Euler's Ratio-Sum Theorem and Generalizations

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Over the centuries, many papers have been written about relations among different parts of a triangle, by well-known mathematicians as well as others. The main aim of this note is to show how asking the right questions can lead to new facts and to far-reaching generalizations that retain an elementary nature and would have been understandable to mathematicians of ages past.

Our starting point is one of the results of Euler's paper [**5**], which shows, in the notation of FIGURE 1, that

$$QB_1/A_1B_1 + QB_2/A_2B_2 + QB_3/A_3B_3 = 1, \qquad (*)$$

where Q is an arbitrary point in the plane of the arbitrary triangle $A_1A_2A_3$ and B_i is the intersection point of the cevian line Q_iQ with the side opposite A_i. Here and throughout, the only restriction is that all the points are well-defined and all the lengths appearing in the denominators are not zero. The lengths are understood as signed lengths; since only ratios of collinear segments are considered, the positive direction on the lines carrying the segments is irrelevant. Euler gives several proofs that use various elementary geometric or trigonometric arguments. We shall provide a simple proof, and show how this result can be generalized in a variety of ways: to analogs of triangles in three and higher dimensions, to polygons with more than three sides and their higher-dimensional analogs, and to other ratio-sums.

We shall first discuss the version of Euler's result that holds for d-dimensional simplices, that is, the simplest polytopes of dimension d — the analogs of the triangles in the plane and tetrahedra in 3-space. We shall tie this with a presentation of similar results for the five other ratio-sums that can be defined using cevians; so far, these seem to have received scant attention. In contrast to Euler's result, formulas with the five other ratio-sums involve the dimension d; in some cases, the formula takes the form of an inequality, with the case of equality precisely identified. The generalizations of these results to more general polygons, polyhedra, and polytopes (higher dimensional relatives of polygons and polyhedra) will then be presented, followed by historical and other comments.

Ratio-sums for simplices For $d \geq 2$, let T^d denote the d-dimensional simplex in Euclidean d-space E^d, with vertices A_i, $0 \leq i \leq d$. Thus T^d can be interpreted as

*Professor Klamkin passed away in the summer of 2004. As a friend and a mathematician he will be sorely missed by many of us. Professor Klamkin was still able to see the referees' comments on our paper and approve the proposed final version of it. A variety of unfortunate circumstances delayed the sending of that version to the Editor. But this had the silver lining contained in part (vii) of the last section, added September 15, 2005. BG

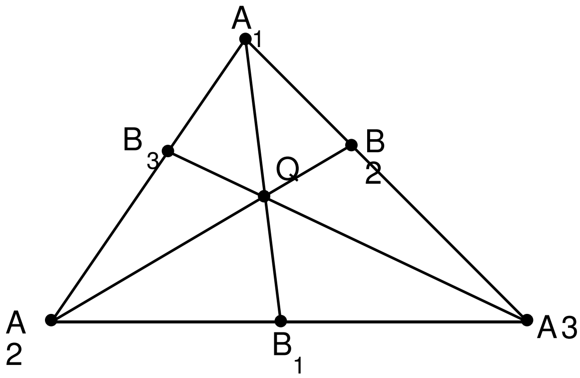

FIGURE 1: An example illustrating the notation used in Euler's theorem and in Theorem 1

the convex hull of $d+1$ points of the Euclidean d-space E^d that are not all contained in a hyperplane of smaller dimension. A good model to illustrate the concept and the following arguments is given by the d-simplex with vertices at the origin and at the unit points of a standard basis of E^d. For notational convenience we shall occasionally use $A_{d+1} = A_0$.

