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RESEARCH PROBLEMS

EDITED BY RICHARD GUY

In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

DO MAXIMAL LINE-GENERATED TRIANGULATIONS OF THE PLANE EXIST?

BRANKO GRUNBAUM AND G. C. SHEPHARD

Let G be any finite set of great circles on a two-dimensional sphere S^2, such that $S^2\backslash G$ is the union of (a finite number of) disjoint open spherical triangles. Then we shall say that G gives rise to a *circle-generated triangulation* of S^2. Many such triangulations are known. For example, we can take

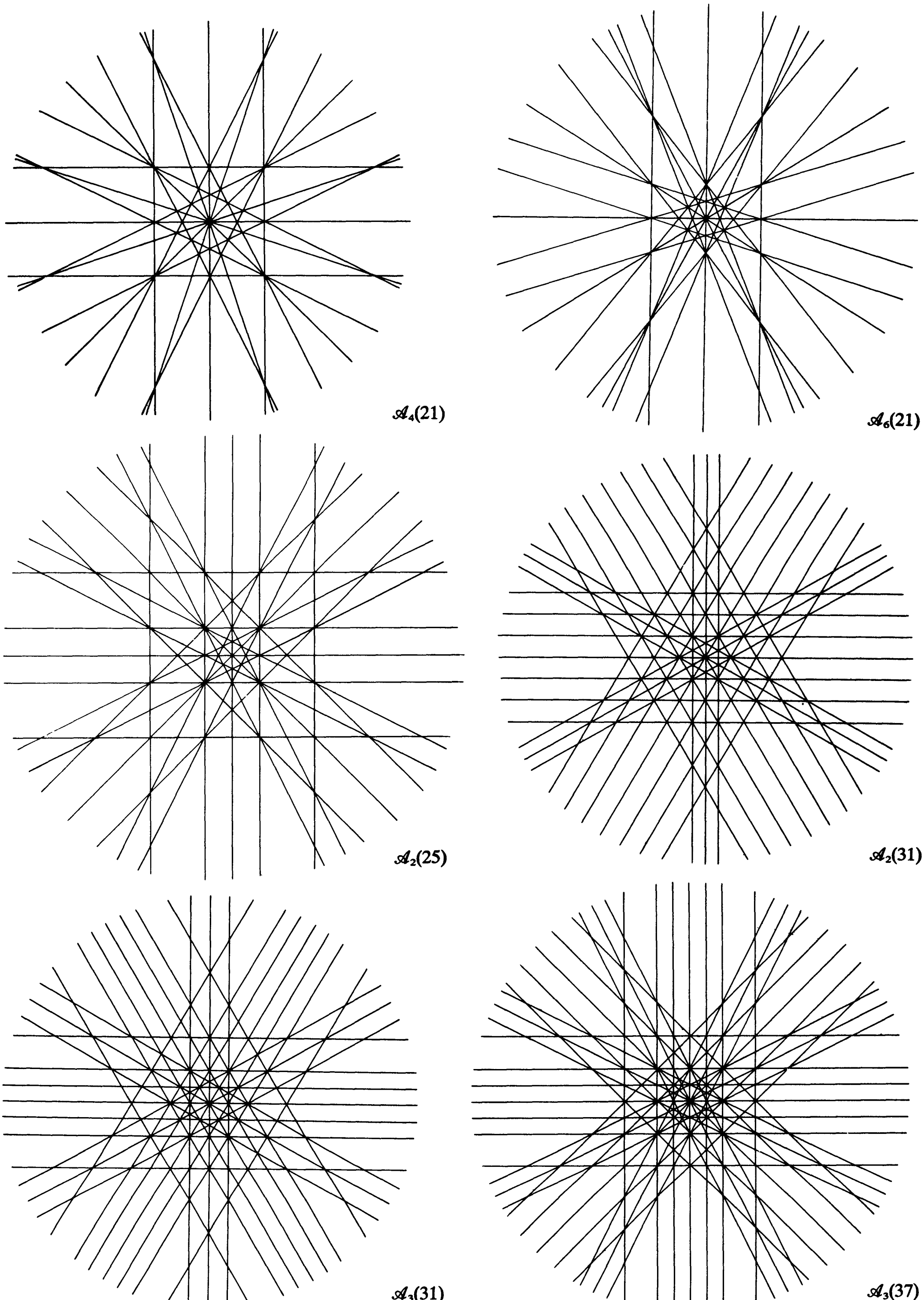


FIG. 1 Six line-generated triangulations of the real projective plane, conjectured to be maximal. The line at infinity is included in each so that there are 21, 21, 25, 31, 31, and 37 lines. The notation is taken from the catalogue [1] of simplicial arrangements. The other four such triangulations that are conjectured to be maximal are there denoted by $\mathcal{A}_7(21)$, $\mathcal{A}_5(25)$, $\mathcal{A}_1(31)$ and $\mathcal{A}_2(37)$.

for G the equator and any finite set of lines of longitude on the surface of the earth, or the set of great circles that result from central projection onto a concentric sphere of the edges of a cuboctahedron or icosidodecahedron.

