Spacecraft Dynamics and Control
- An Introduction
EXERCISES

Anton H.J. de Ruiter, Christopher J. Damaren and James R. Forbes

This document contains exercises to accompany the book. The authors welcome any feedback. Please send any comments, corrections or suggestions to aderuiter@ryerson.ca.

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Chapter 1 Exercises

1. Consider three non-coplanar vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$, as shown in Figure 1.1, which define a parallelepiped.

   Note that all sides are parallelograms, and that opposing sides are parallel. Note that as drawn, the vector $\vec{\mathbf{a}}$ points above the plane defined by vectors $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$. The volume of the parallelepiped is given by the area of any one of its sides, multiplied by the perpendicular distance to the opposing side. In terms of figure 1.1,

$$V = A_1 h_1 \tag{1.1}$$

   (a) Consider the parallelogram defined by vectors $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$, as shown in Figure 1.2. Determine the area A_1 in terms of $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$.

   (b) Referring to figure 1.3, determine the vector $\vec{\mathbf{n}}$ of unit magnitude ($|\vec{\mathbf{n}}| = 1$) that is perpendicular to the parallelogram defined by $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$, and points to the same side of it as the vector $\vec{\mathbf{a}}$.

   (c) Using the result from (b), determine the perpendicular distance from side 1 to the opposing side 4, in terms of the vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{n}}$.

   (d) Using equation (1.1), compute the volume, V of the parallelepiped in terms of the vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$.

   (e) Referring to figure 1.4, compute the volume using the area of side 2 (A_2), multiplied it by the perpendicular distance (h_2) from side 2 to side 5, that is $V = A_2 h_2$.

   (f) Combine the results of (d) and (e).

2. Consider again three non-coplanar vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$, as shown in Figure 1.5, which define a parallelepiped. Note that unlike in Figure 1.1, the vector $\vec{\mathbf{a}}$ points below the plane defined by vectors $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$.

   (a) Repeat parts (a) to (f) of Question 1, using the parallelepiped shown in Figure 1.5.

   (b) Now, consider three co-planar vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$. Compute $\vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} \times \vec{\mathbf{c}})$ and $\vec{\mathbf{b}} \cdot (\vec{\mathbf{c}} \times \vec{\mathbf{a}})$. What can you conclude from this and part (f) in Question 1 and part (a) in this question?

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3. Consider the vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$. In this problem, you are going to find an alternative expression for $\vec{\mathbf{a}} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}})$. It is assumed that $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$ are non-zero, and non parallel (see figure 1.6).

   It has been shown that $\vec{\mathbf{a}} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) = \vec{\mathbf{a}}_\perp \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}})$, where $\vec{\mathbf{a}}_\perp$ is the component of $\vec{\mathbf{a}}$ that is perpendicular to $\vec{\mathbf{b}} \times \vec{\mathbf{c}}$, such that $\vec{\mathbf{a}} = \vec{\mathbf{a}}_\perp + \vec{\mathbf{a}}_{||}$ (where $\vec{\mathbf{a}}_{||}$ is parallel to $\vec{\mathbf{b}} \times \vec{\mathbf{c}}$).

   The vectors $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$ define the plane that is perpendicular to $\vec{\mathbf{b}} \times \vec{\mathbf{c}}$. Therefore, $\vec{\mathbf{a}}_\perp$ must lie in that plane.

   (a) Consider the vectors $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$. Referring to figure 1.7, find $\vec{\mathbf{c}}_{||b}$, the component of $\vec{\mathbf{c}}$ that is parallel to $\vec{\mathbf{b}}$. Using this, find $\vec{\mathbf{c}}_{\perp b}$, the component of $\vec{\mathbf{c}}$ that is perpendicular to $\vec{\mathbf{b}}$. Finally, find $|\vec{\mathbf{c}}_{\perp b}|$ in terms of $|\vec{\mathbf{b}} \times \vec{\mathbf{c}}|$ and $|\vec{\mathbf{b}}|$.

   (b) Since $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}_{\perp b}$ are perpendicular, we may obtain the unit perpendicular vectors

   $$\vec{\mathbf{n}}_1 = \frac{\vec{\mathbf{b}}}{|\vec{\mathbf{b}}|}, \;\; \vec{\mathbf{n}}_2 = \frac{\vec{\mathbf{c}}_{\perp b}}{|\vec{\mathbf{c}}_{\perp b}|}.$$

   These vectors can also be used to define the plane containing $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$.
   Use your solution to part (a) to find $\vec{\mathbf{n}}_2$ in terms of $\vec{\mathbf{b}}$, $\vec{\mathbf{c}}$, $|\vec{\mathbf{b}}|$ and $|\vec{\mathbf{b}} \times \vec{\mathbf{c}}|$.

   (c) The projection of the vector $\vec{\mathbf{a}}$ onto the plane defined by $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$ is given by

   $$\vec{\mathbf{a}}_{proj} = (\vec{\mathbf{a}} \cdot \vec{\mathbf{n}}_1)\vec{\mathbf{n}}_1 + (\vec{\mathbf{a}} \cdot \vec{\mathbf{n}}_2)\vec{\mathbf{n}}_2. \tag{1.2}$$

   Clearly, $\vec{\mathbf{a}}_{proj}$ is perpendicular to $\vec{\mathbf{b}} \times \vec{\mathbf{c}}$, since it lies in the plane defined by $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$.
   Show that the vector $\vec{\mathbf{a}} - \vec{\mathbf{a}}_{proj}$ is perpendicular to the plane, and hence is parallel to $\vec{\mathbf{b}} \times \vec{\mathbf{c}}$. Hint: Check $(\vec{\mathbf{a}} - \vec{\mathbf{a}}_{proj}) \cdot \vec{\mathbf{n}}_1$ and $(\vec{\mathbf{a}} - \vec{\mathbf{a}}_{proj}) \cdot \vec{\mathbf{n}}_2$.
   Conclude from this that the component of $\vec{\mathbf{a}}$ parallel to $\vec{\mathbf{b}} \times \vec{\mathbf{c}}$ is given by $\vec{\mathbf{a}}_{||} = \vec{\mathbf{a}} - \vec{\mathbf{a}}_{proj}$, while the component of $\vec{\mathbf{a}}$ perpendicular to $\vec{\mathbf{b}} \times \vec{\mathbf{c}}$ is given by $\vec{\mathbf{a}}_\perp = \vec{\mathbf{a}}_{proj}$.

   (d) Since $\vec{\mathbf{a}}_\perp$ is perpendicular to $\vec{\mathbf{b}} \times \vec{\mathbf{c}}$, the cross-product $\vec{\mathbf{a}}_\perp \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}})$ is obtained by rotating $\vec{\mathbf{a}}_\perp$ by 90^o about $\vec{\mathbf{b}} \times \vec{\mathbf{c}}$, and then scaling by $|\vec{\mathbf{b}} \times \vec{\mathbf{c}}|$. That is,

   $$\vec{\mathbf{a}}_\perp \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) = \vec{\mathbf{a}}_{\perp rot} |\vec{\mathbf{b}} \times \vec{\mathbf{c}}|, \tag{1.3}$$

   where $\vec{\mathbf{a}}_{\perp rot}$ is the vector $\vec{\mathbf{a}}_\perp$ rotated by 90^o in the direction indicated in Figure 1.8.
   Consider the vectors $\vec{\mathbf{n}}_1$ and $\vec{\mathbf{n}}_2$. What are $\vec{\mathbf{n}}_{1rot}$ and $\vec{\mathbf{n}}_{2rot}$, the vectors obtained by rotating $\vec{\mathbf{n}}_1$ and $\vec{\mathbf{n}}_2$ by 90^o respectively, as indicated in figure 1.9?
   Using this and (1.2), find $\vec{\mathbf{a}}_{\perp rot}$.

   (e) Substitute the results from part (b) into $\vec{\mathbf{a}}_{\perp rot}$ obtained in part (d), and finally obtain the expression for $\vec{\mathbf{a}} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) = \vec{\mathbf{a}}_\perp \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}})$, using (1.3).

4. We have seen that for a right-handed reference frame $\mathcal{F}$, the cross-product of two vectors $\vec{\mathbf{a}} = \vec{\mathcal{F}}^T \mathbf{a}$ and $\vec{\mathbf{b}} = \vec{\mathcal{F}}^T \mathbf{b}$, is

   $$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \vec{\mathcal{F}}^T \mathbf{a}^\times \mathbf{b},$$

where

$$\mathbf{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}, \quad \mathbf{a}^\times \triangleq \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}.$$

Determine the expression for $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$ if the frame $\mathcal{F}$ is left-handed (see figure 1.10). You may start from the expression

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \begin{bmatrix} a_x & a_y & a_z \end{bmatrix} \begin{bmatrix} \vec{\mathbf{x}} \times \vec{\mathbf{x}} & \vec{\mathbf{x}} \times \vec{\mathbf{y}} & \vec{\mathbf{x}} \times \vec{\mathbf{z}} \\ \vec{\mathbf{y}} \times \vec{\mathbf{x}} & \vec{\mathbf{y}} \times \vec{\mathbf{y}} & \vec{\mathbf{y}} \times \vec{\mathbf{z}} \\ \vec{\mathbf{z}} \times \vec{\mathbf{x}} & \vec{\mathbf{z}} \times \vec{\mathbf{y}} & \vec{\mathbf{z}} \times \vec{\mathbf{z}} \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}.$$

5. A five-link robotic manipulator is being designed, with a hand at the end (see Figure 1.11). Each of the links have the same length, given by r. Each of the joints allows a rotation about a single axis. The vector $\vec{\mathbf{r}}_{ij}$ denotes the position of joint j relative to joint i. The vector $\vec{\mathbf{r}}_H$ denotes the position of the hand relative to the base of the robot (joint 1). Refer to Figure 1.11.

   We attach a reference frame to the room, denoted by $\mathcal{F}_r$. This frame is defined with the $\vec{\mathbf{x}}_r$ and $\vec{\mathbf{y}}_r$ axes in the plane of the floor, and the $\vec{\mathbf{z}}_r$ axis pointing vertically upwards. Refer to Figure 1.12.

   It will also be useful to attach a reference frame to each link, denoted by $\vec{\mathcal{F}}_i$, for $i = 1, ..., 5$. The reference frame attached to each link is defined such that the $\vec{\mathbf{z}}_i$ axis points along the length of the link from joint i to joint $i+1$. Refer to Figure 1.13. The joint rotations are:
   - Joint 1 allows a rotation θ_1 about the $\vec{\mathbf{y}}_r$ axis. See figure 1.14.
   - Joint 2 allows a rotation θ_2 about the $\vec{\mathbf{x}}_1$ axis. See figure 1.15.
   - Joint 3 allows a rotation θ_3 about the $\vec{\mathbf{z}}_2$ axis.
   - Joint 4 allows a rotation θ_4 about the $\vec{\mathbf{y}}_3$ axis.
   - Joint 5 allows a rotation θ_5 about the $\vec{\mathbf{z}}_4$ axis.

   (a) Determine the vectors $\vec{\mathbf{r}}_{12}$, $\vec{\mathbf{r}}_{23}$, $\vec{\mathbf{r}}_{34}$, $\vec{\mathbf{r}}_{45}$, $\vec{\mathbf{r}}_{5H}$, in their respective link frames, $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$, $\mathcal{F}_4$, $\mathcal{F}_5$.

   (b) What is the orientation of the hand relative to the room coordinates? (Find the rotational transformation from $\mathcal{F}_r$ to $\mathcal{F}_5$).

   (c) Determine the position of the hand relative to the robot base, $\vec{\mathbf{r}}_H$, in room coordinates $\mathcal{F}_r$. Note: leave your answer in terms of products of principal rotation matrices.

   The results from parts (b) and (c) will allow the robot user to determine the required joint angles θ_1, θ_2, θ_3, θ_4, θ_5 required to pick up an object with a given location and orientation.

6. Earth orbiting spacecraft problems often require the conversion between different reference frames. Three frames that are often used are the Earth-Centered-Inertial (ECI) frame (denoted $\mathcal{F}_G$), the Earth-Centered-Earth-Fixed (ECEF) frame (denoted $\mathcal{F}_F$) and the local Topocentric frame (denoted $\mathcal{F}_T$). The ECI frame is an inertially fixed frame (does not rotate with the earth), with the z-axis is aligned with the earth's

spin axis, and therefore the *x*- and *y*-axes lie in the equatorial plane. The origin of the ECI frame is at the center of the earth. The ECEF frame also has its origin at the center of the earth, and the *z*-axis aligned with the earth's spin axis. The *x*-axis points to the location on the equator with zero longitude. Since the ECEF frame is fixed to the earth, the *x*- and *y*-axes rotate with the earth. The Topocentric frame depends on the location on the surface of the earth, given by longitude λ and latitude δ. The origin is at the surface of the earth, at the location of it's definition. Its *x*-axis points south along the local horizon, and it's *y*-axis points toward the east along the local horizon. This frame is important, because it is within this frame that observations of a satellite are made from the Earth (with a telescope or radar for example).

