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Pairs of edge-disjoint Hamiltonian circuits

Branko Grünbaum and Joseph Malkevitch

Nash-Williams conjectured that each 4-connected 4-valent graph admits a pair of edge-disjoint Hamiltonian circuits. A weaker version of the same conjecture, dealing with planar graphs only, was proposed also in Grünbaum-Zaks.

The two versions of the conjecture have been disproved by Meredith and Martin. One of the aims of the present note is to provide simpler counterexamples. As it turns out, one of the methods used for that purpose also leads to a proof of a conjecture made in Grünbaum-Walter about 'shortness coefficients' of a certain family of graphs (Theorem 2 below).

THEOREM 1. *There exist* 4-*valent* 4-*connected graphs G with the property that each two Hamiltonian circuits in G share at least one edge; moreover, there exist planar graphs with this property.*

We shall give three proofs of Theorem 1. The first consists in pointing out that

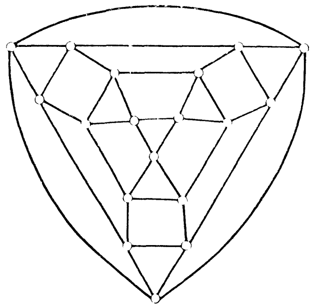

Fig. 1

the planar graph G with 18 vertices shown in Fig. 1 has all the properties required in Theorem 1. This can be verified by exhaustive checking. Due to the symmetry of the graph the checking is actually not very arduous, – but it is an unattractive way of establishing results.

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For our second proof we observe that there are several operations on 4-valent graphs that preserve the existence or non-existence of pairs of edge-disjoint Hamiltonian circuits. Two such operations, for which this assertion is easily checked, are

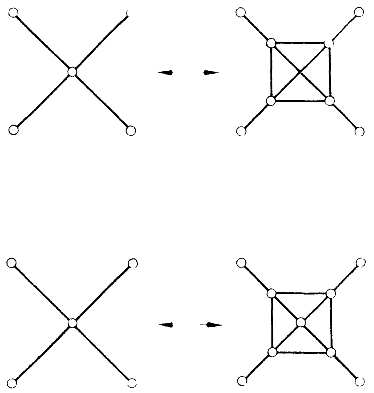

Fig. 2

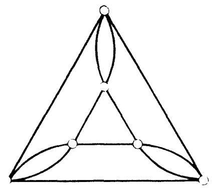

Fig. 3

indicated in Fig. 2; the second clearly preserves planarity as well. Applied to the (multi)graph shown in Fig. 3, the first operation yields a 24-vertex example establishing the first part of Theorem 1, while the second operation yields a 30-vertex planar graph that establishes the second part (Fig. 4).

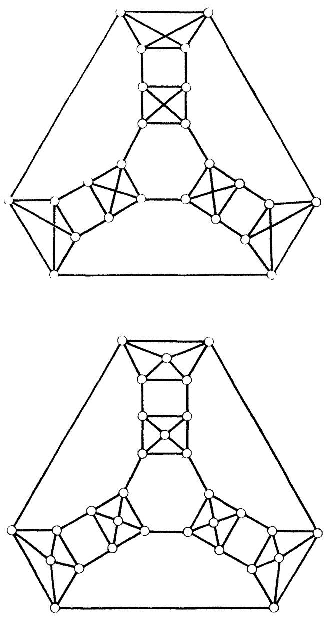

Fig. 4

The third proof is based on the following observation (we recall that a graph is cyclically-k-connected if there is no set of fewer than k edges whose deletion leaves two disjoint subgraphs, each containing a circuit):

(∗) If G is a 3-valent 3-connected cyclically-4-connected graphs, then the *interchange graph* $I(G)$ (Ore [p. 19]) is a 4-valent 4-connected graph. If $I(G)$ has two edge-disjoint Hamiltonian circuits H_1 and H_2 then G has a Hamiltonian circuit H.

