MR598810 (82j:05048) 05B45 (51F15 52A45 82A60)
Grünbaum, Branko; **Shephard, G. C.**
A hierarchy of classification methods for patterns.
Z. Krist. **154** (1981), *no. 3-4*, 163–187.

This important paper presents a method for classification of patterns which provides a context for many previously published classification schemes as well as a coherent setting for future research. The heart of the method is in the spirit of Felix Klein's "Erlangen program", and is a familiar one in classifications of algebraic and topological structures. Roughly, two patterns are of the same "type" if there is a one-to-one correspondence between them which preserves their essential structure (their symmetries).

The authors restrict their discussion to classification of planar patterns, although their technique can be adapted to higher-dimensional Euclidean space or other spaces. A (monomotif) pattern is defined as a set of points $\mathbf{P}$ in E^2 which can be partitioned into a countable family $\{M_i\} = M(\mathbf{P})$ of congruent copies of a single set M (the motif) such that the group of symmetries of $\mathbf{P}$ acts transitively on the set $M(\mathbf{P})$. If $\mathbf{P}$ and $\mathbf{P}^*$ are patterns, and s is a symmetry of $\mathbf{P}$, then s induces a mapping s_M from $M(\mathbf{P})$ to $M(\mathbf{P})$. If Ψ is a one-to-one correspondence between $M(\mathbf{P})$ and $M(\mathbf{P}^*)$, then Ψ is said to be compatible with s if there is a symmetry s^* of $\mathbf{P}^*$ of the same kind as s (e.g., if s is a rotation, then so is s^*) such that the following diagram is commutative:

$$\begin{array}{ccc} M(\mathbf{P}) & \xrightarrow{\psi} & M(\mathbf{P}^*) \\ {\scriptstyle s_M}\big\downarrow & & \big\downarrow{\scriptstyle s_M^*} \\ M(\mathbf{P}) & \xrightarrow{\psi} & M(\mathbf{P}^*) \end{array}$$

Two patterns are defined to be of the same henomeric type if there exists a one-to-one mapping $\Psi\colon M(\mathbf{P}) \to M(\mathbf{P}^*)$ which is compatible with every symmetry of $\mathbf{P}$ and such that Ψ^{-1} is compatible with every symmetry of $\mathbf{P}^*$. This is a natural refinement of the crude classification of planar patterns by symmetry groups alone. Two further refinements of this classification take into account topological and analytic properties of the pattern.

A homeomorphism $\Phi\colon E^2 \to E^2$ which maps a pattern $\mathbf{P}$ onto a pattern $\mathbf{P}^*$ is said to be compatible with a symmetry s of $\mathbf{P}$ if there is a symmetry s^* of $\mathbf{P}^*$ such that the following diagram is commutative:

$$\begin{array}{ccc} E^2 & \xrightarrow{\Phi} & E^2 \\ {\scriptstyle s}\big\downarrow & & \big\downarrow{\scriptstyle s^*} \\ E^2 & \xrightarrow{\Phi} & E^2 \end{array}$$

Two patterns $\mathbf{P}$ and $\mathbf{P}^*$ are homeomeric [resp. diffeomeric] if there is a homeomorphism [resp. diffeomorphism] Φ such that Φ is compatible with every symmetry of $\mathbf{P}$ and Φ^{-1} is compatible with every symmetry of $\mathbf{P}^*$.

The authors present classification results on discrete patterns, and particular results obtained when the type of motif is restricted—a closed topological disk; a single point (dot); a line segment; an open circular disk; an ellipse; a tile. The relation of these classification results to earlier classifications by several mathematicians and crystallographers (including the authors) is clearly pointed out, and corrections to some of these are supplied. The authors point out the subtle difference in definitions of "sets of equivalent points" as given by W. Wollny [*Reguläre Parkettierung der Euklidischen Ebene durch unbeschränkte Bereiche*, Bibliographisches Inst. Mannheim 1969; MR0264512 (41 #9105)] and in the work of N. F. M. Henry and K. Lonsdale (editors) [*International tables for X-ray crystallography, Vol. 1*, Kynoch Press, Birmingham, 1952]; Wollny's definition can be related to homeomeric types of periodic patterns, while Henry and Lonsdale's definition can be related to henomeric types. In the text, the authors carefully correct their earlier claim of 52 henomeric pattern types to 51 types (there are 52 homeomeric types; two of these correspond to one henomeric type). Unfortunately, the patterns in Figure 4 still retain the old (incorrect) labeling; the reader should relabel the last five patterns as instructed by the authors on pp. 172, 176.

Reviewed by *Doris Schattschneider*