The ECEF frame is obtained by a rotation $\theta_{GMT} = \omega_{earth}(t - t_0)$ (called the Greenwich Mean Time) about the ECI *z*-axis. Note that ω_{earth} is the earth's rate of rotation. See Figure 1.17.

The Topocentric frame is obtained by a rotation λ about the ECEF *z*-axis, followed by a rotation $90^o - \delta$ about the transformed *y*-axis. See Figure 1.18.

(a) Determine the rotation matrix defining the transformation from ECEF to ECI coordinates, and from Topocentric to ECI coordinates, that is, determine $\mathbf{C}_{GF}$ and $\mathbf{C}_{GT}$. Hint: You may use the fact that $\mathbf{C}_z(a)\mathbf{C}_z(b) = \mathbf{C}_z(a+b)$.

(b) A ground satellite monitoring station with coordinates λ, δ measures the position of a satellite in local topocentric coordinates as $\vec{\boldsymbol{\rho}} = \vec{\mathcal{F}}_T^T \boldsymbol{\rho}$ (see Figure 1.19). Assuming that the Earth is a sphere with radius R_{earth}, show that the inertial position $\vec{\mathbf{r}} = \vec{\mathcal{F}}_G^T \mathbf{r}$ in ECI coordinates is given by:

$$\mathbf{r} = \begin{bmatrix} x\sin\delta\cos(\lambda+\theta_{GMT}) - y\sin(\lambda+\theta_{GMT}) + (R_{earth}+z)\cos\delta\cos(\lambda+\theta_{GMT}) \\ x\sin\delta\sin(\lambda+\theta_{GMT}) + y\cos(\lambda+\theta_{GMT}) + (R_{earth}+z)\cos\delta\sin(\lambda+\theta_{GMT}) \\ (R_{earth}+z)\sin\delta - x\cos\delta \end{bmatrix}$$

where $\boldsymbol{\rho} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

Hint: You may use the fact that R_{earth}, $\lambda + \theta_{GMT}$ and δ form spherical coordinates for the ground station in ECI coordinates.

(c) The ground station measures the velocity of the satellite relative to the topocentric coordinates as $\vec{\mathbf{v}}_T = \vec{\mathcal{F}}_T^T \dot{\boldsymbol{\rho}}$. Denoting the station position relative to the center of the earth by $\vec{\mathbf{R}}_s$, show that the satellite's inertial velocity $\vec{\mathbf{v}} = \vec{\mathcal{F}}_G^T \dot{\mathbf{r}}$ satisfies

$$\vec{\mathbf{v}} = \vec{\mathbf{v}}_T + \vec{\boldsymbol{\omega}}_{FG} \times \vec{\mathbf{R}}_s + \vec{\boldsymbol{\omega}}_{FG} \times \vec{\boldsymbol{\rho}},$$

where $\vec{\boldsymbol{\omega}}_{FG}$ is the Earth's inertial angular velocity vector, given by

$$\vec{\boldsymbol{\omega}}_{FG} = \vec{\mathcal{F}}_G^T \begin{bmatrix} 0 \\ 0 \\ \omega_{earth} \end{bmatrix}.$$

7. A spacecraft is orbiting the Earth in a circular equatorial orbit. The spacecraft orbit is shown in Figure 1.19, looking down on the orbit from above the north pole (looking down $\vec{\mathbf{z}}_G$).

The spacecraft position relative to the center of the Earth is given by the vector $\vec{\mathbf{r}}$, which makes an angle θ with $\vec{\mathbf{x}}_G$. Note that for a circular orbit, $\dot{\theta} = n = constant > 0$, and $r = |\vec{\mathbf{r}}| = constant$.

(a) What are the coordinates of the spacecraft position $\vec{\mathbf{r}}$ in the ECI frame?

(b) The inertial velocity of the spacecraft, denoted $\vec{\mathbf{v}}$, is the velocity as seen in the ECI frame, that is $\vec{\mathbf{v}} = \dot{\vec{\mathbf{r}}} = \vec{\mathcal{F}}_G^T \dot{\mathbf{r}}_G$. Compute the inertial velocity $\vec{\mathbf{v}}$ in ECI coordinates, and find the angle between $\vec{\mathbf{r}}$ and $\vec{\mathbf{v}}$.

(c) The spacecraft orbital angular momentum vector is given by

$$\vec{\mathbf{h}} = \vec{\mathbf{r}} \times \vec{\mathbf{v}}.$$

Compute $\vec{\mathbf{h}}$ in ECI coordinates. Verify that $\vec{\mathbf{h}}$ is perpendicular to $\vec{\mathbf{r}}$ and $\vec{\mathbf{v}}$.

(d) The magnitude of the orbital angular momentum is $h = |\vec{\mathbf{h}}|$. Compute h in terms of r and n.

(e) The orbital energy is given by

$$\mathcal{E} = \frac{\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}}{2} - \frac{\mu}{r},$$

which is the sum of the kinetic and gravitational potential energy. Note that μ is the Earth's gravitational constant. Given that for a circular orbit $n = \sqrt{\mu/r^3}$, determine the orbital energy.

8. You take your little nephew/niece to the fair. He/she wants to ride on the merry-go-round. You decide to watch from the sideline.

The position of your nephew/niece relative to you may be described by the vector $\vec{\mathbf{r}}_n$. The position of your nephew/niece relative to the center of the merry-go-round is given by the vector $\vec{\mathbf{R}}_n$. The position of the center of the merry-go-round relative to you is given bye $\vec{\mathbf{r}}_m$. The merry-go-round rotates with angular velocity $\omega = \dot{\theta}$.

Attach a reference frame to yourself, labeled $\mathcal{F}_y$, with the $\vec{\mathbf{x}}_y$ and $\vec{\mathbf{y}}_y$ axes parallel to the ground as shown in Figure 1.20 (the $\vec{\mathbf{z}}_y$ axis points vertically upwards). Attach a second reference frame to the merry-go-round, labeled $\mathcal{F}_m$, with the $\vec{\mathbf{x}}_m$ axis pointing from the center of the merry-go-round to your nephew/niece, and the $\vec{\mathbf{y}}_m$ axis perpendicular to $\vec{\mathbf{x}}_m$, in the plane of the merry-go-round, as shown in Figure 1.20. The $\vec{\mathbf{z}}_m$ axis points vertically upwards.

The distance from the center of the merry-go-round to your nephew/niece is $R_n = |\vec{\mathbf{R}}_n|$. The distance from you to the center of the merry-go-round is $r_m = |\vec{\mathbf{r}}_m|$.

(a) Based on Figure 1.20, what are the coordinates in $\vec{\mathbf{F}}_m$ of your nephew/niece relative to the center of the merry-go-round, that is, what is $\mathbf{R}_{n,m}$ such that $\vec{\mathbf{R}}_n = \vec{\mathcal{F}}_m^T \mathbf{R}_{n,m}$?

(b) What is the rotational transformation from your coordinate system $\mathcal{F}_y$ to the merry-go-round coordinate system $\mathcal{F}_m$? (Find $\mathbf{C}_{my}$).

(c) Using your answers to parts (a) and (b), determine the coordinates in $\mathcal{F}_y$ of your nephew/niece relative to you, that is, find $\mathbf{r}_{n,y}$ such that $\vec{\mathbf{r}}_n = \vec{\mathcal{F}}_y^T \mathbf{r}_{n,y}$.

You may take $\vec{\mathbf{r}}_m = \vec{\mathcal{F}}_y^T \begin{bmatrix} x_{m,y} \\ y_{m,y} \\ z_{m,y} \end{bmatrix}$.

(d) Using your answer to part (c), determine the velocity of your nephew/niece as seen by you.

(e) Using your answer to part (a), determine the velocity of your nephew/niece as seen by another person on the merry-go-round.

9. This problem puts some numbers to question 7.

A spacecraft is in a circular equatorial orbit about the earth with altitude 600 km. You will need the following information:
Earth's gravitational parameter: $\mu = 3.986 \times 10^5$ km$^3/s^2$.
Earth's radius: $R_e = 6378$ km.
(Note that altitude means height above the earth's surface).

(a) Referring to part 7(a), what is the spacecraft position in ECI coordinates when $\theta = 30^o$?

(b) Using 7(b), what is the spacecraft velocity as seen in the ECI frame? Note that $\dot{\theta} = n = \sqrt{\mu/r^3}$. Also, compute the orbital speed, $v = |\vec{\mathbf{v}}|$.

(c) Referring to 7(c), compute the spacecraft orbital angular momentum vector in ECI coordinates.

(d) Referring to 7(e), compute the spacecraft orbital energy.

10. The orientation of a reference frame $\mathcal{F}_2$ is obtained from the reference frame $\mathcal{F}_1$ by a 2-3-1 Euler rotation sequence, with angles θ_y, θ_z and θ_x. Specifically, frame $\mathcal{F}_2$ is obtained from frame $\mathcal{F}_1$ by:
- A rotation θ_y about the *y-axis* of frame $\mathcal{F}_1$,
- A rotation θ_z about the *z-axis* of the intermediate frame $\mathcal{F}_i$,
- A rotation θ_x about the *x-axis* of the transformed frame $\mathcal{F}_t$.

(a) Obtain the rotation matrix $\mathbf{C}_{21}$.

(b) Using the result in (a), obtain expressions for computing θ_x, θ_y and θ_z from the rotation matrix.

(c) Where is the singularity of the 2-3-1 rotation sequence? What does this physically mean?

(d) Given that frame $\mathcal{F}_2$ rotates with angular velocity $\vec{\boldsymbol{\omega}}_{21} = \vec{\mathcal{F}}_2^T \boldsymbol{\omega}_{21}$ relative to frame $\mathcal{F}_1$, obtain the kinematical relationship between $\boldsymbol{\omega}_{21}$ and $\begin{bmatrix} \dot{\theta}_x \\ \dot{\theta}_y \\ \dot{\theta}_z \end{bmatrix}$.

(e) Using the result in (d), what happens to the kinematical relationship if the angles θ_x, θ_y and θ_z and rates $\dot{\theta}_x$, $\dot{\theta}_y$ and $\dot{\theta}_z$ are very small? Hints: for a small angle θ, you can set $\sin\theta \approx \theta$ and $\cos\theta \approx 1$. For very small quantities a and b, you can neglect products ab.

11. The orientation of a reference frame $\mathcal{F}_2$ is obtained from the reference frame $\mathcal{F}_1$ by a 3-2-3 Euler rotation sequence, with angles θ_1, θ_2 and θ_3. Specifically, frame $\mathcal{F}_2$ is obtained from frame $\mathcal{F}_1$ by:
- A rotation θ_1 about the *z-axis* of frame $\mathcal{F}_1$,
- A rotation θ_2 about the *y-axis* of the intermediate frame $\mathcal{F}_i$,
- A rotation θ_3 about the *z-axis* of the transformed frame $\mathcal{F}_t$.

    (a) Obtain the rotation matrix $\mathbf{C}_{21}$.

    (b) Given that frame $\mathcal{F}_2$ rotates with angular velocity $\vec{\omega}_{21} = \vec{\mathcal{F}}_2^T \boldsymbol{\omega}_{21}$ relative to frame $\mathcal{F}_1$, obtain the kinematical relationship between $\boldsymbol{\omega}_{21}$ and $\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}$.

    (c) Invert the result in part (b) to obtain $\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}$ as a function of $\boldsymbol{\omega}_{21}$. Hint: For a general block lower triangular matrix,
$$\begin{bmatrix} A & 0 \\ B & C \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & 0 \\ -C^{-1}BA^{-1} & C^{-1} \end{bmatrix}.$$

    (d) Where is the singularity of the 3-2-3 rotation sequence? What does this physically mean?

12. Hooke's joint is shown in Figure 1.21. It can be used in machinery to transmit rotational power when the axis of rotation needs to change slightly. As shown in Figure 1.21, Hooke's joint consists of three components. An input shaft, an output shaft and a cross in the middle. The cross is connected to each shaft such one axis of the cross rotates with the input shaft and the other axis of the cross rotates with the output shaft.