We start by describing the setting of the results. Let Q be a pint of E^d, and F_i the facet (that is, the $(d-1)$-dimensional face) of T^d that is opposite A_i. Let B_i be the point of intersection of the line (the *cevian*) through A_i and Q with the hyperplane H_i that contains F_i. For the defintions, and some of the results, the point Q need not be in the interior of T^d; the only overall restriction on Q is that all points B_i must be well defined, and that the denominators in the various fractions be nonzero. This condition will be assumed throughout, and will not be repeated in the reformulation of our results. We shall be interested in various ratios involving the lengths $a_i = \|A_i - Q\|$, $b_i = \|Q - B_i\|$, $q_i = \|A_i - B_i\|$ of the segments A_iQ, QB_i, A_iB_i. As already mentioned, the lengths in question are to be taken as signed lengths; since we shall always consider ratios of collinear segments, the scale of measurement and the direction chosen as positive on each line are irrelevant. For $d = 2$, one illustration of the possibilities is indicated in FIGURE 1, and another in FIGURE 2.

To begin with, we are interested in the six ratio sums defined as follows:

$$\rho(b,q) = \Sigma_i b_i/q_i, \qquad \rho(a,q) = \Sigma_i a_i/q_i,$$
$$\rho(q,b) = \Sigma_i q_i/b_i, \qquad \rho(q,a) = \Sigma_i q_i/a_i,$$
$$\rho(a,b) = \Sigma_i a_i/b_i, \qquad \text{and} \quad \rho(b,a) = \Sigma_i b_i/a_i,$$

where each sum is over all i, $0 \le i \le d$. We shall prove the following results.

Theorem 1 *With the above notation the following statements are valid for all Q:*

(i) $\rho(b,q) = 1$.

(ii) $\rho(a,q) = d$.

(iii) If $q_i/b_i > 0$ for all i, then $\rho(q,b) \ge (d+1)^2$; equality holds if and only if Q is the centroid of T^d.

(iv) If $q_i/a_i > 0$ for all i, then $\rho(q,a) \ge (d+1)^2$; equality holds if and only if Q is the centroid of T^d.

(v) If $q_i/b_i > 0$ for all i, then $\rho(a,b) \ge d(d+1)$; equality holds if and only if Q is the centroid of T^d.

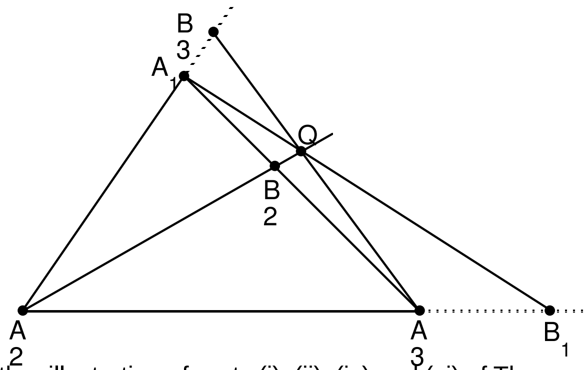

FIGURE 2: Another illustration of parts (i), (ii), (iv) and (vi) of Theorem 1. Parts (iii) and (v) are not applicable to this example since $q_2/b_2 < 0$

(vi) If $q_i/a_i > 0$ for all i, then $\rho(b,a) \geq (d+1)/d$; equality holds if and only if Q is the centroid of T^d.

Before we turn to the proofs, we recall two very useful lemmas.

Let T^d be a d-simplex with vertices A_i, $0 \leq i \leq d$, and let S^d denote the simplex with vertices $Q, A_1, A_2, \cdots, A_d$. Then, denoting by $V(T)$ the signed volume of the simplex T, we have:

Lemma 1 $\|Q - B_0\|/\|A_0 - B_0\| = b_0/q_0 = V(S^d)/V(T^d)$.

This self-evident fact, which was called the “volume principle” in [**11**], has been used without a special name by other authors (see, for example, [**3**, p. 131] for $d = 3$). In case $d = 2$, it has been called the “area principle” in [**10**] and other publications, and it has been used starting at least two centuries ago.

