Some Comments on "Juxtapositions"

by Branko Grünbaum and G.C. Shephard

In connection with the account of the history of space fillings given in the introduction to Baracs' article on "Juxtapositions" (Baracs 1979), several observations can be made.

1. A detailed account of plane tilings (plane fillings) by congruent convex polygons, in which any two polygons are equivalent under some symmetry of the tiling, has recently been given (Grünbaum-Shephard 1978). This paper also gives references to abundant literature, and mentions many open problems. Many additional possibilities arise if one considers tilings by congruent convex polygons that are not necessarily equivalent under symmetries of the tiling. A systematic and very readable account of pentagonal plane fillers of this kind has recently been given by Doris Schattschneider (see Schattschneider 1978, and also the announcement Hirschhorn 1978).

2. Tilings of the three-dimensional space by congruent (or equivalent) copies of a convex polyhedron have also received a great deal of attention — from mathematicians, crystallographers and others. An outline of the theory and an extensive bibliography is given by Nowacki (Nowacki 1976). While it is out of question to present here even a brief account of the achievements, possibly some readers will be induced to follow up the references and to study the applicability of these results. It would appear that several of the questions posed by Baracs can at once be answered using the facts established in the papers mentioned below and the ones to which they refer.

(i) Extensive investigations of the forms of space fillers with relatively few faces have been published in several papers by M. Goldberg. Detailed references can be found in the most recent one (Goldberg 1978).

(ii) The 16-faced space filler shown in **Figure 6** of Baracs 1979 was known already to Föppl (see Föppl 1914). (Michael Goldberg kindly supplied this information.) Physicists continued being interested in space fillers, and Smith described a 20-faced space filler (see Smith 1965). References to many other papers on space fillers that appeared in the physical literature can be found in Smith's article.

(iii) A systematic investigation of **stereohedra**(that is, convex polyhedra that admit a tiling of the space in which each two polyhedra are equivalent by a symmetry of the tiling) was begun by Nowacki (see Nowacki 1935). Among other types of stereohedra, Nowacki found one with 17 faces, and one with 18. In this work, as well as in the papers mentioned in (iv) and (v), the idea is to start from an infinite but discrete set of points, any two of which are equivalent by a symmetry of the whole set, and to obtain the tiling by considering the Dirichlet domains of the points. Nowacki's results were extended by (Löckenhoff-Hellner 1971); an 18-faced stereohedron described there is shown in **Figure 1.**

(iv) Stogrin described 180 different types of stereohedra, among which there are 61 different types with 17 or more faces, including five types with 20 faces (see Stogrin 1973). He also provides many references to the results of Russian workers in this field.

(v) Independently of this "Russian school" a "German school" has achieved even more remarkable results; among these is the discovery of a whole family of 24-faced stereohedra (each with 10 triangles, 4 quadrangles, 4 hexagons, 2 heptagons and 4 dodecagons) along with several families of stereohedra with 18, 20 or 22 faces. See Koch-Fischer 1972 for an account of this result and for references to other papers in this direction. The discovery of a 26-faced stereohedron was announced by W. Fischer and E. Koch at a meeting of the Swiss Crystallographic Society held August 7 - 9, 1979, in Riederalp (Switzerland).

(vi) Some writers restrict the consideration of space fillers to those which admit tilings of the space using **directly congruent** copies only. Many of the results mentioned above apply to this variant as well. For example, the 18-faced stereohedron of Nowacki 1935 mentioned in (iii) tiles space with directly congruent copies, and so does the stereohedron in **Figure 1** as well as all the stereohedra of (Koch-Fischer 1972) discussed in (v).

3. Concerning the application of space fillers to architecture (and to some extent also in industrial design) one wonders about the relevance of such conditions as insisting that the polyhedral space fillers must meet face-to-face, or that they must not only be congruent but equivalent with respect to symmetries of the whole tiling. These conditions seem to be motivated more by a desire for "mathematical neatness" than by any architectural necessity. As with plane tilings, if either of these require-

ments is dropped, there is a dramatic increase in the variety of available shapes and of the possible ways of filling space with them. It seems to us that it is more appropriate to develop the mathematics needed for the treatment of architecturally interesting problems than to impose additional constraints on design purely for mathematical convenience.