    It can be seen from Figure 1.21 that the output shaft has an angle α with respect to the input shaft, with $|\alpha| < 90^o$. We define two fixed (non-rotating) reference frames $\mathcal{F}_i$ and $\mathcal{F}_o$, such that

    - the x-axis of $\mathcal{F}_i$ is parallel to the axis of rotation of the input shaft,
    - the x-axis of $\mathcal{F}_o$ is parallel to the axis of rotation of the output shaft and
    - the y-axes of $\mathcal{F}_i$ and $\mathcal{F}_o$ coincide.

    The rotational angle of the input shaft is labeled θ_i and the rotational angle of the output shaft is labeled θ_o.

    We also attach a reference frame $\mathcal{F}_c$ to the cross, as shown in Figures 1.21 and 1.22.

    (a) Write down the rotation matrix $\mathbf{C}_{oi}$ corresponding to the transformation from $\mathcal{F}_i$ to $\mathcal{F}_o$.

    (b) We further attach reference frames $\mathcal{F}_2$ and $\mathcal{F}_3$ to the input and output shafts respectively, as shown in Figures 1.23 and 1.24. Write down the rotation matrices $\mathbf{C}_{2i}$ and $\mathbf{C}_{3o}$. You may assume that $\vec{\mathcal{F}}_i = \vec{\mathcal{F}}_2$ when $\theta_i = 0$ and that $\vec{\mathcal{F}}_o = \vec{\mathcal{F}}_3$ when $\theta_o = 0$.

(c) What is the rotational transformation from $\mathcal{F}_3$ to $\mathcal{F}_c$? Write down the corresponding rotation matrix $\mathbf{C}_{c3}$.

(d) By considering the rotational transformations $\mathcal{F}_c \rightarrow \mathcal{F}_3 \rightarrow \mathcal{F}_o$ and $\mathcal{F}_c \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_i \rightarrow \mathcal{F}_o$, obtain two expressions for the z-axis of $\mathcal{F}_c$ in $\mathcal{F}_o$ coordinates. Hint: You may use the fact that $\vec{\mathbf{z}}_c = \vec{\mathbf{z}}_2$.

(e) Using the expressions obtained in part (d), show that the input and output rotational angles are related by

$$\sin\theta_o = \frac{\sin\theta_i}{\left(1 - \sin^2\alpha\cos^2\theta_i\right)^{\frac{1}{2}}},$$

$$\cos\theta_o = \frac{\cos\alpha\cos\theta_i}{\left(1 - \sin^2\alpha\cos^2\theta_i\right)^{\frac{1}{2}}}.$$

13. Show that for any unit column matrix $\mathbf{a} \in R^3$ with $\mathbf{a}^T\mathbf{a} = 1$,

$$\mathbf{a}^\times\mathbf{a}^\times\mathbf{a}^\times = -\mathbf{a}^\times.$$

14. Making use of the scalar-triple product and vector-triple product identities, show that

$$\left(\vec{\mathbf{a}} \times \vec{\mathbf{b}}\right) \cdot \left(\vec{\mathbf{c}} \times \vec{\mathbf{d}}\right) = \left(\vec{\mathbf{a}} \cdot \vec{\mathbf{c}}\right)\left(\vec{\mathbf{b}} \cdot \vec{\mathbf{d}}\right) - \left(\vec{\mathbf{a}} \cdot \vec{\mathbf{d}}\right)\left(\vec{\mathbf{b}} \cdot \vec{\mathbf{c}}\right).$$

15. Consider the axis-angle parameters, $\mathbf{a}$ and ϕ.

(a) Starting with the expression for the rotation matrix given in (1.26) in the book, obtain an approximate expression for the rotation matrix when the angle of rotation ϕ is very small. The quantity $\boldsymbol{\phi} = \mathbf{a}\phi$ will be useful (this is sometimes called the *rotation vector*).

(b) Show that the axis-angle parameters $(\mathbf{a}_3, \phi_3)$ equivalent to successive rotations $(\mathbf{a}_1, \phi_1)$ followed by $(\mathbf{a}_2, \phi_2)$ are given by

$$\mathbf{a}_3 = \frac{1}{\sin(\phi_3/2)}\left[\sin\frac{\phi_1}{2}\cos\frac{\phi_2}{2}\mathbf{a}_1 + \sin\frac{\phi_2}{2}\cos\frac{\phi_1}{2}\mathbf{a}_2 + \sin\frac{\phi_1}{2}\sin\frac{\phi_2}{2}\mathbf{a}_1^\times\mathbf{a}_2\right],$$

$$\cos\frac{\phi_3}{2} = \cos\frac{\phi_1}{2}\cos\frac{\phi_2}{2} - \sin\frac{\phi_1}{2}\sin\frac{\phi_2}{2}\mathbf{a}_1^T\mathbf{a}_2.$$

(c) Show that their kinematical equations are given by

$$\dot{\mathbf{a}} = \frac{1}{2}\left(\mathbf{a}^\times - \frac{1}{\tan(\phi/2)}\mathbf{a}^\times\mathbf{a}^\times\right)\boldsymbol{\omega},$$

$$\dot{\phi} = \mathbf{a}^T\boldsymbol{\omega}.$$

(d) Show that the inverse kinematics are given by

$$\boldsymbol{\omega} = \left(\sin\phi\mathbf{1} - (1 - \cos\phi)\mathbf{a}^\times\right)\dot{\mathbf{a}} + \mathbf{a}\dot{\phi}.$$

What does this reduce to when the rotation is about a fixed axis?

(e) Is there a singularity associated with these parameters?

(f) Starting with the inverse kinematics obtained in part (d), obtain an approximate expression for $\boldsymbol{\omega}$ when the angle of rotation ϕ is very small. The quantity $\boldsymbol{\phi} = \mathbf{a}\phi$ will again be useful.

16. Consider the rotation vector $\boldsymbol{\phi} = \mathbf{a}\phi$, introduced in Question 15.

(a) Show that the rotation matrix associated with $\boldsymbol{\phi}$ is given by

$$\mathbf{C} = \mathbf{1} + \frac{(1-\cos\phi)}{\phi^2}\boldsymbol{\phi}^\times\boldsymbol{\phi}^\times - \frac{\sin\phi}{\phi}\boldsymbol{\phi}^\times,$$

where $\phi = \|\boldsymbol{\phi}\|$.

(b) Show that the kinematical equations for the rotation vector are

$$\dot{\boldsymbol{\phi}} = \left(\mathbf{1} + \frac{\boldsymbol{\phi}^\times}{2} + \frac{1}{\phi^2}\left[1 - \frac{\phi/2}{\tan(\phi/2)}\right]\boldsymbol{\phi}^\times\boldsymbol{\phi}^\times\right)\boldsymbol{\omega}.$$

(c) Show that the inverse kinematics are given by

$$\boldsymbol{\omega} = \left(\mathbf{1} - \frac{(1-\cos\phi)}{\phi^2}\boldsymbol{\phi}^\times + \frac{(\phi-\sin\phi)}{\phi^3}\boldsymbol{\phi}^\times\boldsymbol{\phi}^\times\right)\dot{\boldsymbol{\phi}}.$$

(d) Where is the singularity associated with these parameters? Careful, it is not at $\boldsymbol{\phi} = \mathbf{0}$.

17. Consider the following parameterization of the rotation matrix

$$\mathbf{p} = \mathbf{a}\tan\frac{\phi}{2}.$$

These are called the Euler-Rodrigues parameters.

(a) Show that the rotation matrix associated with $\mathbf{p}$ is given by

$$\mathbf{C} = \frac{(1-\mathbf{p}^T\mathbf{p})\mathbf{1} + 2\mathbf{p}\mathbf{p}^T - 2\mathbf{p}^\times}{1+\mathbf{p}^T\mathbf{p}}.$$

(b) Starting with the expression for the rotation matrix obtained in part (a), obtain an approximate expression for the rotation matrix when $\mathbf{p}$ is very small.

(c) Show that the Euler-Rodriguez parameters may be obtained from a rotation matrix by

$$\mathbf{p} = \frac{1}{1 + C_{11} + C_{22} + C_{33}}\begin{bmatrix} C_{23} - C_{32} \\ C_{31} - C_{13} \\ C_{12} - C_{21}\end{bmatrix}.$$

(d) Show that the rotation $\mathbf{p}_3$ equivalent to successive rotations $\mathbf{p}_1$ followed by $\mathbf{p}_2$ is given by

$$\mathbf{p}_3 = \frac{\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_1^\times\mathbf{p}_2}{1 - \mathbf{p}_1^T\mathbf{p}_2}.$$

(e) Show that the kinematical equations for the Euler-Rodrigues parameters are

$$\dot{\mathbf{p}} = \frac{1}{2}(\mathbf{1} + \mathbf{p}\mathbf{p}^T + \mathbf{p}^\times)\boldsymbol{\omega}.$$

(f) Show that the inverse kinematics are given by

$$\boldsymbol{\omega} = \frac{2(\mathbf{1} - \mathbf{p}^\times)\dot{\mathbf{p}}}{1 + \mathbf{p}^T\mathbf{p}}.$$

(g) Starting with inverse kinematics obtained in part (f), obtain an approximate expression for $\boldsymbol{\omega}$ when $\mathbf{p}$ and $\dot{\mathbf{p}}$ are very small.

(h) Where is the singularity associated with these parameters?

18. Consider the following parameterization of the rotation matrix

$$\boldsymbol{\sigma} = \mathbf{a}\tan\frac{\phi}{4}.$$

These are called the modified Euler-Rodrigues parameters.

(a) Show that the rotation matrix associated with $\boldsymbol{\sigma}$ is given by

$$\mathbf{C} = \mathbf{1} + \frac{8\boldsymbol{\sigma}^\times\boldsymbol{\sigma}^\times - 4(1 - \boldsymbol{\sigma}^T\boldsymbol{\sigma})\boldsymbol{\sigma}^\times}{(1 + \boldsymbol{\sigma}^T\boldsymbol{\sigma})^2}.$$

(b) Starting with the expression for the rotation matrix obtained in part (a), obtain an approximate expression for the rotation matrix when $\boldsymbol{\sigma}$ is very small.

(c) Setting $s = C_{11} + C_{22} + C_{33}$, show that the modified Euler-Rodrigues parameters may be obtained from a rotation matrix by

$$\boldsymbol{\sigma} = \frac{1}{1 + s + 2\sqrt{1+s}}\begin{bmatrix} C_{23} - C_{32} \\ C_{31} - C_{13} \\ C_{12} - C_{21} \end{bmatrix},$$

when $\|\boldsymbol{\sigma}\| < 1$,

$$\boldsymbol{\sigma} = \frac{1}{1 + s - 2\sqrt{1+s}}\begin{bmatrix} C_{23} - C_{32} \\ C_{31} - C_{13} \\ C_{12} - C_{21} \end{bmatrix},$$

when $\|\boldsymbol{\sigma}\| > 1$, and $\boldsymbol{\sigma} = \mathbf{a}$ (the principal axis of rotation) when $\|\boldsymbol{\sigma}\| = 1$.

(d) Show that the rotation $\boldsymbol{\sigma}_3$ equivalent to successive rotations $\boldsymbol{\sigma}_1$ followed by $\boldsymbol{\sigma}_2$ is given by

$$\boldsymbol{\sigma}_3 = \frac{(1 - \boldsymbol{\sigma}_2^T\boldsymbol{\sigma}_2)\boldsymbol{\sigma}_1 + (1 - \boldsymbol{\sigma}_1^T\boldsymbol{\sigma}_1)\boldsymbol{\sigma}_2 + 2\boldsymbol{\sigma}_1^\times\boldsymbol{\sigma}_2}{1 + (\boldsymbol{\sigma}_1^T\boldsymbol{\sigma}_1)^2(\boldsymbol{\sigma}_2^T\boldsymbol{\sigma}_2)^2 - 2\boldsymbol{\sigma}_1^T\boldsymbol{\sigma}_2}.$$

(e) Show that the kinematical equations for the modified Euler-Rodrigues parameters are

$$\dot{\boldsymbol{\sigma}} = \frac{1}{4}\left((1 - \boldsymbol{\sigma}^T\boldsymbol{\sigma})\mathbf{1} + 2\boldsymbol{\sigma}\boldsymbol{\sigma}^T + 2\boldsymbol{\sigma}^\times\right)\boldsymbol{\omega}.$$

(f) Show that the inverse kinematics are given by

$$\boldsymbol{\omega} = \frac{4}{(1+\boldsymbol{\sigma}^T\boldsymbol{\sigma})^2}\left((1-\boldsymbol{\sigma}^T\boldsymbol{\sigma})\mathbf{1} + 2\boldsymbol{\sigma}\boldsymbol{\sigma}^T - 2\boldsymbol{\sigma}^\times\right)\dot{\boldsymbol{\sigma}}.$$

(g) Starting with inverse kinematics obtained in part (f), obtain an approximate expression for $\boldsymbol{\omega}$ when $\boldsymbol{\sigma}$ and $\dot{\boldsymbol{\sigma}}$ are very small.

