Discrete Mathematics
Lecture-16 Communication Security

November 17, 2012
Question

Mr. Nguyen sells expensive jewelry. He has an interesting idea for a business model. Each customer will have access to boxes with a combination lock. Once a person grabs a box he can set his own private combination lock. An open box can be closed by anyone, but only the owner knows the combination and can open it. The content of any open box sent between persons will be stolen.
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How can we accomplish this?
Discussion

This is exactly how business transactions are being conducted on the Internet today, except that the boxes are virtual boxes. Closing a box is accomplished by encrypting the message. So while the message is traveling on the Internet, being exposed to hackers and others, it is encrypted using a “key”. Only the owner of the key knows how to open the box and retrieve its content.
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- Participants can securely exchange messages over an “open” system.
- Messages can be sent to “Bob” so only Bob will be able to understand.
- Transactions can be “signed.”
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In 1976 Rivest, Shamir and Adelman proposed the public key cryptosystem: RSA.
The RSA public key system

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  \[ S = M^e \mod k \]
  and sends \( S \).
- Decryption: the receiver calculates \( S^d \mod k \) and retrieves \( M \)
  where \( d = e^{(-1)} \mod (p - 1)(q - 1) \).
Decryption

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Security experts all over the world are trying hard to devise methods to factor large integers quickly. So far their efforts have not succeeded.
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We shall devote the rest of our time to take a quick glimpse at factoring.
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Note: we assume that everyone can intercept the message $S$. Furthermore, everyone knows exactly how $S$ was calculated, everyone knows $k$ and $e$, so why can’t they retrieve $M$?

After all, all they need to do is calculate $d = e^{-1} \mod (p-1)(q-1)$ and in order to do it they just need to factor $k$.

Our goal is to understand how this system works, why it is considered secure and other applications of this system.
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To understand it we need to study some very mathematically interesting topics in modular arithmetic.
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Prime and not primes, some key facts

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An element $\alpha \in GF(q)$ is **primitive** if
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- A polynomial $p(x)$ of degree $k$ over $GF(q)$ has at most $k$ roots.
Primality testing

To build an RSA cryptosystem we need to be able to test whether given numbers are prime and to "manufacture" large primes. We shall start by testing.

Question:
Can Fermat's theorem be used for testing primality?

Answer:
Unfortunately not. There are numbers for which the chances for finding an integer \( a < n \) such that \( a^{(n-1)} \mod n \neq 1 \) are very slim. For instance if \( n = (6k+1)(12k+1)(18k+1) \) and \( (6k+1), (12k+1), (18k+1) \) are prime, then if \( \gcd(a, n) = 1 \) then \( a^{n-1} \mod n = 1 \).
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Let \( N \) be an integer. By Fermat’s theorem if \( N \) is prime then \( a^{N-1} \mod N = 1 \). This calculation can be executed very fast on integers with a few thousand digits. This means that if for some \( 1 < a < N - 1 \); \( a^{N-1} \mod N \neq 1 \) then \( N \) is definitely not a prime number.
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But what can we conclude if $a^{N-1} \mod N = 1$?

Answer: NOTHING! $N$ may be prime and it may be composite! At best, we can try another integer $a$.

Example

As we noted in our drill, $k^{1728} \mod 1729 = 1$ for all $k$, $\gcd(k, 1729) = 1$. Our chances to randomly select $k$ such that $\gcd(k, 1729) > 1$ are very slim.
The Miller-Rabin Primality Test

Comment

Positive integers $N$ for which $a^{N-1} \mod N = 1$ for all $a$ such that $\gcd(a, N) = 1$ are called **Carmichael numbers**. There are infinitely many Carmichael numbers.
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- Let $N$ be an odd positive integer, $N - 1 = 2^m \cdot (2k + 1)$. 

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  - Or: $w^{2^i \cdot (2k+1)} \mod (N) = 1$ and $w^{2k+1} = \pm 1$. 
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In other words, the test fails to determine whether \( N \) is composite. (Do you know another example of a “failing” test?)
Proof.

If $N$ is prime then $w^{N-1} \mod N = 1$ (Fermat).

So $w^{(N-1)/2} \mod N = \pm 1$.

If $w^{(N-1)/2} \mod N = -1$ the test stops. It is inconclusive.

If $w^{(N-1)/2} \mod N = 1$ we calculate $w^{(N-1)/4} \mod N = \pm 1$.

As long as the results of $w^{(N-1)/2^i} \mod N = 1$ we continue until we reach $w^{2^k+1}$.

If $w^{2^k+1} \mod N \neq \pm 1$ then $N$ is definitely composite.