A second well-known tool is the elementary

Lemma 2 *For all $x > 0$, $x + 1/x \geq 2$, with equality if and only if $x = 1$.*

We shall frequently apply this lemma in the form $1/x \geq 2 - x$.

Proof of Theorem 1: We shall give here proofs for only the first three parts of Theorem 1, to serve as warm-up for the generalizations presented in Theorem 2.

The result of part (i) follows at once from Lemma 1, upon noticing that the signed volumes of the simplices with common apex Q that are spanned by the $d+1$ facets of T^d add up precisely to the signed volume of T^d. For part (ii) it is enough to note that

$$\rho(a,q) = \sum_i a_i/q_i = \sum_i (1 - b_i/q_i)$$
$$= (d+1) - \sum_i b_i/q_i = d + 1 - \rho(b,q) = d + 1 - 1 = d.$$

For part (iii), using Lemma 2, we have

$$\rho(q,b)/(d+1) = \sum_i q_i/\left((d+1)b_i\right) \geq 2(d+1) - \sum_i \left((d+1)b_i\right)/q_i$$
$$= 2(d+1) - (d+1)\rho(b,q) = d+1,$$

which is equivalent to the inequality of (iii). Equality holds if and only if $\left((d+1)b_i\right)/q_i = 1$ for every i, which is a characterization of Q as the centroid of T^d. ■

Further generalizations The role that the simplex T^d plays in the above theorems will become clearer as we move to generalize of Theorem 1. It is convenient to introduce appropriate notation.

Let P denote a fixed polytope of dimension d in Euclidean d-space E^d. It is simplest to think of P as a convex polytope, that is, as a generalization of convex polygons in the plane or convex solids in 3-space, which one could call polytopes of dimension $d = 2$ or 3—the reader is welcome to use these to picture the generalizations. However, the restriction to convex polytopes is in no way necessary. For $d = 2$ and $d = 3$, we can admit polygons and polyhedra in the generality described in [**8**] and [**9**], that is, self-intersecting polygons, and self-intersecting polyhedra with possibly self-intersecting faces. For $d \geq 4$ we admit the obvious generalizations of these kinds of polygons and polyhedra. We shall use the term *polytope* for all dimensions $d \geq 2$.

We impose the following restrictions on the polytopes considered here: The polytopes must be orientable, and the d-polytopes and all their facets must have nonzero content (volume in dimension d or $d-1$, respectively). The content of a d-polytope P will be denoted by $V(P)$. The d-pyramid determined by a $(d-1)$-polytope F and point X will be denoted $F(X)$.

Polytopes satisfying these conditions shall be called *star-like*. The traditional Kepler-Poinsot polyhedra—that is, the nonconvex analogs of the Platonic regular solids—are star-like both in our sense and visually. So are many (but not all) of the uniform polyhedra presented in [**4**], and beautifully illustrated by photos of models in [**19**]. Many other examples appear in [**8**] and [**9**].

Let P be a star-like d-polytope. The f facets of P are labeled $F_1, F_2, \ldots, F_f$ in an arbitrary order. Let A_j, $1 \leq j \leq f$, be a collection of points of E^d such that for suitable points B_j, with B_j in the hyperplane determined by F_j, the line $L_j = A_jB_j$ is well defined, intersects F_j only in B_j, and all lines L_j pass through a common point Q.

In analogy to the notation for Theorem 1, we put:

$$\rho(b,q;W) = \Sigma_j w_j b_j/q_j, \qquad \rho(q,b;W) = \Sigma_j q_j/(w_j b_j),$$
$$\rho(a,q;W) = \Sigma_j w_j a_j/q_j, \qquad \rho(q,a;W) = \Sigma_j q_j/(w_j a_j),$$
$$\rho(a,b;W) = \Sigma_j w_j a_j/(w_j b_j), \qquad \text{and} \quad \rho(b,a;W) = \Sigma_j b_j/(w_j a_j),$$

where $W = (w_1, w_2, \ldots, w_f)$ is an ordered f-tuple of suitable weights specified below; these weights depend on P and the points A_j, but are independent of Q. All summations are for $j = 1, 2, \ldots, f$. In all parts of Theorem 2 it is understood that P is a star-like d-polytope with f facets, the points Q and A_j satisfy the above condition, and the weights W are given by $w_j = V\left(F_j(A_j)\right)/V(P)$. We abbreviate $w = \Sigma_j w_j$ and $w^* = \Sigma_j 1/w_j$. FIGURE 3 illustrates the notation.