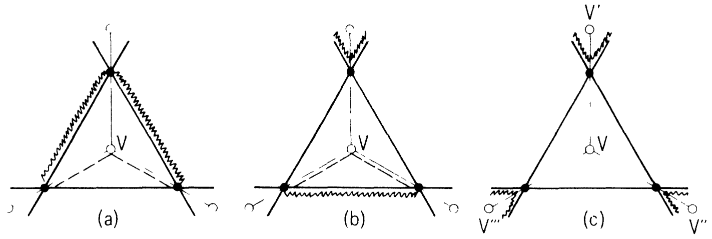

Fig. 5

In order to establish the validity of the second sentence of (∗) it is sufficient to consider the three essentially different ways in which a Hamiltonian circuit H_1 of $I(G)$ can visit the 3 vertices of $I(G)$ that correspond to edges incident with a vertex V of G. In Fig. 5 the graph G is indicated by the thin lines, $I(G)$ by heavy lines; the wavy line indicates part of H_1. If $I(G)$ has a Hamiltonian circuit H_2 that shares no edge with H_1 then the situation in Fig. 5(c) does not arise. But in each of the other two cases it is possible to deform H_1 to a part of a circuit in G that passes through V (dashed lines in Fig. 5).

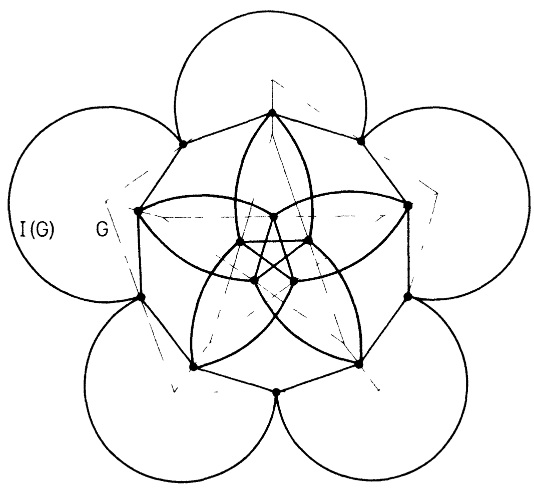

Fig. 6

In view of (∗), examples establishing Theorem 1 are provided by graphs $I(G)$ whenever G is a 3-valent 3-connected cyclically-4-connected non-Hamiltonian graph. The smallest such graph G is the Petersen graph (G in Fig. 6), while the smallest known planar graph with that property (having 42 vertices and due to Grinberg; see also Grünbaum [Section 1.4], Honsberger [p. 90] and Faulkner-Younger, is shown in Fig. 7.

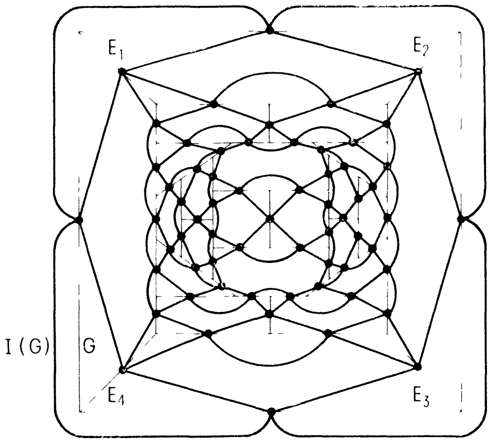

Fig. 7

This completes our proofs of Theorem 1.

The interchange-graph technique establishes also one half of the result below, in which $\mathscr{C}(4)$ denotes the family of all 3-valent 3-connected cyclically-4-connected planar graphs, and ϱ is the 'shortness coefficient'. We recall (see Grünbaum-Walther) that if $\mathscr{G}$ is a family of graphs and if $v(G)$ and $h(G)$ denote the numbers of vertices in G and in the longest simple circuit in G, then $\varrho(\mathscr{G})$ is defined by $\varrho(\mathscr{G}) = \liminf_{\substack{G_n \in \mathscr{G} \\ v(G_n) \to \infty}} h(G_n)/vG_n$.