(h) Where is the singularity associated with these parameters? What are the advantages of the modified Euler-Rodrigues parameters compared to the Euler-Rodrigues parameters (introduced in Question 17)?

19. In this question, we consider a reference frame $\mathcal{F}_1$ defined by three non-coplanar unit vectors, $\vec{\mathbf{1}}_1$, $\vec{\mathbf{1}}_2$ and $\vec{\mathbf{1}}_3$, which are not necessarily orthogonal. Note that we reserve the notation $\vec{\mathbf{x}}$, $\vec{\mathbf{y}}$ and $\vec{\mathbf{z}}$ for the basis vectors of orthogonal right-handed frames of reference, which is why we do not use them here. Just as in Section 1.2 of the book, since the basis vectors, $\vec{\mathbf{1}}_1$, $\vec{\mathbf{1}}_2$ and $\vec{\mathbf{1}}_3$ are non-coplanar, we may represent every physical vector $\vec{\mathbf{r}}$ as

$$\vec{\mathbf{r}} = \vec{\mathcal{F}}_1^T\mathbf{r}_1,$$

where

$$\vec{\mathcal{F}}_1 = \begin{bmatrix} \vec{\mathbf{1}}_1 \\ \vec{\mathbf{1}}_2 \\ \vec{\mathbf{1}}_3 \end{bmatrix},$$

is the vectrix containing the basis vectors defining frame $\mathcal{F}_1$, and

$$\mathbf{r}_1 = \begin{bmatrix} r_{1,1} \\ r_{2,1} \\ r_{3,1} \end{bmatrix},$$

is the column matrix containing the coordinates of $\vec{\mathbf{r}}$ in frame $\mathcal{F}_1$. Note that for two physical vectors $\vec{\mathbf{a}} = \vec{\mathcal{F}}_1^T\mathbf{a}_1$ and $\vec{\mathbf{b}} = \vec{\mathcal{F}}_1^T\mathbf{b}_1$, it is easy to verify that

$$\vec{\mathbf{a}} + \vec{\mathbf{b}} = \vec{\mathcal{F}}_1^T(\mathbf{a}_1 + \mathbf{b}_1),$$

and that for any scalar c,

$$c\vec{\mathbf{a}} = \vec{\mathcal{F}}_1^T(c\mathbf{a}_1).$$

(a) Show that for any two physical vectors $\vec{\mathbf{a}} = \vec{\mathcal{F}}_1^T\mathbf{a}_1$ and $\vec{\mathbf{b}} = \vec{\mathcal{F}}_1^T\mathbf{b}_1$, the scalar product obeys

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \mathbf{a}_1^T\mathbf{W}_1\mathbf{b}_1,$$

where

$$\mathbf{W}_1 = \begin{bmatrix} \vec{\mathbf{1}}_1\cdot\vec{\mathbf{1}}_1 & \vec{\mathbf{1}}_1\cdot\vec{\mathbf{1}}_2 & \vec{\mathbf{1}}_1\cdot\vec{\mathbf{1}}_3 \\ \vec{\mathbf{1}}_2\cdot\vec{\mathbf{1}}_1 & \vec{\mathbf{1}}_2\cdot\vec{\mathbf{1}}_2 & \vec{\mathbf{1}}_2\cdot\vec{\mathbf{1}}_3 \\ \vec{\mathbf{1}}_3\cdot\vec{\mathbf{1}}_1 & \vec{\mathbf{1}}_3\cdot\vec{\mathbf{1}}_2 & \vec{\mathbf{1}}_3\cdot\vec{\mathbf{1}}_3 \end{bmatrix}.$$

Furthermore, verify that the matrix $\mathbf{W}_1$ is symmetric and positive definite.

(b) Show that for a physical vector $\vec{\mathbf{r}} = \vec{\mathcal{F}}_1^T \mathbf{r}_1$, the coordinates $\mathbf{r}_1$ may be found according to

$$\mathbf{r}_1 = \mathbf{W}_1^{-1} \mathbf{r}_1^{proj},$$

where $\mathbf{W}_1$ is defined in part (a), and

$$\mathbf{r}_1^{proj} = \vec{\mathcal{F}}_1 \cdot \vec{\mathbf{r}} = \begin{bmatrix} \vec{\mathbf{1}}_1 \cdot \vec{\mathbf{r}} \\ \vec{\mathbf{1}}_2 \cdot \vec{\mathbf{r}} \\ \vec{\mathbf{1}}_3 \cdot \vec{\mathbf{r}} \end{bmatrix},$$

which is recognized to be the column matrix containing the orthogonal projections of $\vec{\mathbf{r}}$ onto the basis vectors defining frame $\mathcal{F}_1$.

(c) Show that the cross-product between two physical vectors obeys

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \vec{\mathcal{F}}_1^T (d_1 \mathbf{W}_1^{-1} \mathbf{a}_1^{\times} \mathbf{b}_1),$$

where $\mathbf{W}_1$ is defined in part (a), and

$$d_1 = \vec{\mathbf{1}}_1 \cdot (\vec{\mathbf{1}}_2 \times \vec{\mathbf{1}}_3) = \vec{\mathbf{1}}_2 \cdot (\vec{\mathbf{1}}_3 \times \vec{\mathbf{1}}_1) = \vec{\mathbf{1}}_3 \cdot (\vec{\mathbf{1}}_1 \times \vec{\mathbf{1}}_2).$$

20. Continuing from Question 19, consider now a second reference frame $\mathcal{F}_2$, with unit-length basis vectors $\vec{\mathbf{2}}_1$, $\vec{\mathbf{2}}_2$ and $\vec{\mathbf{2}}_3$, which are also not necessarily orthogonal.

(a) Consider an arbitrary physical vector $\vec{\mathbf{a}}$, with representations in frames $\mathcal{F}_1$ and $\mathcal{F}_2$, given by

$$\vec{\mathbf{a}} = \vec{\mathcal{F}}_1^T \mathbf{a}_1 = \vec{\mathcal{F}}_2^T \mathbf{a}_2.$$

Show that the two sets of coordinates satisfy

$$\mathbf{a}_2 = \mathbf{T}_{21} \mathbf{a}_1,$$

where

$$\mathbf{T}_{21} = \mathbf{W}_2^{-1} \bar{\mathbf{T}}_{21},$$

with $\mathbf{W}_2$ being defined as in Question 19 (a), and

$$\bar{\mathbf{T}}_{21} = \begin{bmatrix} \vec{\mathbf{2}}_1 \cdot \vec{\mathbf{1}}_1 & \vec{\mathbf{2}}_1 \cdot \vec{\mathbf{1}}_2 & \vec{\mathbf{2}}_1 \cdot \vec{\mathbf{1}}_3 \\ \vec{\mathbf{2}}_2 \cdot \vec{\mathbf{1}}_1 & \vec{\mathbf{2}}_2 \cdot \vec{\mathbf{1}}_2 & \vec{\mathbf{2}}_2 \cdot \vec{\mathbf{1}}_3 \\ \vec{\mathbf{2}}_3 \cdot \vec{\mathbf{1}}_1 & \vec{\mathbf{2}}_3 \cdot \vec{\mathbf{1}}_2 & \vec{\mathbf{2}}_3 \cdot \vec{\mathbf{1}}_3 \end{bmatrix}.$$

Note that we label the transformation matrix by $\mathbf{T}$, since it is not in general a rotation matrix, which we denote by $\mathbf{C}$.

(b) Show that

$$\mathbf{T}_{21} = \begin{bmatrix} \mathbf{1}_{1,2} & \mathbf{1}_{2,2} & \mathbf{1}_{3,2} \end{bmatrix},$$

where

$$\vec{\mathbf{1}}_1 = \vec{\mathcal{F}}_2^T \mathbf{1}_{1,2}, \quad \vec{\mathbf{1}}_2 = \vec{\mathcal{F}}_2^T \mathbf{1}_{2,2}, \quad \vec{\mathbf{1}}_3 = \vec{\mathcal{F}}_2^T \mathbf{1}_{3,2},$$

are the coordinate representations in $\mathcal{F}_2$ of the basis vectors defining $\mathcal{F}_1$. How does this compare to the expression for the rotation matrix given in equation (1.18) in the book.

(c) Noting that $\bar{\mathbf{T}}_{21} = \bar{\mathbf{T}}_{12}^T$, show that

$$\mathbf{T}_{21} = \mathbf{W}_2^{-1}\mathbf{T}_{12}^T\mathbf{W}_1.$$

(d) Let $\mathbf{a}_2 = \mathbf{T}_{21}\mathbf{a}_1$. Show that

$$\mathbf{a}_2^\times = \left(\frac{d_1}{d_2}\right)\mathbf{T}_{12}^T\mathbf{a}_1^\times\mathbf{T}_{12}.$$

where d_1 and d_2 are defined as in Question 19 (c). How does this compare to equation (1.21) in the book?

21. Define the derivative of a physical vector $\vec{\mathbf{r}} = \vec{\mathcal{F}}_1^T\mathbf{r}_1$ as seen in a not-necessarily orthogonal frame $\mathcal{F}_1$ by

$$\overset{\cdot 1}{\vec{\mathbf{r}}} \triangleq \vec{\mathcal{F}}_1^T\dot{\mathbf{r}}_1.$$

Show that the rules for differentiation obtained in Section 1.4 of the book hold in this case also. Note that we assume that the basis vectors of $\mathcal{F}_1$ are fixed relative to each other.

22. Consider the frames $\mathcal{F}_1$ and $\mathcal{F}_2$ from question 19. Define right-handed orthogonal frames $\mathcal{F}_{1o}$ and $\mathcal{F}_{2o}$ such that $\mathcal{F}_1$ is fixed in $\mathcal{F}_{1o}$, and $\mathcal{F}_2$ is fixed in $\mathcal{F}_{2o}$. Define the angular velocity $\vec{\boldsymbol{\omega}}_{21}$ of frame $\mathcal{F}_2$ relative to frame $\mathcal{F}_1$ by

$$\vec{\boldsymbol{\omega}}_{21} = \vec{\boldsymbol{\omega}}_{2o,1o}.$$

Note that this is well defined, since any choice of orthogonal right-handed frames $\mathcal{F}_{1o}$ and $\mathcal{F}_{2o}$ would lead to the same angular velocity vector. Indeed, let $\mathcal{F}_{1o'}$ and $\mathcal{F}_{2o'}$ be another pair of right-handed orthogonal reference frames such that $\mathcal{F}_1$ is fixed in $\mathcal{F}_{1o'}$, and $\mathcal{F}_2$ is fixed in $\mathcal{F}_{2o'}$. Then, $\vec{\boldsymbol{\omega}}_{1o',1o} = \vec{\boldsymbol{\omega}}_{2o',2o} = \vec{\mathbf{0}}$, which leads to $\vec{\boldsymbol{\omega}}_{2o,1o} = \vec{\boldsymbol{\omega}}_{2o',1o'}$.

(a) Consider $\vec{\mathbf{r}} = \vec{\mathcal{F}}_1^T\mathbf{r}_1 = \vec{\mathcal{F}}_{1o}^T\mathbf{r}_{1o}$. Show that

$$\overset{\cdot 1}{\vec{\mathbf{r}}} = \overset{\cdot 1o}{\vec{\mathbf{r}}}.$$

(b) Making use of the Transport Theorem for right-handed orthogonal reference frames (equation (1.52) in the book), and part (a), show that the Transport Theorem also holds for non-orthogonal reference frames. That is, show

$$\overset{\cdot 1}{\vec{\mathbf{r}}} = \overset{\cdot 2}{\vec{\mathbf{r}}} + \vec{\boldsymbol{\omega}}_{21} \times \vec{\mathbf{r}}.$$

(c) Let $\vec{\boldsymbol{\omega}}_{21} = \vec{\mathcal{F}}_2^T\boldsymbol{\omega}_{21}$. Show that the rotational kinematics are given by

$$\dot{\mathbf{T}}_{21} = -d_2\mathbf{W}_2^{-1}\boldsymbol{\omega}_{21}^\times\mathbf{T}_{21},$$

where $\mathbf{T}_{21}$ was introduced in Question 20 (a), and d_2, $\mathbf{W}_2$ were defined in Question 19. How does this compare to Poisson's equation (equation (1.55) in the book)?