Theorem 2 *With the above notation the following statements are valid:*

(i) $\rho(b,q;W) = 1$.

(ii) $\rho(a,q;W) = w - 1$.

(iii) If $q_j/b_j > 0$ *and* $w_j > 0$ *for all* j*, then* $\rho(q,b;W) \geq w(2f-1)$*. Equality holds if and only if* $q_j/(w_jb_j) = w$ *for all* $j = 1, 2, \ldots, f$.

(iv) If $q_j/a_j > 0$ *and* $w_j > 0$ *for all* j*, then* $\rho(q,a;W) \geq f^2/(w-1)$*, with equality if and only if* $q_j/(w_ja_j) = f/(w-1)$ *for all* $j = 1, 2, \ldots, f$.

(v) If $q_j/b_j > 0$ *and* $w_j > 0$ *for all* j*, then* $\rho(a,b;W) \geq w(2f-w) - w^*$*, with equality if and only if* $q_j/(w_jb_j) = w$ *for all* $j = 1, 2, \ldots, f$.

(vi) If $q_j/a_j > 0$ *and* $w_j > 0$ *for all* j*, then* $\rho(b,a;W) \geq f^2/(w-1) - w^*$*, with equality if and only if* $q_j/(w_ja_j) = f/(w-1)$ *for all* $j = 1, 2, \ldots f$.

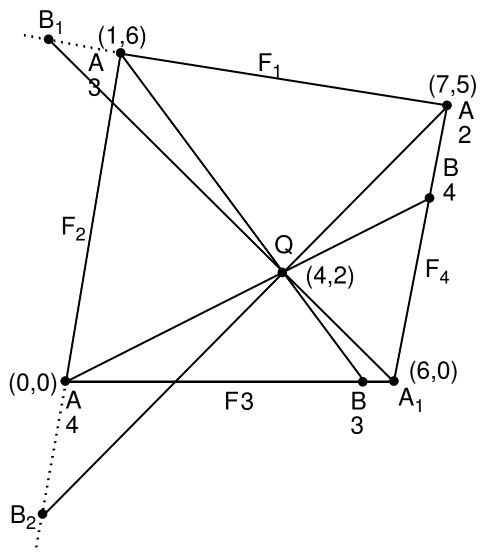

FIGURE 3: An example of the situation covered by Theorem 2. As is easily verified, in this example the weights are given by $W = (31/67, 37/67, 36/67, 30/67)$, hence $w = 2$

Proof of Theorem 2: For part (i), we note that an easy generalization of what we called the "volume principle" shows that $b_j/q_j = V\left(F_j(Q)\right)/V\left(F_j(A_j)\right)$. Hence we have: $\rho(b,q;W) = \Sigma_j w_j b_j/q_j = \Sigma_j w_j V\left(F_j(Q)\right)/V\left(F_j(A_j)\right) = \Sigma_j V\left(F_j(Q)\right)/V(P) = 1$, since the sum of the volumes of the pyramids with apex Q equals the volume of P.

For part (ii), in analogy to the above and using Theorem 1(i), we have

$$\begin{aligned}\rho(A,q;W) &= \sum_j w_j a_j/q_j = \sum_j w_j(1 - b_j/q_j)\\ &= \sum_j w_j - \sum_j w_j V\left(F_j(Q)\right)/V\left(F_j(A_j)\right) = (\sum_j w_j) - 1 = w - 1.\end{aligned}$$

For part (iii), in analogy to the proof of Theorem 1(iii), we have

$$\rho(q,b;W)/w = \sum_j q_j/(ww_jb_j) \geq \sum_j (2 - ww_jb_j/q_j) = 2f - w.$$

The equality criterion follows from Lemma 2.