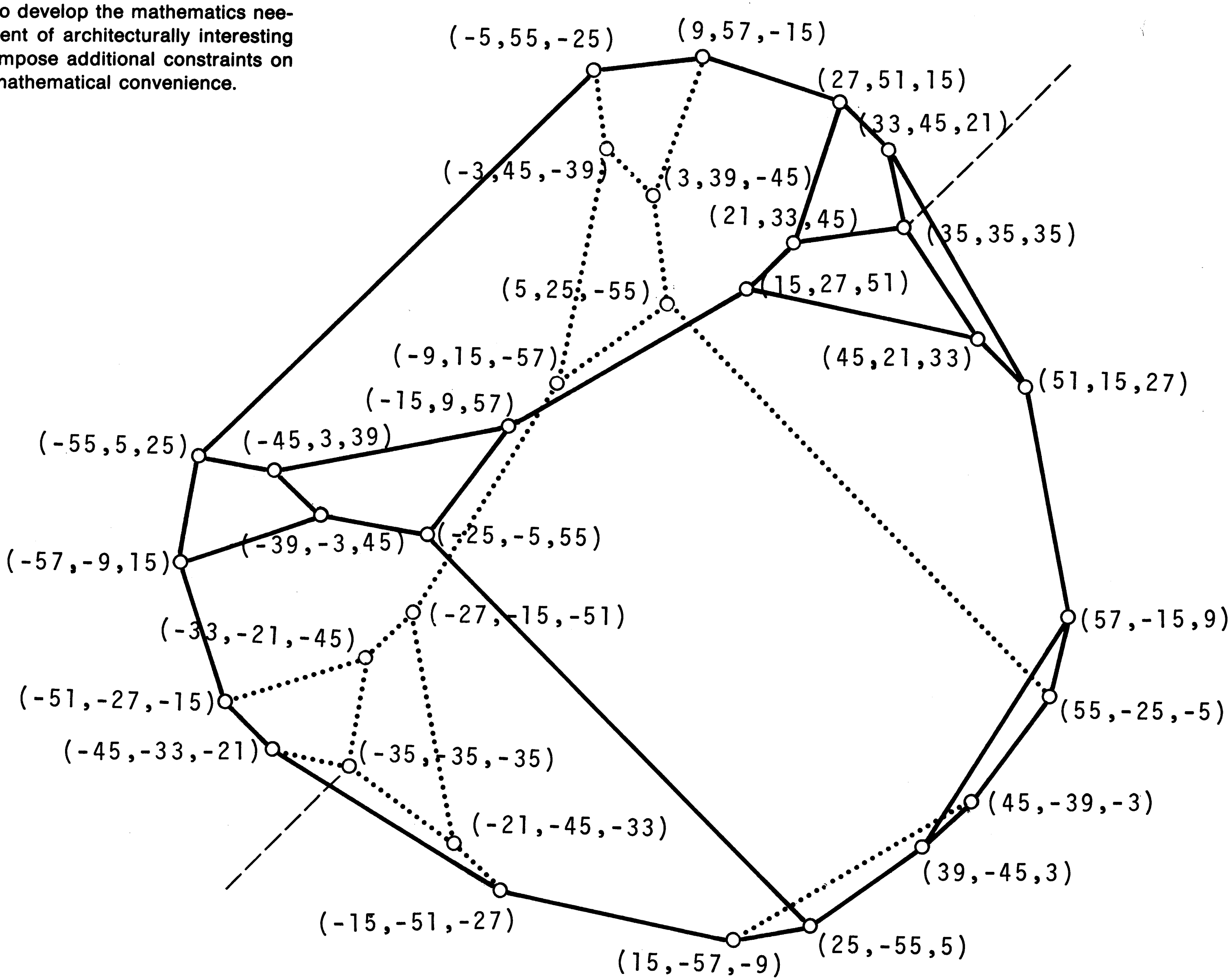

Figure 1. An 18-faced stereohedron, from (Löckenhoff-Hellner 1971). The coordinates of each vertex are indicated, with the x-axis perpendicular to the paper. The dashed line indicates an axis of 3-fold rotational symmetry; the stereohedron also possesses three axes of 2-fold rotational symmetry.

Nowacki 1935 Werner Nowacki A—ME—J	**Homogene Raumteilung und Kristallstruktur.** E.T.H. thesis, Zürich 1935.	The first systematic study of stereohedra. Examples with 17 and 18 faces.
Nowacki 1976 Werner Nowacki A—ME—J	**Uber allgemeine Eigenschaften von Wirkungsbereichen.** Zeitschrift für Kristallographie 143(1976), 360 - 368.	Outline of the theory of tilings of three-dimensional space, with an extensive bibliography.
Schattschneider 1978 Doris Schattschneider A—M—J	**Tiling the plane with congruent pentagons.** Mathematics Magazine 51(1978), 29 - 44.	A systematic and very readable account of plane filling by congruent pentagons.
Smith 1965 F.W. Smith A—ME—J	**The structure of aggregates — a class of 20-faced space-filling polyhedra.** Canadian Journal of Physics 43(1965), 2052 - 2055.	Written by a physicist, this article gives an example of a space-filling polyhedron with 20 faces. References to many other papers on space-filling, from the physics literature.
Stogrin 1975 M.I. Stogrin A—M—J	**Regular Dirichlet-Voronoi partitions for the second triclinic group.** (In Russian) Proceedings of the Steklov Institute of Mathematics, 123(1973). English translation: Proceedings of the Steklov Institute of Mathematics 123(1973), American Mathematical Society, Providence, Rhode Island, 1975.	Describes 180 different types of stereohedra. References to results by Russian researchers.