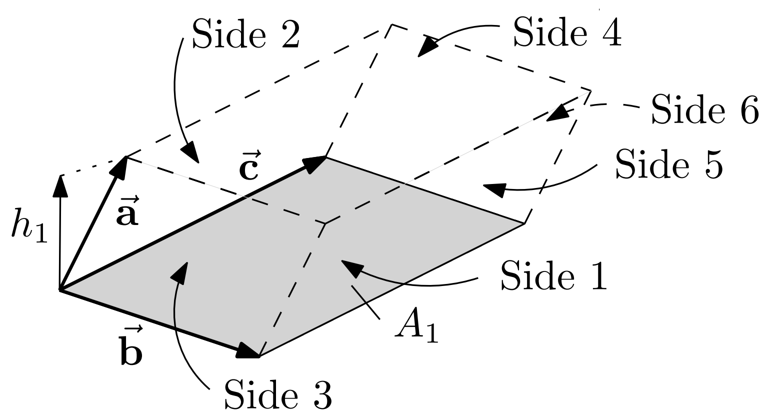

Figure 1.1 Parallelepiped defined by $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$

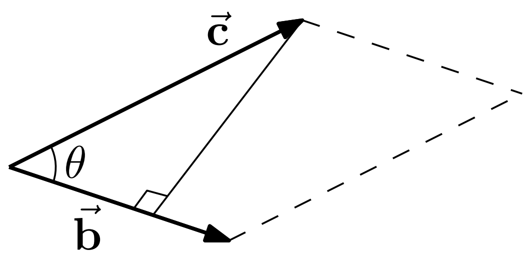

Figure 1.2 Parallelogram defined by $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$

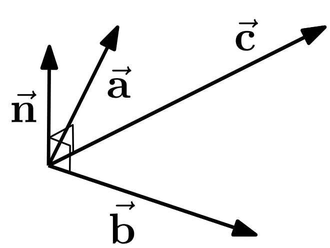

Figure 1.3 Unit normal vector to $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$

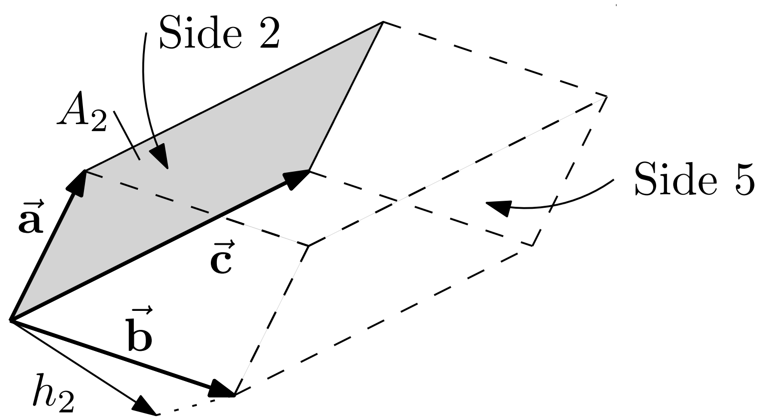

Figure 1.4 Parallelepiped defined by $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$

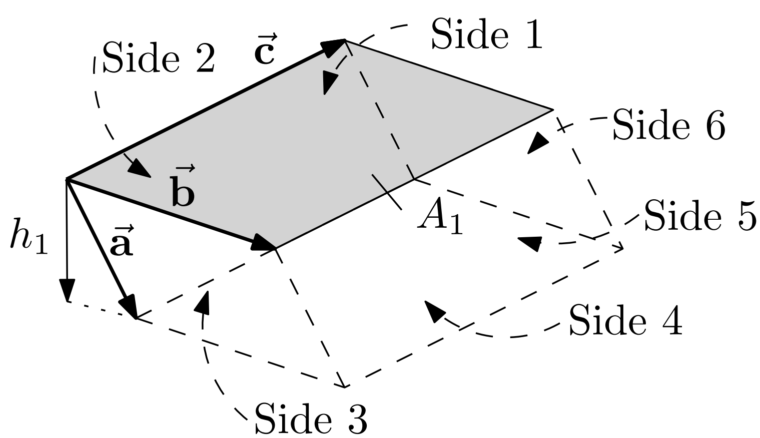

Figure 1.5 Parallelepiped defined by $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$