For part (iv) we have

$$\begin{aligned}&(w-1)\rho(q,a;W)/f\\ &\quad= \sum_j (w-1)qZ_j/(fw - ja_j) \geq \sum_j \left(2 - fw_ja_j/\left((w-1)q_j\right)\right) =\\ &\quad= \sum_j 2 - (f\sum_j w_ja_j/q_j)/(w-1) = 2f - f(w-1)/(w-1) = f,\end{aligned}$$

which is equivalent to the claim. The equality condition is again a consequence of Lemma 2.

For part (v), in analogy to the above, and using Lemma 2 and part (i) of the theorem, we have

$$\begin{aligned}\rho(a,b;W) &= \sum_j a_j/(w_j b_j) = \sum_j q_j/(w_j b_j) - \sum_j 1/w_j \\ &= w\sum_j q_j/(ww_j b_j) - \sum_j 1/w_j \\ &\geq w\sum_j (2 - ww_j b_j/q_j) - w^* = 2wf \geq Vw_2 \sum_j w_j b_j/q_j - w^* \\ &= 2wf - w_2 - w^*.\end{aligned}$$

Equality holds if and only if it holds in part (iii) of the theorem.

For part (vi), in analogy to the above, and using part (iv) of the theorem:

$$\begin{aligned}&\rho(b,a;W) \\ &\quad = \sum_j b_j/(w_j a_j) = \sum_j q_j/(w_j a_j) - \sum_j 1/w_j \\ &\quad = (f/(w-1))\sum_j (w-1)q_j/(fw_j a_j) - w^* \\ &\quad \geq (f/(w-1))\left(\sum_j 2 - \sum_j fw_j a_j/\left((w-1)q_j\right)\right) - w^* \\ &\quad = 2f^2/(w-1) - \left(f^2/(w-1)^2\right)\sum_j w_j a_j/q_j - w^* \\ &\quad = 2f^2/(w-1) - \left(f^2/(w-1)^2\right)(w-1) - w^* = f^2/(w-1) - w^*,\end{aligned}$$

with equality if and only if equality holds in part (iv).

This completes the proof of all parts of Theorem 2. ■

Historical and other comments.

(i) For $d = 2$, Theorem 1(i) contains Euler's result. Euler's theorem has been rediscovered by several authors; first among them is Gergonne [**6**]. Very few of these mention Euler—even the authoritative work of Zacharias [**21**] mentions only Gergonne. Surprisingly, the detailed survey of pre-20th century geometry by Simon [**18**] (which has references to well over 2000 authors!) does not mention the result at all. Without any attribution, Euler's result appears in [**1**, p. 162]. The extension to higher dimensions is also not new. For $d = 3$ the earliest mention we are aware of is by Gergonne [**6**]. Parts (i) and (ii) of Theorems 1 appear in texts [**2**, page 115] and [**3**, page 131]. For general d, our Theorem 1(i) appears in [**13**] and probably in several other places; it was also mentioned in a letter from Prof. H. Gülicher in 1998.

(ii) We are not aware of any mention of parts (iii) to (vi) of Theorem 1 in the literature. The fact that equality holds in these cases for Q at the centroid is obvious. In each of them, the characterization of Q as the centroid in case of equality is due to Klamkin [**14**].

(iii) It is easy to verify that the results of Theorem 2 reduce to those of Theorem 1 in the special case that P is the d-simplex T^d and the points A_i are the vertices of T^d. However, even if P is T^d the results of Theorem 2 are more general since they do not restrict the points A_i to be vertices of T^d.