We skip the important part of the proof: more than 50% of the integers $a < N$ are composite-witnesses. So, to test whether an integer $p$ is prime, randomly select 100 integers $a < p$, apply to them the Miller-Rabin test. If the test fails, we assume that $p$ is prime. The probability that we made a mistake, that is declared $p$ is prime while it is not, is less than $(1/2)^{100}$ which is far less than the probability that the computer will make a mistake.
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- If $w^{(N-1)/2} \mod N = -1$ the test stops; it is inconclusive.
- If $w^{(N-1)/2} \mod N = 1$ we calculate $w^{(N-1)/4} \mod N = \pm 1$.
- As long as the results of $w^{(N-1)/2^i} \mod N = 1$ we continue until we reach $w^{2k+1}$.
- If $w^{2k+1} \mod N \neq \pm 1$ then $N$ is definitely composite.
Proof.

- If \( N \) is prime then \( w^{N-1} \mod N = 1 \) (Fermat).
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We skip the important part of the proof: more than 50% of the integers \( a < N \) are composite-witnesses. So, to test whether an integer \( p \) is prime, randomly select 100 integers \( a < p \), apply to them the Miller-Rabin test. If the test fails, we assume that \( p \) is prime. The probability that we made a mistake, that is declared \( p \) is prime while it is not, is less than \((\frac{1}{2})^{100}\) which is far less than the probability that the computer will make a mistake.
Example

1729 is a composite integer.

\[ 1728 = 2^6 \cdot 3^3. \]

\[ 2^3 \cdot 3^3 \equiv 1 \mod 1729 \]

for \( 1 \leq i \leq 6 \).

But \( 2^3 = 64 \), proving that 1729 is composite.

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Factoring large integers

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- The RSA corporation published many integers and challenged the public to factor them.

RSA-1024 (2^{1024} bits or 309 decimal digits) has not been factored. There is a $100,000 USD prize offered for its factorization. It is of particular interest as this is the current size used in applications.

R-2048 (617 decimal digits) has a prize of $200,000 USD for its factors.

The largest RSA number factored so far is RSA-768 (2^{32} decimal digits). RSA-200 was factored in 2009. The CPU time spent by computers working in parallel on this factorization was equivalent to about 75 years of CPU time on a 2.2GHz single processor.
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Square roots and factoring

Most integers are not perfect squares. Finding the square root or identifying that it is not a perfect square is very easy. Yet in modular arithmetic the situation is drastically different.

**Definition**

$r \in \mathbb{GF}^*(q)$ is a quadratic-residue mod $q$ if there is an $s \in \mathbb{GF}(q)$ such that $s^2 = r$.

We shall start with the easy task: finding $\sqrt{n} \mod p$, $p$ is prime.

Half the positive integers mod a prime number $p$ are quadratic residues. While finding their square roots is not difficult it is a bit trickier than finding the square root of an integer.
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Testing whether an integer $n$ is a quadratic residue mod $p$ is easy:

Calculate $n^{(p-1)/2} \mod p$. 

Note: $n = \alpha m$ where $\alpha$ is a primitive number mod $p$.

$n$ is a quadratic residue mod $p$ if and only if $m = 2^k$. 

$(\sqrt{n} \mod p, p = 4k + 3)$

Calculate $a = n^{2k+1} \mod p$.

If $a = -1$ stop, $n$ does not have a square root mod $p$.

$n^{2k+1} = 1 \mod p \Rightarrow n^{2k+2} \mod p = n$. 

$\sqrt{n} \mod p = n^{k+1} \mod p$. 
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Examples

Let $p = 337639$. This is a prime. $337639 = 4 \cdot 84409 + 3$. 

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- If we can find an odd integer such that \( a^{2s+1}b^{2t} \mod p = 1 \) then \( \sqrt{a} \mod p = a^{s+1} \cdot b^t \mod p \).
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This can be accomplished as follows:
While \( a^{2d(2m+1)} \mod p = 1 \) do: \( d = d - 1 \).
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We can repeat reducing the exponent by a factor of 2, multiplying by $b$ to make sure that the product will remain 1 until we reach $a^{2^{k+1}}b^{2^j} \mod p = 1$. 
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$a^{2^d(2m+1)} b^{2^{k-1}(2m+1)} \mod p = 1$

We can repeat reducing the exponent by a factor of 2, multiplying by $b$ to make sure that the product will remain 1 until we reach $a^{2^{k+1}} b^j \mod p = 1$.

$\sqrt{n} \mod p = a^{k+1} b^j \mod p$.

For an example see the SAGE sample in the supplements folder.
$\sqrt{n \mod p \cdot q}$

See the file factoring.pdf