Counting-Basics

Ngày 16 tháng 11 năm 2012

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Rule (The Sum Rule)

If a task can be performed either in m distinct ways **or** *in k other distinct ways and both ways are mutually disjoint then there are $m + k$ distinct ways to perform the task.*

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Suppose that a task has to be performed in two steps, where the first step can be performed in m different ways **and** *the second step in k different ways, then there are $m \times k$ different ways to perform the task.*

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More product rule examples

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How many distinct functions $f : \{1, 2, 3, 4, 5, 6, 7, 8, 9, 0\} \rightarrow \{1, 2, 3, 4\}$ *are there?*

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Each function is built in 10 steps: choose a value for $f(1), f(2), \ldots, f(0)$.
Each step can be performed in 4 *different ways.*
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Answer: Each function requires 3 steps: select a value for $f(a)$ then $f(b)$ and $f(c)$. $f(a)$ can be chosen in 10 different ways, $f(b)$ in 9 and $f(c)$ in 8. So the total number of functions is 720.

The Inclusion-Exclusion Principle

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There are 2^9 bit strings that begin with a 1. There are 2^8 bit strings that end with 10. There are 2^7 bit strings that start with 1 and end with 10. Therefore the number of bitstrings of length 10 that start with a 1 or end with 10 is $2^9 + 2^8 - 2^7$.

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6. $|A_{1729}| = 1728 - \frac{1729}{7} - \frac{1729}{13} - \frac{1729}{19} + \frac{1729}{7 \cdot 13} + \frac{1729}{7c19} + \frac{1729}{13 \cdot 19} = 1296.$

The Inclusion-Exclusion General Principle

Theorem

For a finite family of finite sets $\{A_1, A_2, \ldots A_n\}$ *we have:*
$|\cup_{i=1}^{n} A_i| = \sum_{\emptyset \neq I \subset \{1,2,\ldots,n\}} (-1)^{|I|-1} |\cap_{i \in I} A_i|$.

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2. Let $x \in \cap_{j=1}^k A_{i_j}$.
3. Since x belongs to every set A_{i_j}, it contributes:

$$\sum_{\emptyset \neq I \subset \{1,2,\ldots k\}} (-1)^{|I|-1} |\cap_{j \in I} A_{i_j}| = \sum_{j=1}^{k} (-1)^{j-1} \binom{k}{j} = 1$$

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$$\prod_{i=1}^{n}(1 + x_i) = \sum_{A \subset \{1,2,\ldots,n\}} (\prod_{i \in A} x_i)$$

Two counting problems "saved" by the inclusion-exclusion principle

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We shall count the number of permutations for which $f(i) = i$ for some i.

Continued

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$$D_n = n! - \sum_{j=1}^{n} (-1)^{j-1} \cdot \frac{n!}{j!} = n! \cdot \sum_{j=0}^{n} (-1)^j \frac{1}{j!}$$

Euler's function $\phi(n)$

Euler's function is very important in many applications, in particular in computer security applications.

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Our goal is to calculate $\phi(n)$.

Calculating $\phi(n)$

Theorem

For $n = p_1^{r_1} \cdot p_2^{r_2} \dots p_k^{r_k} \quad \phi(n) = n(1 - \frac{1}{p_1})(1 - \frac{1}{p_2}) \dots (1 - \frac{1}{p_k})$

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Chứng minh.

Let $A_i = \{s |\ 1 < s < n,\ p_i | s\}$. Then:

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Recall that:

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Recall that:

$$|\cup_{i=1}^{k} A_i| = \sum_{\substack{I \subset \{1,2,\dots,k\} \\ I \neq \emptyset}} (-1)^{|I|-1} |\cap_{i \in I} A_i|.$$

$A_i \cap A_j$ is the set of all integers $\leq n$ that are divisible by p_i *and* p_j that is divisible by $p_i \cdot p_j$. It follows that $|A_i \cap A_j| = \frac{n}{p_i p_j}$. □

continued.

Similarly,

$$|\cap_{i\in I\subset\{1,2,\ldots,k\}} A_i| = n/\prod_{i\in I} p_i$$

Hence:

$$\phi(n) = n \;-\sum_{\substack{I\subset\{1,2,\ldots,k\}\\ I\neq\emptyset}} (-1)^{|I|-1}|\cap_{i\in I} A_i| =$$

n - $\sum_{\substack{I\subset\{1,2,\ldots,k\}\\ I\neq\emptyset}}(-1)^{|I|-1}(n/\prod_{i\in I} p_i) = n(1-\frac{1}{p_1})\ldots(1-\frac{1}{p_k})$ □

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The last equality is an instance of the general useful identity that embodies the Sum-Product rule:

$\prod_{i=1}^{n}(1+x_i) = \sum_{A\subset\{1,2,\ldots,n\}}(\prod_{i\in A} x_i)$

Tree Diagrams

How many bead strings of length four, composed of green and blue beads without two consecutive green beads can be constructed?