(iv) Parts (i) and (ii) of Theorems 1 and 2 are somewhat analogous to the classical theorems of Ceva and Menelaus, and the new results on self-transversality (see [**11**]). These earlier results deal with product of ratios, while here we are concerned with sums of ratios. However, our other results seem not to have any multiplicative analogs.

(v) The ratios a_j/b_j for cevians of a triangle appear in Euler's paper [**5**], in the following result, formulated in the notation of Section 1:

$$\begin{aligned} & A_1Q/QB_1 + A_2Q/QB_2 + A_3Q/QB_3 + 2 \\ = \; & (A_1Q/QB_1)\cdot(A_2Q/QB_2)\cdot(A_2Q/QB_2). \end{aligned} \qquad (**)$$

This nonlinear relation seems to have been largely forgotten. it has been established in a simple way and its validity extended in the recent paper [**17**]. An analog of this result is due to Euler [**5**]; it deals with ratios of lengths of the segments into which each side of a triangle is partitioned by parallels to the other sides. It was independently found by Gülicher [**12**].

(vi) The idea to use weights attached to the ratios originated with Shephard [**16**]; he kindly sent a preprint of this paper to one of us. For polygons in the plane Shephard establishes in [**16**] a restricted version of part (i) of our Theorem 2. Weights were also assigned to ratios in [**7**], for ratio-sums of a slightly different kind.

(vii) Part (iii) of Theorem 1, as well as some results found in the literature, can be generalized so that, instead of cevians, we use arbitrary segments, which need not have a common point. Let T^d, A_i, Q, F_i, and H_i have the same meaning as in the discussion leading to Theorem 1. Let B_i be an arbitrary point of H_i, and let C_i denote the point of H_i such that the segment QC_i is parallel to the segment A_iB_i. FIGURE 4 illustrates a case with $d = 2$. Let q_i and c_i denote the signed lengths of the segments A_iB_i and QC_i, let f and f_i denote the d-dimensional volumes of T^d and of the simplices with basis F_i, and let u and v be nonnegative reals. Using the obvious generalization of the volume principle we have

$$\begin{aligned} \sum_i q_i/(uc_i + vq_i) &= \sum_i 1/\left(u(c_i/q_i) + v\right) = \sum_i 1/\left(u(f_i/f) + v\right) \\ &\geq (d+1)^2/\sum_i \left(u(f_i/f) + v\right) \\ &= (d+1)^2/\left(u + v(d+1)\right). \end{aligned}$$

Here we used the fact that the arithmetic mean is greater than or equal to the harmonic mean, and that $\sum_i f_i/f = 1$. One could also add the less interesting generalization of part (i) of Theorem 1, namely $\sum_i (uc_i + vq_i)/q_i = u + v(d+1)$. The special case $d = 3$, with B_i the foot of the altitude from A_i, and Q the incenter appears in [**20**] and [**15**]. In the former, $u = 3$ and $v = 1$, so that the lower bound is $16/7$; Murray Klamkin was one of the solvers of [**20**]. In [**15**], $v = 0$ and $u = 1$, hence the lower bound is 16.

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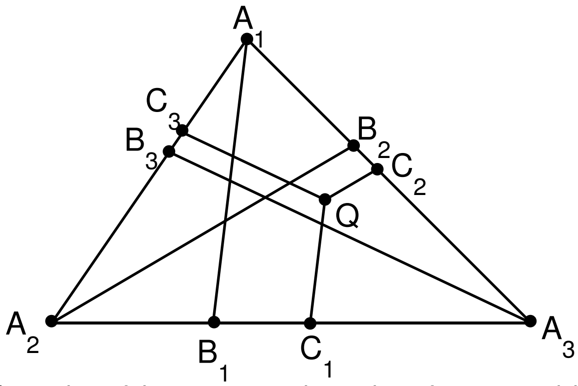

FIGURE 4: An illustration of the content and notation of comment (vii)

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