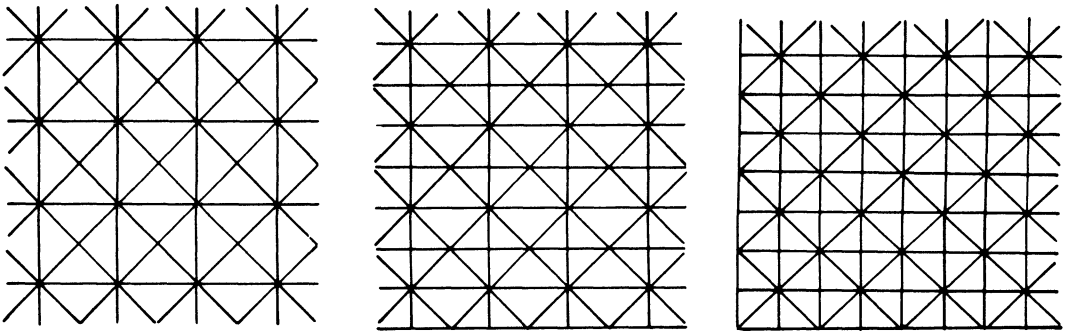
The circle-generated triangulations of the sphere are clearly in one-to-one correspondence with the *line-generated triangulations* of the real projective plane, that is, with triangulations $\mathcal{T}$ in which each edge of $\mathcal{T}$ belongs to a straight line composed entirely of edges of $\mathcal{T}$. Because of easier representability we shall henceforth discuss the latter, and for simplicity of expression we shall often say "triangulation" instead of "line-generated triangulation." Examples of such triangulations appear in Figure 1.

Two triangulations are said to be *equivalent* if they are combinatorially isomorphic, and a triangulation $\mathcal{T}_1$ is called a *refinement* of a triangulation $\mathcal{T}_2$ if $\mathcal{T}_1$ is equivalent to a subdivision of $\mathcal{T}_2$. In other words, triangulations equivalent to all refinements of $\mathcal{T}_2$ can be obtained by drawing additional lines in a diagram representing $\mathcal{T}_2$. Clearly the relation of being a refinement is a partial ordering on the set of all line-generated triangulations of the real projective plane.

In **[1]** and **[2]** it was conjectured that, apart from three well defined infinite families of very special triangulations, there exist only a finite number of line-generated triangulations of the projective plane. If that conjecture were established, it could be shown that there exist *maximal* triangulations, that is to say, triangulations for which no further refinement is possible.

CONJECTURE 1. *The ten line-generated triangulations of the projective plane listed in Figure* 1 *are maximal.*

CONJECTURE 2. *Any maximal line-generated triangulation of the projective plane is equivalent to one of the* 10 *listed in Figure* 1.



(a) (b) (c)

FIG. 2. Line-generated triangulations of the Euclidean plane. The triangulation in (b) is a refinement of (a), that in (c) is a refinement of (b) (and of (a)), and is equivalent to (a).

If we extend these ideas to Euclidean or affine planes, the situation becomes entirely different. We restrict attention to locally finite triangulations; examples are given in Figures 2, 3 and 4. It is easy to see that every such triangulation necessarily contains an infinite number of lines, and that an uncountable infinity of such triangulations exist. The relation of being a refinement is no longer a partial ordering. This is illustrated by the triangulations of Figures 2(a) and 2(b) which are not equivalent, yet each is equivalent to a refinement of the other. Also, a triangulation may be equivalent to a proper refinement of itself, as shown by the examples in Figures 2(a) and 2(c). Consequently, there is no reason why any maximal triangulations should exist. In fact, we propose

CONJECTURE 3. *There are no maximal line-generated triangulations of the Euclidean plane.*

A proof of this conjecture would probably be facilitated if one could establish

CONJECTURE 4. *Every line-generated triangulation of the Euclidean plane has a proper refinement which is a periodic line-generated triangulation.*

This conjecture is non-trivial even if the starting triangulation is itself periodic. It appears that in this case the following stronger statement is true:

CONJECTURE 5. *Every periodic line-generated triangulation of the Euclidean plane is equivalent to a proper refinement of itself.*

As illustrations of this situation we mention that for the triangulation in Figure 3 a contraction with center at the marked vertex in the ratio 2:1 produces a proper subdivision of the original triangulation. Analogously, in the case shown in Figure 4, except that here a contraction in ratio 3:1 or 5:1 is needed.