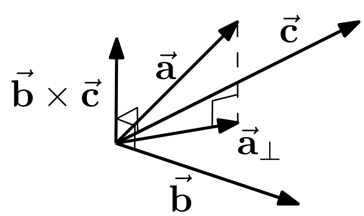

Figure 1.6 Question 3 scenario

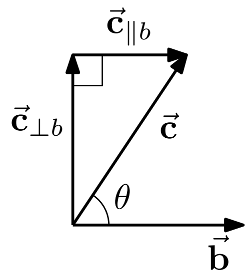

Figure 1.7 Components of $\vec{\mathbf{c}}$ relative to $\vec{\mathbf{b}}$

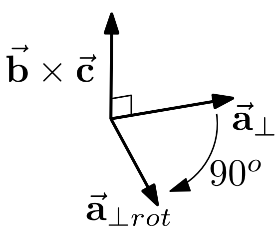

Figure 1.8 Rotation of $\vec{\mathbf{a}}_{\perp}$

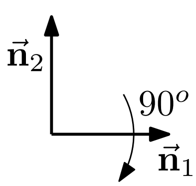

Figure 1.9 Rotation of $\vec{\mathbf{n}}_1$ and $\vec{\mathbf{n}}_2$

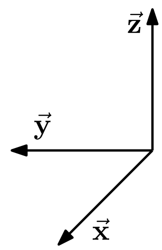

Figure 1.10 Left-handed reference frame

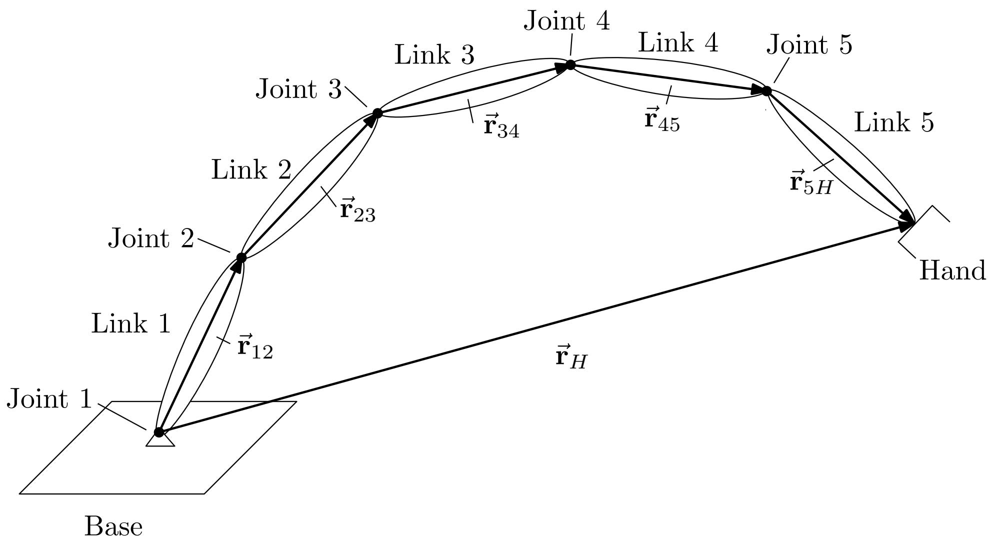

Figure 1.11 5-link robotic manipulator

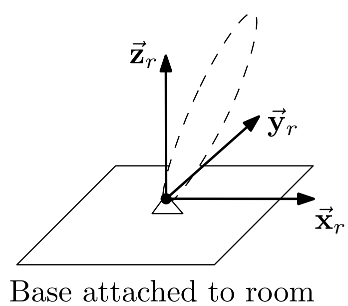

Figure 1.12 Room Frame Definition

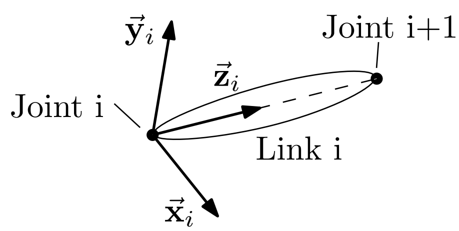

Figure 1.13 Robot link frame definition

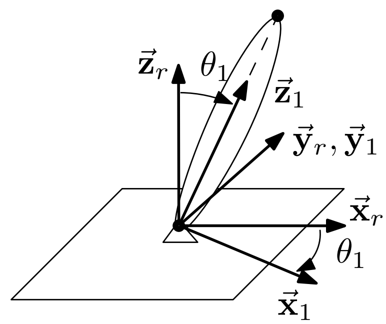

Figure 1.14 Joint 1 rotation

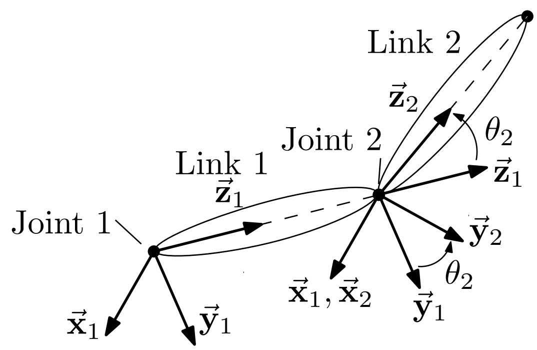

Figure 1.15 Joint 2 rotation

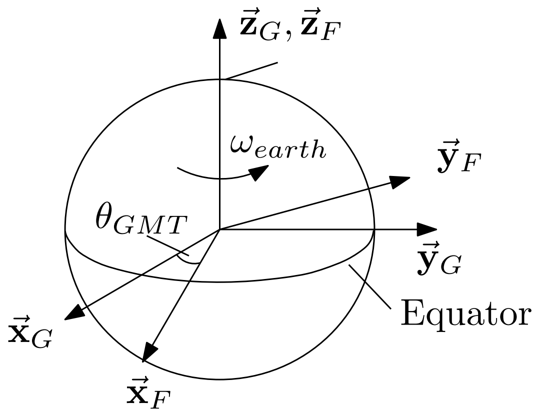

Figure 1.16 ECI and ECEF frame definitions

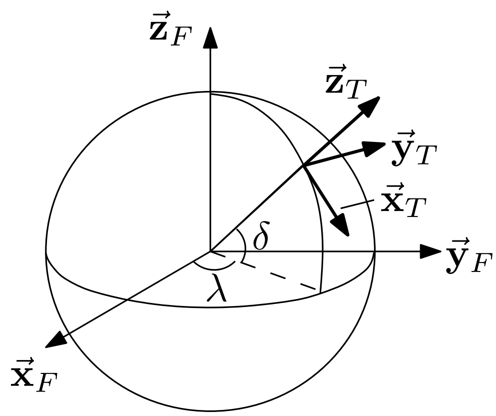

Figure 1.17 Topocentric frame definition

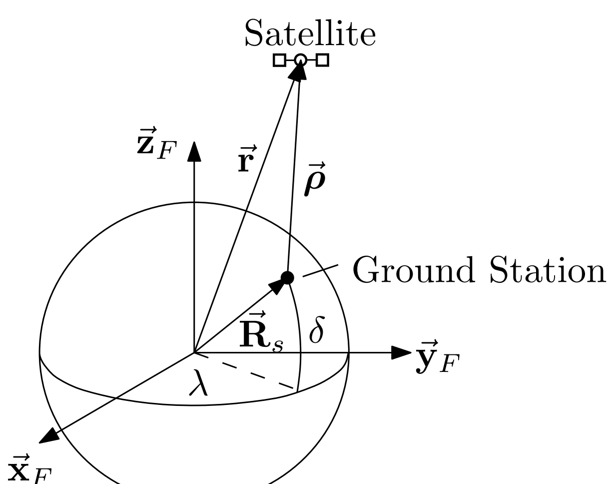

Figure 1.18 Ground station and satellite geometry

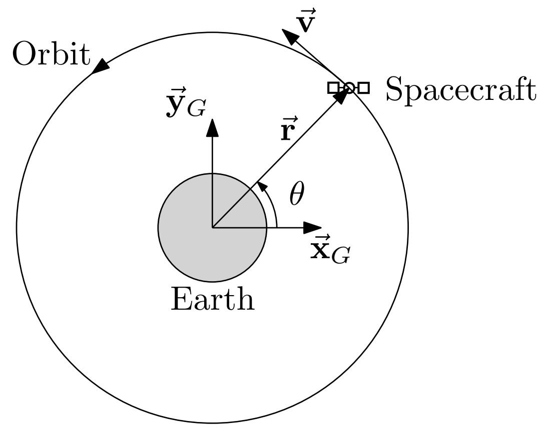

Figure 1.19 Circular Equatorial Orbit

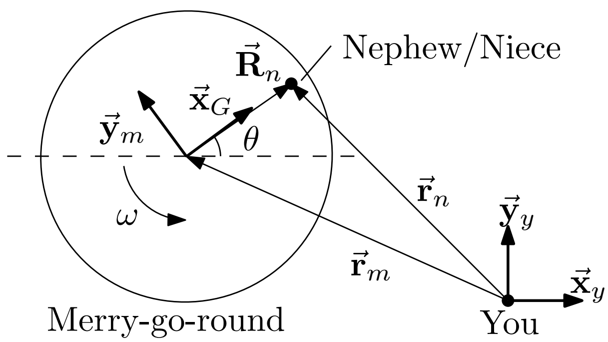

Figure 1.20 Merry-go-round scenario

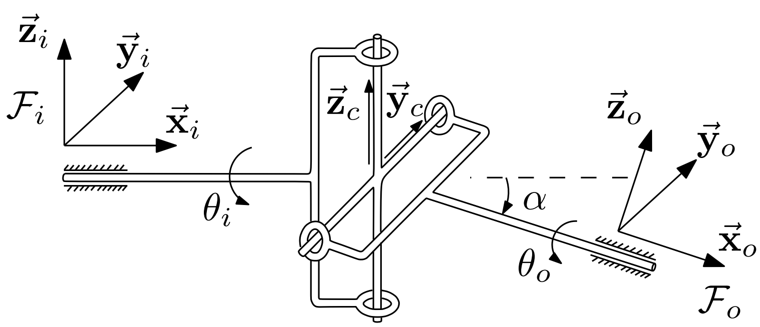

Figure 1.21 Hooke's joint

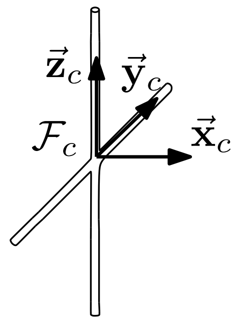

Figure 1.22 Hooke's joint cross

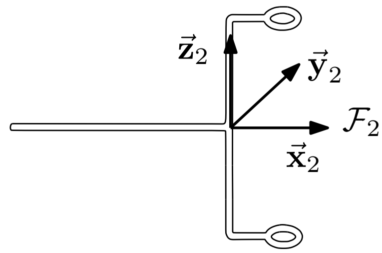

Figure 1.23 Hooke's joint input shaft

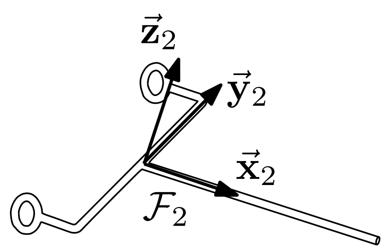

Figure 1.24 Hooke's joint output shaft

2

Chapter 2 Exercises

1. This question makes use of the Earth-Centered-Inertial (ECI) $\mathcal{F}_G$, Earth-Centered-Earth-Fixed (ECEF) $\mathcal{F}_F$ and Topocentric $\mathcal{F}_T$ reference frames. The figures defining the frames are reproduced in Figures 2.1 and 2.2. Figure 2.3 shows the location of a ground station, which is tracking a satellite. For Earth-Orbiting satellites, the ECI frame can be considered to be inertial.

   The position of the satellite relative to the center of the earth is given by the vector $\vec{\mathbf{r}}$. The position of the satellite relative to the ground station is given by the vector $\vec{\boldsymbol{\rho}}$. The position of the ground station relative to the center of the earth is given by the vector $\vec{\mathbf{R}}_s$. The Topocentric frame has it's origin at the ground station location. The angular velocity of the Earth (angular velocity of frame $\mathcal{F}_F$ relative to $\mathcal{F}_G$) is given by the vector $\vec{\boldsymbol{\omega}}_{FG}$, and is constant as seen in the ECI and ECEF frames.

   The mass of the satellite is m, and the sum of all external forces acting on the satellite is given by the vector $\vec{\mathbf{F}}$.

   (a) Show that the vector equation of motion of the satellite as seen by an observer at the ground station (as seen in $\mathcal{F}_T$) is given by

   $$m \overset{\circ\circ}{\vec{\boldsymbol{\rho}}} = -2m\vec{\boldsymbol{\omega}}_{FG} \times \overset{\circ}{\vec{\boldsymbol{\rho}}} - m\vec{\boldsymbol{\omega}}_{FG} \times (\vec{\boldsymbol{\omega}}_{FG} \times \vec{\mathbf{R}}_s) - m\vec{\boldsymbol{\omega}}_{FG} \times (\vec{\boldsymbol{\omega}}_{FG} \times \vec{\boldsymbol{\rho}}) + \vec{\mathbf{F}},$$

   where $(\circ)$ denotes time differentiation as seen in frame $\mathcal{F}_T$.

   (b) Given the coordinates of the $\vec{\boldsymbol{\rho}}$ in $\mathcal{F}$, the coordinates of $\vec{\boldsymbol{\omega}}_{FG}$, $\vec{\mathbf{R}}_s$ an $\mathcal{F}_F$, and the coordinates of $\vec{\mathbf{F}}$ in $\mathcal{F}_G$, that is, given

   $$\vec{\boldsymbol{\rho}} = \vec{\mathcal{F}}_T^T \boldsymbol{\rho}, \quad \vec{\boldsymbol{\omega}}_{FG} = \vec{\mathcal{F}}_F^T \boldsymbol{\omega}_{FG}, \quad \vec{\mathbf{R}}_s = \vec{\mathcal{F}}_F^T \mathbf{R}_s, \quad \vec{\mathbf{F}} = \vec{\mathcal{F}}_G^T \mathbf{F},$$

   show that in topocentric coordinates, the equation of motion obtained in part (a) is given by

   $$m\ddot{\boldsymbol{\rho}} = -2m\mathbf{C}_{TF}\boldsymbol{\omega}_{FG}^{\times}\mathbf{C}_{TF}^T\dot{\boldsymbol{\rho}} - m\mathbf{C}_{TF}\boldsymbol{\omega}_{FG}^{\times}\boldsymbol{\omega}_{FG}^{\times}\mathbf{R}_s - m\mathbf{C}_{TF}\boldsymbol{\omega}_{FG}^{\times}\boldsymbol{\omega}_{FG}^{\times}\mathbf{C}_{TF}^T\boldsymbol{\rho} + \mathbf{C}_{TG}\mathbf{F},$$

   where $\mathbf{C}_{TF}$ is the transformation from $\mathcal{F}_F$ to $\mathcal{F}_T$ coordinates and $\mathbf{C}_{TG}$ is the transformation from $\mathcal{F}_G$ to $\mathcal{F}_T$ coordinates.

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2. Consider the moment of inertia matrix

$$\mathbf{J} = \begin{bmatrix} J_{xx} & J_{xy} & J_{xz} \\ J_{xy} & J_{yy} & J_{yz} \\ J_{xz} & J_{yz} & J_{zz} \end{bmatrix}.$$

   Show that for any three-dimensional body

$$J_{xx} - J_{yy} + J_{zz} > 0$$

3. In this question you will derive the expression for the angular momentum of a dual-spin spacecraft.

   (a) Consider a wheel with moment of inertia matrix about the center of mass as evaluated in a principal body frame $\mathcal{F}_p$ given by

$$\mathbf{I}_p = \begin{bmatrix} I_t & 0 & 0 \\ 0 & I_t & 0 \\ 0 & 0 & I_s \end{bmatrix},$$

   where I_s is the moment of inertia about the spin axis, and I_t is the transverse moment of inertia. Clearly in this case, the spin axis is the principal z-axis, which is given by

$$\vec{\mathbf{z}}_p = \vec{\mathcal{F}}_p^T \mathbf{e}_z,$$

   where

$$\mathbf{e}_z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

   Consider any other frame $\mathcal{F}_b$. Show that the inertia matrix about the wheel center of mass as evaluated in frame $\mathcal{F}_b$ is given by

$$\mathbf{I}_w = I_t \mathbf{1} + (I_s - I_t)\mathbf{a}\mathbf{a}^T,$$

   where $\mathbf{a}$ are the coordinates of the axis of symmetry in frame $\mathcal{F}_b$. Hint: first show that $\mathbf{I}_p = I_t \mathbf{1} + (I_s - I_t)\mathbf{e}_z \mathbf{e}_z^T$.

   (b) Consider a rigid wheel as shown in figure 2.4, which has inertial angular velocity $\vec{\boldsymbol{\omega}}_w$. The vector $\vec{\mathbf{r}}_w$ locates the wheel's center of mass from the point c, and the vector $\vec{\boldsymbol{\rho}}$ locates the mass element dm from the wheel's center of mass w. By definition, the wheel's angular momentum about its center of mass is given by

$$\vec{\mathbf{h}}_w = \int_B \vec{\boldsymbol{\rho}} \times \dot{\vec{\boldsymbol{\rho}}} dm,$$

   where $\dot{\vec{\boldsymbol{\rho}}}$ denotes the inertial time-derivative of $\vec{\boldsymbol{\rho}}$. Starting with the definition for the wheel's angular momentum about point c,

$$\vec{\mathbf{h}}_{wc} = \int_B (\vec{\mathbf{r}}_w + \vec{\boldsymbol{\rho}}) \times \left(\dot{\vec{\mathbf{r}}}_w + \dot{\vec{\boldsymbol{\rho}}}\right) dm,$$

show that the wheel's angular momentum about the point c is given by

$$\vec{\mathbf{h}}_{wc} = m_w \vec{\mathbf{r}}_w \times \dot{\vec{\mathbf{r}}}_w + \vec{\mathbf{h}}_w,$$

where m_w is the total mass of the wheel.

(c) Consider the dual-spin spacecraft as shown in figure 2.5. We divide the spacecraft into two parts: the wheel (labeled W), and the rest of the spacecraft, called the platform (labeled P). The point c denotes the center of mass of the spacecraft (the combined platform and wheel). The vector $\vec{\mathbf{r}}_w = \vec{\mathcal{F}}_b^T \mathbf{r}_w$ locates the center of mass of the wheel from the center of mass of the spacecraft. Let $\mathcal{F}_b$ be a body-fixed reference frame attached to the platform. The platform has inertial angular velocity $\vec{\boldsymbol{\omega}} = \vec{\mathcal{F}}_b^T \boldsymbol{\omega}$. The wheel has angular velocity $\vec{\boldsymbol{\omega}}_s = \omega_s \vec{\mathcal{F}}_b^T \mathbf{a}$ relative to the platform, where $\vec{\mathbf{a}} = \vec{\mathcal{F}}_b^T \mathbf{a}$ is the wheel spin axis, and ω_s is the wheel spin-rate relative to the platform. The wheel moment of inertia about the spin axis is labeled I_s, and the wheel transverse moment of inertia is labeled I_t.

Show that the wheel angular momentum vector about the wheel center of mass is given by

$$\vec{\mathbf{h}}_w = \vec{\mathcal{F}}_b^T \left[\mathbf{I}_w \boldsymbol{\omega} + h_s \mathbf{a}\right],$$

where $h_s = I_s \omega_s$ is the wheel relative angular momentum, and

$$\mathbf{I}_w = I_t \mathbf{1} + (I_s - I_t)\mathbf{a}\mathbf{a}^T.$$

(d) Given that the wheel has mass m_w, use the result in part (b) to show that the wheel angular momentum about the spacecraft center of mass c is given by

$$\vec{\mathbf{h}}_{wc} = \vec{\mathcal{F}}_b^T \left[\mathbf{J}_{wc} \boldsymbol{\omega} + h_s \mathbf{a}\right],$$

where $\mathbf{J}_{wc} = \mathbf{I}_w - m_w \mathbf{r}_w^\times \mathbf{r}_w^\times$ is the wheel moment of inertia matrix about the spacecraft center of mass c evaluated in $\mathcal{F}_b$.

(e) Finally, show that the total angular momentum of the spacecraft (platform plus wheel) about the spacecraft center of mass is given by

$$\vec{\mathbf{h}}_c = \vec{\mathcal{F}}_b^T \left[\mathbf{I} \boldsymbol{\omega} + h_s \mathbf{a}\right],$$

where $\mathbf{I} = \mathbf{J}_{pc} + \mathbf{J}_{wc}$ is the moment of inertia matrix of the spacecraft about the center of mass c, and $\mathbf{J}_{pc}$ is the platform moment of inertia about the spacecraft center of mass c.

4. Consider again the dual-spin satellite from Question 3, shown in Figure 2.5. Let $\vec{\mathbf{v}}_c = \vec{\mathcal{F}}_b^T \mathbf{v}_c$ be the inertial velocity of point c, which as we recall is the center of mass of the spacecraft (combined platform and wheel).