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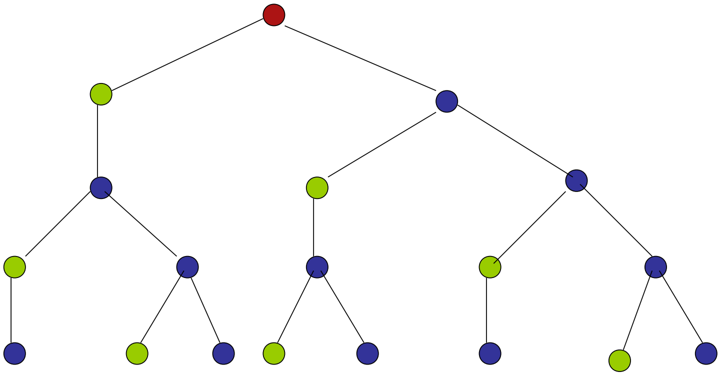

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The four rules are simple, self explanatory and obvious, Yet they exhibit a surprising power to solve some intricate counting problems.

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The four rules are simple, self explanatory and obvious, Yet they exhibit a surprising power to solve some intricate counting problems. We shall next visit some examples.

Example

In a previous exercise you were asked to produce an integer n *and find an integer* k *such that* $n \cdot k = 111\ldots1$.
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Did any one bother to ask whether the program will ever stop?

Theorem

For any odd positive integer n that is relatively prime to 5 one can find an integer k such that $n \cdot k = 11\ldots1$.

Chứng minh.

□

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7. Since n is odd, and $GCD(n, 5) = 1$ we conclude that $1^{\{j-m\}}$ is a multiple of n

□

The Chinese Reamainder theorem

Theorem

If $a_1, a_2, \ldots, a_k$ are relatively prime, and $0 \leq m_i < a_i$ then there is a unique integer $m < M = a_1 \cdot a_2 \cdot \ldots \cdot a_k$ such that $m \bmod a_i = m_i$.

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3. It remains to prove that s is unique.

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CRT-continued.

To prove uniqueness we use the pigeonhole principle.

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4. Each integer $t < M$ produces a k-tuple $t_i = t \bmod a_i, i = 1, \ldots, k$ that will be placed in h_t.

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5. Each hole contains a $k-tuple$. The number of k-tuples is equal to the number of holes.

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4. Each integer $t < M$ produces a k-tuple $t_i = t \bmod a_i, i = 1, \ldots, k$ that will be placed in h_t.
5. Each hole contains a $k - tuple$. The number of k-tuples is equal to the number of holes.
6. Conclusion: each hole contains exactly one item, or the uniqueness is established.

□

Two more examples

Question (Example number 1)

In the ASEAN Cầu lông championship held in Hanoi, Linh won first place. The championship lasted 21 days. Linh played 35 matches, playing at least one match every day. Prove that there is a span of consecutive days in which Linh played exactly 6 matches.

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4. x_i is also monotonically increasing and $x_{21} = 41$.

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5. $\{m_i\}$ and $\{x_i\}$ together have 42 integers.

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6. But the largest integer is 41, so at least one integer must appear twice.

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7. Since $m_i < m_j, \ and \ x_i < x_j \ if \ i < j$ we must have $m_i = x_j$ for some i and j.

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7. Since $m_i < m_j, \ and \ x_i < x_j \ if \ i < j$ we must have $m_i = x_j$ for some i and j.
8. But this means that between days j and i Linh played exactly 6 matches.

□

Second example

Question

To commemorate Vua Le's defeat of the Chinese invaders, he decided to mint 11 commemerative gold coins. He gave a large amount of gold to a jeweler.
When the jeweler returned the coins, Vua Le suspected that the jeweler stole some gold and replaced it with cheaper metals. Vua Le, knew that the jeweler will not dare to tinker with more than one coin. The only way to identify the fake coin is to weigh coins on a balanced scale.

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Vua Le decided to test the intelligence of his chief adviser. He ordered him to design a weighing scheme to detect the fake coin, decide whether it is heavier or lighter by using no more than three weighings.

It is your mission to help the adviser by designing the weighing scheme.