Leichtweiss body.*

Proof. By Theorem 1 and (iii) of the Introduction, (a) implies (b). Let us now assume that (b) is satisfied. Then an inspection of the proof of Theorem 1 shows that there exists a family of cells $\{x_i + \alpha_i S\}$ corresponding to case (i) of the proof, and such that equality holds in relations (11) and (8).

Now, since $4n + \delta/(2n + 2 + \delta) < 2n/(n+1)$ for $\delta > 0$ (and $n \geq 2$) it follows that (b) implies

$$(15) \qquad\qquad \alpha_0 = \alpha_1 = \cdots = \alpha_n$$

in other words, (a) holds.

On the other hand, (b) implies that $y \in \text{Front } (x_0 + \alpha_0 S)$ and that the corresponding relation must hold for the points analogous to y on the other cells. But this implies, because of (15), that S is a Leichtweiss body with $2P$ equal to the convex hull of the set $\{x_0, x_1, \cdots, x_n\}$, and therefore (c) follows from (b).

We end the proof of the Theorem 2 by proving that (c) implies (a). Let the simplex P be the convex hull of the points $\{x_i; 0 \leq i \leq n\}$ and let $S_P^* \subset S \subset S_P^{**}$. Then $\left\{x_i + \dfrac{1}{2} S; 0 \leq i \leq n\right\}$ is a family of mutually intersecting cells, since $\left\{x_i + \dfrac{1}{2} S_P^*; 0 \leq i \leq n\right\}$ is such a family, but

$$\bigcap_{i=0}^{n} \left(x_i + \frac{1}{2} \mu S \right) = \phi$$

for any positive $\mu < 2n/(n+1)$ since

$$\bigcap_{i=0}^{n} \left(x_i + \frac{1}{2} \, \mu S_P^{**} \right) = \phi$$

for any such μ.

Thus all the assertions of Theorem 2 are proved.

REMARK. Theorem 2 generalizes the symmetric case of Leichtweiss [8] Satz 2.

4. **Expansion constants of Euclidean spaces.** In this section we shall strengthen Jung's [7] result by proving the following theorem.

THEOREM 3. $E_{E^n} = J_{E^n} = \left(\dfrac{2n}{n+1} \right)^{1/2}$.

Proof. Let $\{x_i + \alpha_i S ; 0 \le i \le n\}$ be any family of mutually intersecting cells in E^n with $\alpha_i > 0$, and let μ be the least positive number such that

(16)
$$\bigcap_{i=0}^{n} (x_i + \mu \alpha_i S) \ne \phi .$$

We shall prove Theorem 3 by constructing a family $\{y_i + S, 0 \le i \le n\}$ of mutually intersecting cells of equal radius, and such that μ is least positive number for which

$$\bigcap_{i=0}^{n} (y_i + \mu S) \ne \phi .$$

Without loss of generality we may assume that (16) contains only one point z which, moreover, belongs to the boundary of each $x_i + \mu \alpha_i S$, for $0 \le i \le n$. Let C_i denote the cone with vertex z generated by $x_i + \alpha_i S$; then $C_i \cap C_j$ contains a ray starting at z, and all the cones C_i are congruent. Now, if $y_i = z + (x - z)/\alpha_i$, the cell $y_i + S$ also generates the cone C_i and therefore $(y + S)(y_j + S) \ne \phi$. But obviously $\bigcap_{i=0}^{n} (y_i + \mu S) = \{z\}$, and Theorem 3 is proved.

REMARK. It is well-known that Jung's constant of Hilbert space equals $\sqrt{2}$ (see, e.g., Routledge [10]); it may be shown that its expansion constant also equals $\sqrt{2}$.

5. **Concluding remarks.** In all the cases discussed above (spaces X with $E_x = 1$ or $2n/(n+1)$ and Euclidean spaces) we found that the value of the expansion constant equals the value of Jung's constant. Nevertheless, the two constants are different in general; the simplest example to that effect is perhaps the following:

Let Y denote the Minkowski plane (with points (x, y)) whose unit cell is the hexagon with vertices $(\pm 1, 0)$, $(\pm \alpha, 1)$, $(\pm \alpha, -1)$ for some α satisfying $0 < \alpha < \dfrac{1}{2}$. We recall the following result (Grünbaum [3]): For any Minkowski plane X, if ρ is the minimum of the lengths of sides of equilateral triangles with vertices on the boundary of the unit cell and containing the origin, then $J_X = 2/\rho$. It is easily checked that the points $(0; -1)$, $(\alpha/2; (2 - \alpha)/(2 - 2\alpha))$ and $(\alpha/2; -(2 - \alpha)/(2 - 2\alpha))$ of Y, whose mutual distances are $2 - \alpha$, determine such a minimal triangle for H. Therefore $J_Y = 2/(2 - \alpha)$. On the other hand, considering the mutually intersecting cells $(1, 0) + H$, $(-1, 0) + H$ and $(0, 1/\alpha + \dfrac{1 - \alpha}{\alpha} H$,

it is immediate that $E_Y \geq 1/(1 - \alpha + \alpha^2)$. (It is not difficult to establish that the equality sign holds.) Our assertion then results from the obvious inequality $1/(1 - \alpha + \alpha^2) > 2/(2 - \alpha)$.

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