(a) Show that the kinetic energy of the wheel is given by

$$T_w = \frac{1}{2} m_w v_c^2 + \mathbf{v}_c^T \boldsymbol{\omega}^\times (m_w \mathbf{r}_w) + \frac{1}{2} \boldsymbol{\omega}^T \mathbf{J}_{wc} \boldsymbol{\omega} + I_s \omega_s \mathbf{a}^T \boldsymbol{\omega} + \frac{1}{2} I_s \omega_s^2,$$

where $v_c = \|\mathbf{v}_c\|$, and all other quantities are defined in Question 3.

(b) Show that the platform kinetic energy is given by

$$T_p = \frac{1}{2} m_p v_c^2 + \mathbf{v}_c^T \boldsymbol{\omega}^\times \mathbf{c}_p + \frac{1}{2} \boldsymbol{\omega}^T \mathbf{J}_{pc} \boldsymbol{\omega},$$

where $\mathbf{c}_p = \int_p \boldsymbol{\rho} dm$ is the platform first moment of mass about point c, and m_p is the total platform mass.

(c) Combining the results from parts (a) and (b), show that the total spacecraft kinetic energy is given by

$$T = T_t + T_r,$$

where

$$T_t = \frac{1}{2} m v_c^2,$$

is the spacecraft translational kinetic energy, and $m = m_p + m_w$ is the total spacecraft mass, and

$$T_r = \frac{1}{2} \boldsymbol{\omega}^T \mathbf{I} \boldsymbol{\omega} + I_s \omega_s \mathbf{a}^T \boldsymbol{\omega} + \frac{1}{2} I_s \omega_s^2,$$

is the spacecraft rotational kinetic energy. Hint: $m_p \mathbf{r}_w$ is the first moment of mass of the wheel about the point c.

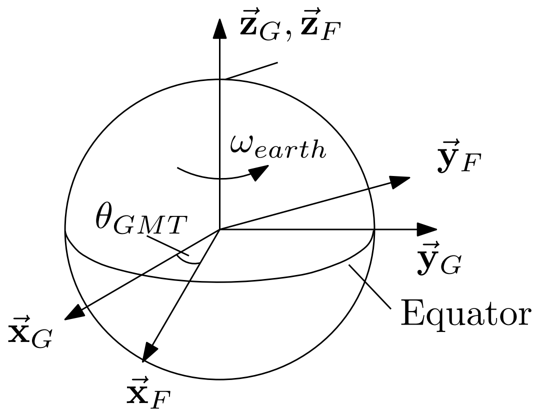

Figure 2.1 ECI and ECEF frame definitions

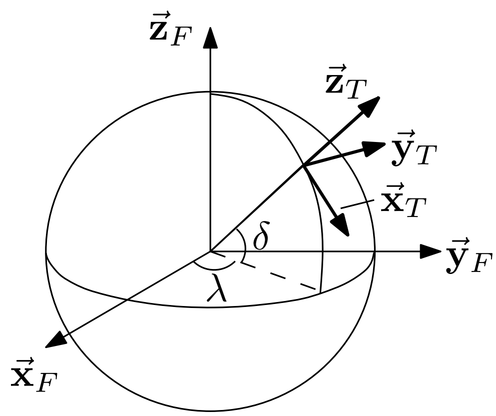

Figure 2.2 Topocentric frame definition

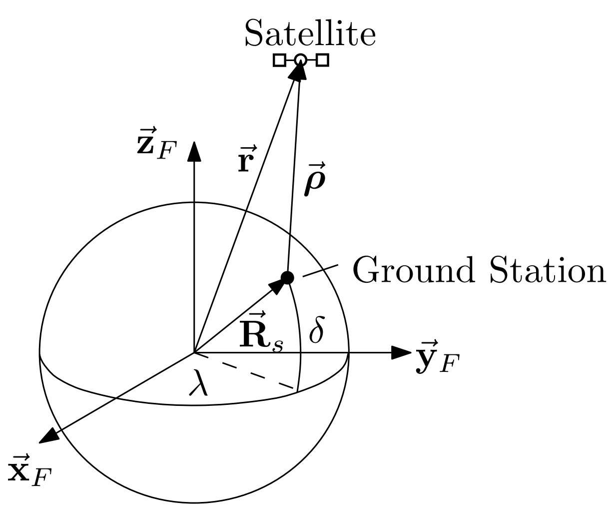

Figure 2.3 Ground station and satellite geometry

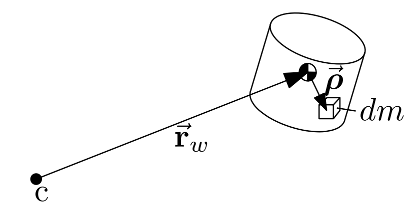

Figure 2.4 Wheel

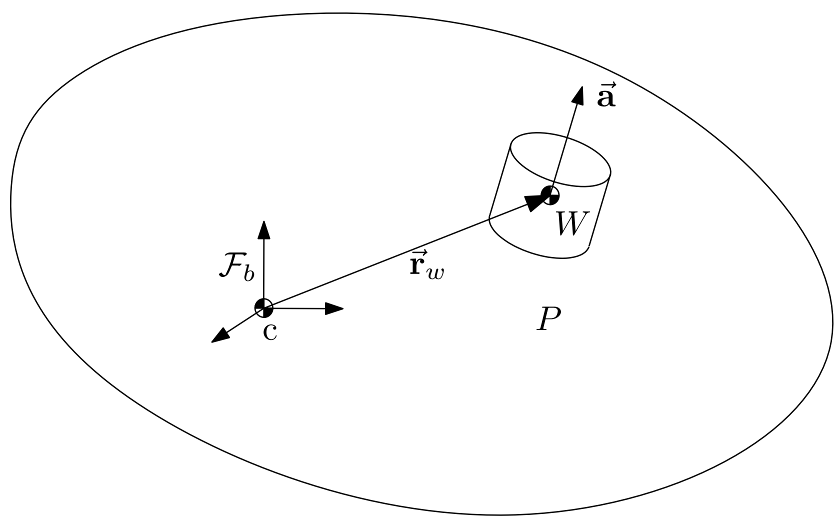

Figure 2.5 Dual-spin satellite

3

Chapter 3 Exercises

For some of the following questions, you will need the Earth's gravitational constant

$$\mu_{\oplus} = 3.986 \times 10^5 \text{ km}^3/\text{s}^2,$$

the Sun's gravitational constant

$$\mu_{\odot} = 1.3271244 \times 10^{11} \text{ km}^3/\text{s}^2,$$

and

$$1 \text{ AU} = 1.4959787 \times 10^8 \text{ km}.$$

1. A spacecraft is observed with inertial position and velocity vectors relative to the center of the Earth, given in ECI coordinates by

$$\vec{\mathbf{r}} = \vec{\mathcal{F}}_G^T = \begin{bmatrix} 1670.6319 \\ 1670.6319 \\ 6491.2735 \end{bmatrix} \text{ km}, \quad \vec{\mathbf{v}} = \vec{\mathcal{F}}_G^T = \begin{bmatrix} -5.3429 \\ -5.3429 \\ 3.3788 \end{bmatrix} \text{ km/s}.$$

The Earth's gravitational constant is given by $\mu_{\oplus} = 3.986 \times 10^5$ km^3/s^2.

   (a) Compute the orbital angular momentum vector $\vec{\mathbf{h}}$ in ECI coordinates.
   (b) Compute the orbital energy, $\mathcal{E}$. What type of orbit is it?
   (c) Compute the eccentricity vector $\vec{\mathbf{e}}$ in ECI coordinates.
   (d) Compute the eccentricity, e and the semi-latus rectum p.
   (e) Compute the true anomaly, θ, noting that θ is measured positive from $\vec{\mathbf{e}}$ as a right-hand rotation about $\vec{\mathbf{h}}$.
   (f) Compute the radius at periapsis r_{min}.
   (g) Compute the spacecraft position vector at periapsis in ECI coordinates.
   (h) Compute the orbital speed at periapsis.
   (i) Compute the angle, i, between the orbital plane and the Earth's equatorial plane, noting that $\vec{\mathbf{h}}$ is a vector normal to the orbit, and $\vec{\mathbf{z}}_G$ is a vector normal to the equator.

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2. A *geosynchronous* orbit has semi-major axis and eccentricity:

$$\begin{aligned} a &= 42241.08007 \text{ km}, \\ e &= 0. \end{aligned}$$

   (a) Compute the orbital period in hours. What can you conclude about a satellite in a geosynchronous orbit?
   (b) A *geostationary* orbit is a geosynchronous orbit with zero inclination $i = 0$. What is the plane of the orbit? What can you conclude about a satellite in a geosynchronous orbit in relation to an observer on the ground?

3. Halley's comet last passed perihelion on February 9, 1986. It has a semimajor axis of 17.9564 AU and eccentricity $e = 0.967298$ (one AU is the semimajor axis of the earth's orbit around the sun). Predict the date of its next return.

4. A satellite is in a geocentric Keplerian (two-body) orbit with a period of 270 minutes and eccentricity $e = 0.5$. It has passed perigee and is now at a point in which the orbital radius is the same as the semi-latus rectum. How much time (in minutes) has elapsed since perigee passage?

5. An earth-orbiting spacecraft has classical orbital elements

$$\begin{aligned} a &= 8000 \text{ km}, \\ e &= 0.1, \\ i &= 45^o, \\ \omega &= 0^o, \\ \Omega &= 90^o. \end{aligned}$$

   The spacecraft currently has true anomaly $\theta = 30^o$.

   (a) Determine the spacecraft position and velocity vectors in perifocal coordinates.
   (b) Determine the transformation from perifocal to ECI coordinates $\mathbf{C}_{Gp}$.
   (c) Determine the spacecraft position and velocity vectors in ECI coordinates.

6. At time $t = 0$, the position and velocity vectors for an earth-orbiting satellite are given in ECI coordinates as:

$$\vec{\mathbf{r}} = \vec{\mathcal{F}}_G^T \begin{bmatrix} -3718.8 \\ 1602.9 \\ 6517.7 \end{bmatrix} \text{ km},$$

$$\vec{\mathbf{v}} = \vec{\mathcal{F}}_G^T \begin{bmatrix} -4.8991 \\ -5.4428 \\ -0.6659 \end{bmatrix} \text{ km/s}.$$

   (a) Find the classical orbital elements
   (b) Thirty minutes later, what are $\vec{\mathbf{r}}$ and $\vec{\mathbf{v}}$? (Express your answers in ECI coordinates)

7. An earth-orbiting satellite has orbital radius and speed at perigee

$$r_p = 7000 \text{ km}, \quad v_p = 8 \text{ km/s}.$$

(a) Determine the orbital period, T in minutes.

(b) Determine the orbital speed twenty minutes after perigee passage.

8. At time $t = 0$, the position and velocity vectors for an earth-orbiting satellite are given in ECI coordinates as:

$$\vec{\mathbf{r}} = \vec{\mathcal{F}}_G^T \begin{bmatrix} 1.703 \times 10^5 \\ 0.0426 \times 10^5 \\ 0.638 \times 10^5 \end{bmatrix} \text{ km},$$

$$\vec{\mathbf{v}} = \vec{\mathcal{F}}_G^T \begin{bmatrix} 0.0972 \\ 1.271 \\ 1.465 \end{bmatrix} \text{ km/s}.$$

(a) Find the classical orbital elements

(b) Twenty minutes later, what are $\vec{\mathbf{r}}$ and $\vec{\mathbf{v}}$? (Express your answers in ECI coordinates)

9. By starting with the polar solution for an orbit, and the equation for the orbital angular momentum, show that the time-of-flight equation for a parabolic orbit is given by

$$6\sqrt{\frac{\mu}{p^3}}(t - t_0) = 3\tan\frac{\theta}{2} + \tan^3\frac{\theta}{2},$$

where $t - t_0$ denotes the time since periapsis passage.

10. In this question you are going to derive the time-of-flight equation for a hyperbolic orbit.

First, we need to discuss hyperbolic functions. The hyperbolic sine, cosine and tangent are defined as

$$\sinh x \triangleq \frac{e^x - e^{-x}}{2} \text{ and } \cosh x \triangleq \frac{e^x + e^{-x}}{2}, \quad \tanh x \triangleq \frac{\sinh x}{\cosh x},$$

respectively. From these definitions, the following property can readily be shown.

$$\cosh^2 x - \sinh^2 x = 1.$$

The derivatives are readily obtained as

$$\frac{d}{dx}\sinh x = \cosh x, \quad \frac{d}{dx}\cosh x = \sinh x, \quad \frac{d}{dx}\tanh x = \frac{1}{\cosh^2 x}.$$

Similar to the trigonometric functions, the following "double-angle" formulae can also readily be found

$$\cosh x = 2\cosh^2\frac{x}{2} - 1, \quad \sinh x = 2\sinh\frac{x}{2}\cosh\frac{x}{2}.$$

Consider the hyperbola satisfying

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \text{ with } a < 0, \quad b < 0.$$

As shown in figure 3.1, the hyperbola has two branches. The left-hand branch of the hyperbola can be represented parametrically by

$$x = a\cosh P, \;\; y = -b\sinh P$$

We now consider a hyperbolic orbit with eccentricity $e > 1$, $a < 0$ and $b = a\sqrt{e^2-1}$. Since a hyperbolic orbit corresponds to the left-hand branch of a hyperbola, we can represent the x and y components of the orbital position in perifocal coordinates by

$$x_p = -ae + a\cosh H, \;\; y_p = -b\sinh H,$$

where we call H the "hyperbolic eccentric anomaly".