THEOREM 2. $\frac{3}{4} \leqslant \varrho(\mathscr{C}(4)) \leqslant \frac{76}{77}$.

To prove the first inequality we first observe that if $G \in \mathscr{C}(4)$ then $I(G)$ is a 4-connected planar graph and therefore by a theorem of Tutte [1956] (see also Ore [1967]) there is a Hamiltonian circuit H_1 in $I(G)$. Referring again to Fig. 5 we amplify (∗) by noting that if the behaviour of H_1 in the vicinity of a vertex V of G is as indicated in Fig. 5(c) (so that no local change of H_1 leads to a circuit passing through V) then the behaviour of H_1 around the vertices V', V'', V''', as well as around their neighbors other than V, is not of that type. Therefore, the simple circuit H in G derived from H_1 by the procedure indicated in Figs. 5(a) and 5(b) contains at least $\frac{3}{4}$ of the vertices of G, establishing $\varrho(\mathscr{C}(4)) \geqslant \frac{3}{4}$. On the other hand, if we consider any circuit H in the Grinberg graph G of Fig. 7, such that H contains all the vertices of the 'inside' part of G, then (since G has no Hamiltonian circuit) H must reach the 'outer' square via a pair of diagonally opposite edges (i.e., either E_1 and E_3, or E_2 and E_4). Therefore, if we omit the outer four edges of G to obtain a graph G^*, and then combine four copies of G^* as indicated in Fig. 8 (in which each shaded area

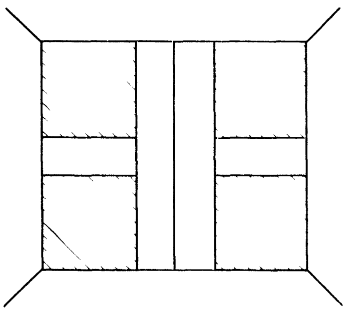

Fig. 8

represents a copy of the 'inside' of G), we obtain a graph G^{**} with the following property: If G^{**} is a subgraph of a larger graph, and if a simple circuit in the large graph visits G^{**} (once or twice), then the circuit contains at most 152 of the 154 vertices of G^{**}. Hence by replacing all the vertices in any 4-valent 4-connected planar graph with copies of G^{**} we obtain a graph $G' \in \mathscr{C}(4)$, with $h(G')/v(G') \leqslant \frac{76}{77}$. This completes the proof of Theorem 2.

Remarks. The non-planar case of Theorem 1 is obviously a consequence of the following result of Meredith: For each $n \geqslant 3$ there exist n-valent n-connected graphs that are non-Hamiltonian. This result solves in the negative Nash-Williams' conjecture concerning the case $n=4$. However, the question whether the graph of every simple 4-dimensional polytope (which is automatically 4-valent and 4-connected) has a Hamiltonian circuit is still open; some relevant results have been obtained by Rosenfeld-Barnette.

Our third proof of Theorem 1 was independently obtained by Martin, who established the following result stronger than assertion (∗) above: A 3-valent 3-connected graph G is Hamiltonian if and only if $I(G)$ has two edge-disjoint Hamiltonian circuits. We are indebted to Prof. Martin for a prepublication copy of his paper, received after the submission of our manuscript to 'Aequationes Mathematicae'.

Other open questions related to the above results are:

Does every 4-valent 4-connected planar graph G with even $v(G)$ have a Hamiltonian circuit the complement of which is composed of circuits of even length? Or is every such G at least edge-4-colorable?

Does every 5-connected planar graph have two edge-disjoint Hamiltonian circuits?

Let $d(G)$ denote the minimal number of edges of the 4-valent 4-connected graph G that are shared by any pair of Hamiltonian circuits of G. What is lim sup$d(G)/v(G)$ over all such graphs, or over the planar ones among them?

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Branko Grünbaum,
Department of Mathematics,
University of Washington,
Seattle, Washington 98105,
U.S.A.

J. Malkevitch,
Department of Mathematics,
York College (CUNY),
Jamaica, New York 11432,
U.S.A.