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What groups are present in the Alhambra?

On the occasion of the approaching International Congress of Mathematicians (Madrid 2006) it is appropriate to renew the enjoyment of the arts — modern as well as historical — that grace many locations in Spain. The cover of the February 2006 issue, and the article by Allyn Jackson (starting on p. 218) are helpful, as is the note of Bill Casselman (on p. 213). Two sentences in the latter made me curious. Casselman states that "The geometric nature of Islamic design, incorporating complex symmetries, has been well-explored from a mathematical point of view. A fairly sophisticated discussion, referring specifically to the Alhambra, can be found in the book *Classical Tessellations and Three-manifolds* by José Maria Montesinos." I had visited the Alhambra more than twenty years ago, and have seen Montesinos' book soon after it appeared; that's a long time ago, and I had forgotten the details. I was about to get the book from our library, but before that I checked the Math Reviews. There I found an assertion which ran counter to my memories; so I eagerly started looking at the book itself, and recovering old papers and notes on the topic.

The question which of the seventeen wallpaper groups[2] are represented in the fabled ornamentation of the Alhambra has been raised and discussed quite often, with widely diverging answers. The first to investigate it was Edith Müller in her 1944 PhD thesis at the Zürich University, written under the guidance of Andreas Speiser[3]. In her thesis [7] Müller documents the appearance of 12 wallpaper groups among the ornaments of Alhambra[4]. (She also investigates other kinds of groups, but this is not relevant for our discussion at this time.) Due to a misunderstanding of Müller's comment that minor changes would have yielded two additional groups, several writers claimed that she found examples of 14 groups. Some later writers gave examples they claimed show the presence of one or the other of the missing groups, while others just stated that all 17 are present in the Alhambra. This latter phenomenon can be most readily explained as authors copying from authors who copied from others — all without any actual investigation. An exception to this is part of Jose Maria Montesinos' book [6], in which he argues that the photos he presents show the appearance of all 17 groups in the Alhambra. This was stressed in the review of [6] by Roger Fenn [1] that startled me: "... Incidentally, for the benefit of Hispanophiles, this book produces photographic evidence once and for all that all 17 plane symmetry patterns appear in the Alhambra." But does it really? My memory contradicts this.

Before justifying my standpoint, let us briefly consider the situation in which one is asked to count the number of trees in a forest. Clearly, for the effort to mean anything one needs to know where to count them — in the whole forest, or a certain square mile, or some other part. But it also has to be decided

1. What kinds of trees to count;
2. What is a tree? Is a sapling a tree? Should a minimum of 3" diameter be required? If so, where is it to be measured (just above ground, 3 feet above ground or some other way)?

3. What about dead but standing trees? What about fallen trees, possibly decomposed to the extent that no visual examination can determine their kind?

In the light of this metaphor we may agree that Montesinos considers all mosaics, paintings and plasterworks present in the whole Alhambra complex. But then we encounter problems:

There is no explanation concerning what is being considered in an ornament: Do we count the symmetries of the underlying tiling, without taking into account the colors of the tiles, or do we insist on color-preserving symmetries? (Similar questions about interlaces. The interlace in Fig. 1[5] has 4-fold rotational symmetries with no reflections if the interlacing is considered, but has mirrors if it is not.) In fact, Montesinos counts whatever he finds convenient. In one case he replaces all non-white colors by black in order to find an example with 3-fold rotational symmetry and no reflections (see Fig. 2). But one could equally well abstract color altogether, and get an example with 6-fold symmetry. In another case (not illustrated) he does disregard colors altogether.

There is no explanation as to what is the size or extent of an ornament that is sufficient to accept it as a representative of a certain group. In one case a single decorated tile is considered, while in another case miniature copies of the pattern shown in Fig. 3 are claimed to represent the group with 3-fold rotations and mirrors through all the rotation centers — although the sets of four triangles are in a pattern with 4-fold symmetry, and these are again arranged in a larger pattern with 4-fold symmetry, the whole just part of a decoration on the back of a chair.

Several of the ornaments shown are deteriorated to such an extent that it is impossible to see the pattern. Montesinos states that for several of these ornaments better examples can be found within the Alhambra, but does not show them.