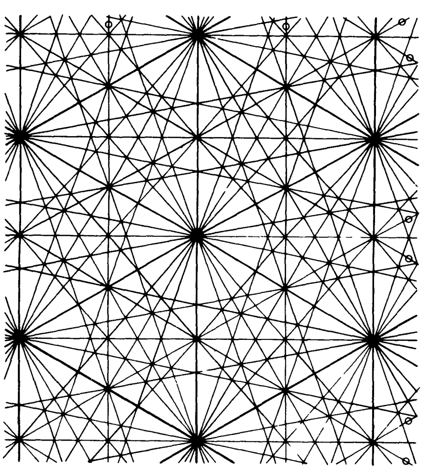


FIG. 3. A line-generated periodic triangulation of the Euclidean plane. Contraction in ratio 2:1 towards the vertex marked * yields a proper refinement of the triangulation. The marked vertex belongs to 12 lines of the triangulation. The deletion of any of the lines marked by a small circle leads to a triangulation.

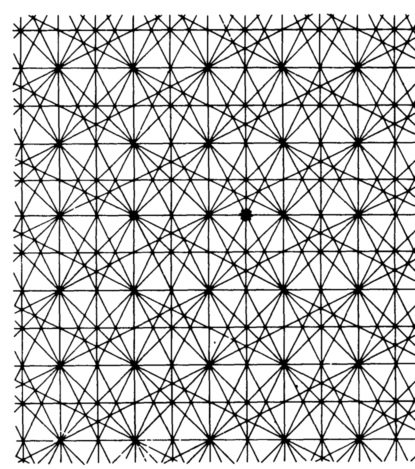


FIG. 4. A line-generated periodic triangulation of the Euclidean plane. Contraction in ratio 3:1 towards the vertex marked * as well as in ratio 5:1 towards the vertex marked *, yields a proper refinement.

In the periodic triangulation $\mathcal{T}$ of Figure 3 the removal of any among the lines marked by small circles leads to a triangulation that is equivalent to a proper refinement of $\mathcal{T}$; hence $\mathcal{T}$ has uncountably many inequivalent refinements. Possibly every triangulation has uncountably many refinements, although this seems unlikely; even for the triangulation of Figure 4 we were unable to settle this question.

Although the triangulations appear to be intricate in many respects, they seem all rather simple in the following sense:

CONJECTURE 6. *Each vertex of a line-generated triangulation of the Euclidean plane belongs to at most* 12 *lines.*

The example in Figure 3 shows that 12 cannot be replaced by any smaller number.

There are many possible variations of these ideas and conjectures. We mention *pseudoline-generated triangulations* of the real projective plane; these coincide with the simplicial arrangements of pseudolines discussed in [1] and [2]. The corresponding notions in the Euclidean plane also deserve investigation. Then there are analogous problems about *plane-generated triangulations* of Euclidean or projective space of 3 or more dimensions. Another possibility is to consider triangulations on 2-manifolds, generated by simple closed curves any two of which are either disjoint or cross each other precisely twice. Beyond the fact that non-trivial triangulations exist in each case, nothing appears to be known on these topics.

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CLASSROOM NOTES

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AN IMPORTANT FUNCTOR IN ANALYSIS AND TOPOLOGY

Donald Hartig

1. The language and viewpoint of categories have become firmly implanted in the mathematics curriculum at the graduate level, and following the normal trend, have now begun to filter into the undergraduate program. First contact with the unifying and generalizing concepts of this theory is usually made in a course in algebra where the relevant mappings become **morphisms** and the student is expected to develop a facility with and (we hope) an appreciation of the new art of commutative diagrams, universal elements and "natural mappings." Surely at this level the most important aspect of Category Theory is its universal applicability and its power to highlight the analogies that exist among structures that arise in so many contexts.

Modern Functional Analysis has evolved from a shift from the study of sets of points to the study of sets of functions (which are mappings between sets of points). Category Theory represents a similar step from the study of sets of functions to a study of mappings between sets of functions. Such mappings are called **functors** and are the morphisms of Category Theory. In what follows we shall investigate some properties of one of the most important functors in analysis.

2. If A is a closed subset of a compact (includes Hausdorff) space X, then we call the ordered pair (X,A) a **compact pair**. The compact pair $(X,\varnothing)$ is referred to simply as X. A morphism $\alpha\colon (X,A)\to(Y,B)$ is a continuous function taking X into Y so that $\alpha(A)\subset B$. Such mappings constitute the category of compact pairs, denoted by *CompPr* for short. Contained in this category is the family of continuous maps $\alpha\colon X\to Y$ (X and Y compact) which we denote by *Comp*.

Given a compact pair (X,A) we let $C(X,A)$ denote the set of all continuous scalar-valued