As pointed out in a private communication from John Jaworski, it is easy to verify that by assigning appropriate colors, just two of the Alhambra mosaics could yield all seventeen groups. The ornaments shown in Figs. 5 and 6 are suitable for that purpose; photos of the same ornaments have been used by Jaworski in his very interesting work [5].

Due to these objections, and similar ones that could be made concerning some other publications, Fenn's enthusiasm seems premature. Moreover, during several days in 1983 of examining the decorations in the Alhambra, I found representatives of the 12 wallpaper groups listed by Müller and one she missed; it is illustrated in Fig. 4 (part of which is Montesinos' #4). A more detailed consideration of the difficulties in consistently counting the groups in the Alhambra, and what other kinds of groups (color symmetry, interlace symmetry, ...) might be more appropriate for some of the ornaments, appears in [4].

In view of the above discussion, one may wonder whether it is at all possible to arrive at a final, generally accepted count of the groups present in the Alhambra. The answer must be affirmative, but only if the counting is based on actual examination of the ornaments, presented in a consistent and well-explained manner, and following explicit criteria. It is possible that the presence of thousands of mathematicians at the International Congress may lead some of them to visit the Alhambra and be sufficiently taken by its splendor to invest their time and energy in such a count.

On the other hand, one may well ask why anybody would wish to do this, and what — if anything — would be the significance of the result. It seems to me that there is no more meaning to the determination of that number than, say, to the parity of the number of attendees of the ICM. Groups of symmetry had no relevance to artists and artisans who decorated the Alhambra. They certainly could have produced equally attractive ornamentation in any of the symmetry groups had anybody wished them to do so. Naturally, nobody did, since nobody knew about symmetry groups for the next five centuries. Thus it is only our infatuation with the idea that any attractive ornamentation must be explained in group-theoretic terms that leads us to try to find them there. It is probably worth mentioning that the analogous infatuation of crystallographers with groups crashed with the discovery of quasicrystals.

Does this mean that there is no role for mathematics in the study of ornaments, in the Alhambra or anywhere else? I feel very strongly that there is, provided we approach the task in a way consistent with the culture we are trying to understand and interpret. Thus, we have to think, or at least try to think, in terms that people creating the artifacts would understand and follow.

As an example, there is a lot of what could be called symmetry in the tiling in Fig. 4. Given the shape of the tiles, they are arranged in the only possible way; it entails periodicity. On this, the designer imposed coloration rules: Half the tiles are white; of the other half, half are black and the remainder are equally divided between green, blue and brown tiles. This is a way of looking that would have been understood by the Moorish artisans, and well may have been their intention. We could say that we find there an example of color symmetries (some horizontal mirrors preserve the white, black and green tiles, while interchanging blue and brown ones, while other mirrors and glide axes lead to other permutations of colors) — but this would have been totally extraneous to the thinking of people 500 years ago, hence it is entirely irrelevant. There are many additional examples of similar assignments of colors, in the Alhambra as well as in the ornamentation of other cultures. For example, Figure 7 shows an example in which one half of the tiles is white, one quarter black, and the last quarter evenly divided between tan and green. A mathematical investigation of the possibilities would appear to be both interesting and doable, and possibly even useful to anthropologists. On the other hand, in many cases there is no such orderliness in the colors of the tiles; one has the feeling that the artists destroyed the symmetries to make the tilings less monotonous.

In decorations on pottery, as well as other surfaces, patterns that are not discrete often appear. Circles around pots and vases, straight lines and strips on flat surfaces, are examples of (admittedly rather minimal) decorations. They are not approachable through the study of discrete groups — but their historic development within a culture still can be of interest.

In the study of the exquisite textiles from ancient Peru discrete patterns are common, and quite orderly. The motifs in some of them form one orbit under isometric symmetries, but in others this is not the case; often the colors "spoil" any symmetry. Moreover, just as in the case of Moorish ornamentation, investigation of the symmetry groups of the patterns is totally irrelevant. On the other hand it can be shown that taking into account the structure of the "fabric plane" in which the patterns are imbedded, one can devise (see [3]) an explanation for the orderliness of the patterns, that could have been understood and transmitted among the illiterate weavers of long ago. As it turns out, there is only a finite number of possibilities.

There are probably many other situations in which a more flexible approach of mathematical interpretation would be not only more productive but also more relevant. In particular, this applies to the beautiful ornamentation in the Alhambra, but also those in Sevilla and other locations.

References

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Fig. 1.

Fig. 2.

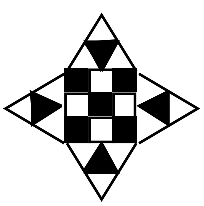

Fig. 3.

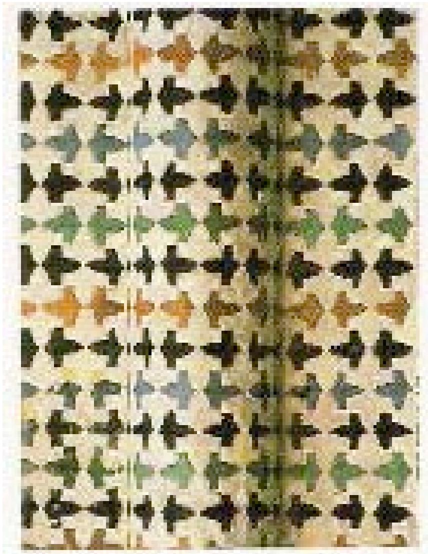

Fig. 4.

Fig. 5.

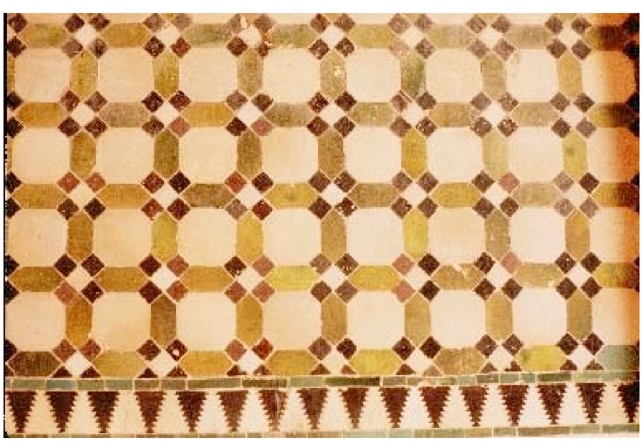

Fig. 6.

Fig. 7

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[2] Classes of discrete groups of isometries, with two independent translations.

[3] Speiser's 1922 text on the theory of groups deals extensively with the investigation of symmetry groups of ornaments, and illustrates the topic with several patterns from ancient Egypt. His rather biased opinion about Egyptian decorations is best seen from his assertion that "... Egypt, which is the source of all later ornamentation" ("... Ägypten, denn hier is the Quelle aller späteren Ornamentik." He also quotes approvingly (in the original English) the opinion of Flinders Petrie that "Practically it is very difficult, or almost impossible, to point out decoration which is proved to have originated independently, and not to have been copied from the Egyptian stock."

[4] She also investigated other kinds of groups as well, in particular the 80 groups of the two-sided Euclidean plane. These are the topic of a short note Müller [8], which includes also illustrations mosaics from the Alhambra representing 9 different wallpaper groups. Despite her mathematical beginnings, Müller became a well-known astronomer, and was for several years the General Secretary of the International Mathematica Union. In a letter from 1984 she mentioned that there is continuing interest in her thesis, and that there is a plan to republish it. Regrettably, this seems not to have happened.

[5] All photos were taken by the author in 1983, during a visit to the Alhambra while on a sabbatical from the University of Washington, and with the support of a Guggenheim fellowship.

[6] This interdisciplinary journal, with a blue-ribbon Advisory Board, was started after meticulous preparations, and its first issue contained contributions by A. L. Mackay, A. L. Loeb, M. J. Wenninger, C. A. Pickover, V. Vasarely and others. The publication of the journal was cancelled after the first issue, by its publisher VCH Publishers, due to low rate of subscriptions. So much for investment in the interdisciplinary approach.

[7] Unfortunately, the dedication of the paper to Heinrich Heesch was omitted, and the colored illustrations have been rendered in black-and-white.