(a) Show that the orbital radius satisfies

$$r = a(1 - e\cosh H).$$

(b) Show that

$$\cos\theta = \frac{a\left[\cosh H - e\right]}{r},$$

and

$$\sin\theta = \frac{-a\sqrt{e^2-1}\sinh H}{r}.$$

(c) By applying trigonometric and hyperbolic double angle formulae to the results in part (b), show that

$$\cos^2\frac{\theta}{2} = \frac{-a(e-1)\cosh^2\frac{H}{2}}{r},$$

and

$$\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \frac{-a\sqrt{e^2-1}\sinh\frac{H}{2}\cosh\frac{H}{2}}{r}.$$

(d) Using the results from part (c), show that the true anomaly and the hyperbolic eccentric anomaly are related by

$$\tan\frac{\theta}{2} = \sqrt{\frac{e+1}{e-1}}\tanh\frac{H}{2}.$$

(e) By differentiating the result in part (d), and making use of the first result in part (c), show that

$$\frac{d\theta}{dH} = -\frac{a\sqrt{e^2-1}}{r}.$$

(f) Let t_0 be the time of periapsis passage. Evaluate the integral

$$\int_{t_0}^{t} h\,d\tau = \int_0^{\theta} r^2\,d\theta$$

to obtain the hyperbolic form of Kepler's equation

$$e\sinh H - H = \sqrt{\frac{\mu}{-a^3}}\left(t - t_0\right).$$

11. The inertial position and velocity of a spacecraft over the Earth are observed in the ECI frame to be

$$\vec{\mathbf{r}} = \vec{\mathcal{F}}_G^T \begin{bmatrix} 9000 \\ 9000 \\ 0 \end{bmatrix} \text{ km}, \quad \vec{\mathbf{v}} = \vec{\mathcal{F}}_G^T \begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix} \text{ km/s}.$$

Calculate:

(a) The angular momentum vector $\vec{\mathbf{h}}$.
(b) The inclination i,
(c) the right ascension of the ascending node, Ω.

12. Consider the circle of radius a as shown in figure 3.2. The wedge bounded by a radius with angle E from the x-axis and the x-axis itself may be divided into two parts: a triangular part with area A_t and the remaining part with area A_o. Therefore, the area of the wedge is given by

$$A_w = A_t + A_o.$$

(a) Show that the area A_o is given by

$$A_o = \frac{1}{2}a^2 \left[E - \sin E \cos E\right].$$

(b) Referring to figure 3.3, it can be seen that the area of an orbit swept out by the radius vector from periapsis at time t_0 to the current time t, can be divided into two parts: a triangular part with area A_2 and the remaining part with area A_1. Show that

$$A_2 = \frac{ab}{2}\left[\sin E \cos E - e \sin E\right],$$

where b is the semi-minor axis, and e is the eccentricity.

(c) Given that $A_1 = (b/a)A_0$, where A_0 was found in part (a), show that the area swept out by the radius vector is given by

$$A(t) = \frac{ab}{2}\left[E - e \sin E\right].$$

(d) Using the result from part (c), make use of Kepler's second law to derive Kepler's equation.

13. A spacecraft is in a geocentric Keplerian orbit. It has passed perigee, and is currently at a position where the orbital radius is equal to the semi-latus rectum. The current orbital radius and speed are

$$r = 7000 \text{ km}, \quad v = 7.5555 \text{ km/s}.$$

How much time has elapsed since perigee passage?

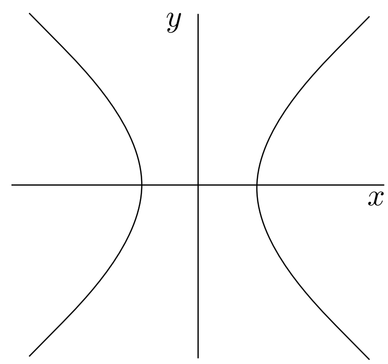

Figure 3.1 Left and right branches of a hyperbola

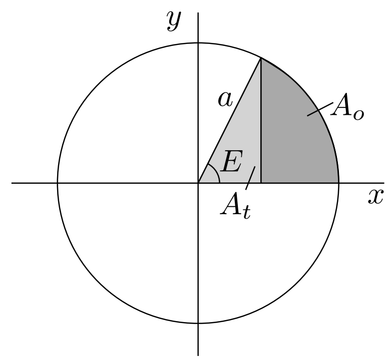

Figure 3.2 Segment of bounding circle

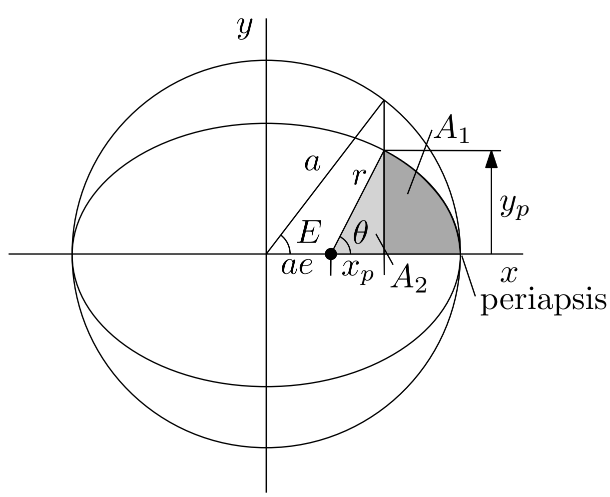

Figure 3.3 Area swept out by orbital radius

4

Chapter 4 Exercises

For some of the following questions, you will need the Earth's gravitational constant

$$\mu_{\oplus} = 3.986 \times 10^5 \text{ km}^3/\text{s}^2,$$

the Sun's gravitational constant

$$\mu_{\odot} = 1.3271244 \times 10^{11} \text{ km}^3/\text{s}^2,$$

Earth's heliocentric orbital radius

$$R_{\oplus} = 1.49598023 \times 10^8 \text{ km},$$

and Mars' heliocentric orbital radius

$$R_{Mars} = 2.27939186 \times 10^8 \text{ km}.$$

1. Radar observations have provided the following successive position vectors of an object orbiting the earth:

$$\vec{\mathbf{r}}_1 = \vec{\mathcal{F}}_G^T \begin{bmatrix} 7000 \\ 0 \\ 0 \end{bmatrix} \text{ km},$$

$$\vec{\mathbf{r}}_2 = \vec{\mathcal{F}}_G^T \begin{bmatrix} 5846.8 \\ 5846.8 \\ 0 \end{bmatrix} \text{ km},$$

$$\vec{\mathbf{r}}_3 = \vec{\mathcal{F}}_G^T \begin{bmatrix} 0 \\ 14700 \\ 0 \end{bmatrix} \text{ km}.$$

   (a) Determine whether the orbit is elliptic, parabolic or hyperbolic.
   (b) Determine the radius at perigee.
   (c) Determine the orbital speed at perigee.

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2. Radar observations have provided the following successive position vectors of an object orbiting the earth:

$$\vec{\mathbf{r}}_1 = \vec{\mathcal{F}}_G^T \begin{bmatrix} 3467.3 \\ 3467.3 \\ 4903.5 \end{bmatrix} \text{ km},$$

$$\vec{\mathbf{r}}_2 = \vec{\mathcal{F}}_G^T \begin{bmatrix} 0 \\ 0 \\ 7425.0 \end{bmatrix} \text{ km},$$

The time between observations is $t_2 - t_1 = 740.6$ seconds. You may assume that the object is in an elliptical orbit. For simplicity, you may take $\eta = \eta_H$ as the sector-triangle area ratio.

   (a) Determine the orbital period.
   (b) Determine the eccentricity of the orbit.

3. It is desired to perform an interplanetary transfer from Earth to Mars. It is determined that a Hohmann transfer requires too much time. Assume that the Earth and Mars both possess coplanar circular orbits. At time $t = 0$, the Earth has true anomaly $\theta_E(0) = 0$, and Mars has true anomaly $\theta_M(0) = 30^o$. The spacecraft is desired to arrive at Mars when Mars has a true anomaly $\theta_M = 45^o$. See Figure 4.1.

   (a) Determine the time of flight of the transfer in days.
   (b) Determine the required heliocentric velocity vector for the spacecraft upon departing the Earth's sphere of influence. Use the coordinate system shown in Figure 4.1. You may take $\eta = \eta_H$ for the sector to triangle area ratio of the transfer orbit.

4. Radar observations have provided the following successive position vectors of an object orbiting the earth:

$$\vec{\mathbf{r}}_1 = \vec{\mathcal{F}}_G^T \begin{bmatrix} -1568.3998 \\ 4895.6516 \\ 4570.7746 \end{bmatrix} \text{ km},$$

$$\vec{\mathbf{r}}_2 = \vec{\mathcal{F}}_G^T \begin{bmatrix} -3090.7866 \\ 3963.6107 \\ 4988.2121 \end{bmatrix} \text{ km},$$

$$\vec{\mathbf{r}}_3 = \vec{\mathcal{F}}_G^T \begin{bmatrix} -5431.0755 \\ 1739.9314 \\ 5070.6676 \end{bmatrix} \text{ km}.$$

Determine the semi-major axis, eccentricity and radius of perigee of the orbit.

5. Suppose that you are an astronaut onboard the International Space Station. You receive a radio message from Canadian Space Surveillance (CSS) that a previously undetected asteroid is on a collision course with the Earth, and will likely impact somewhere near Ottawa. You are asked to fire a missile (which is kept onboard for such emergencies)

at the asteroid, which will break it into pieces small enough to burn up upon entry into the atmosphere. CSS informs you that the last point on the trajectory of the asteroid that such an intercept is possible has ECI coordinates

$$\vec{\mathbf{r}}_2 = \vec{\mathcal{F}}_G^T \begin{bmatrix} -2102.02476 \\ 528.32428 \\ 6941.38176 \end{bmatrix} \text{ km},$$

which is where the asteroid will be in precisely 12 minutes time. It will take you 2 minutes to prepare the missile, at which time your location in ECI coordinates will be

$$\vec{\mathbf{r}}_1 = \vec{\mathcal{F}}_G^T \begin{bmatrix} 1668.39097 \\ 3624.99549 \\ 5163.39798 \end{bmatrix} \text{ km}.$$

What inertial velocity vector should the missile have upon being fired, in order to intercept the asteroid at $\vec{\mathbf{r}}_2$? Express your result in ECI coordinates.

Note: Upon firing, the missile has an impulsive (instantaneous) thrust to give it the required velocity, after which it is in free orbital flight until intercept with the target.

6. Radar observations have provided the following successive position vectors of an object orbiting the earth:

$$\vec{\mathbf{r}}_1 = \vec{\mathcal{F}}_G^T \begin{bmatrix} 1955.2948 \\ 4646.0121 \\ 5227.4178 \end{bmatrix} \text{ km},$$

$$\vec{\mathbf{r}}_2 = \vec{\mathcal{F}}_G^T \begin{bmatrix} 107.8848 \\ 3469.9455 \\ 6531.6767 \end{bmatrix} \text{ km},$$

$$\vec{\mathbf{r}}_3 = \vec{\mathcal{F}}_G^T \begin{bmatrix} -2316.9737 \\ 1373.0632 \\ 7168.8558 \end{bmatrix} \text{ km}.$$

   (a) Determine the position vector at perigee $\vec{\mathbf{r}}_p$ in ECI coordinates.
   (b) Determine the velocity vector at perigee $\vec{\mathbf{v}}_p$ in ECI coordinates.
   (c) Determine the time since perigee passage $t_2 - t_0$ for $\vec{\mathbf{r}}_2$.

7. Verify the velocity vector at perigee obtained in 6(b), by solving Lambert's problem given $\vec{\mathbf{r}}_p$ obtained in 6(a), $\vec{\mathbf{r}}_2$ and the time of flight $t_2 - t_0$ obtained in 6(c).

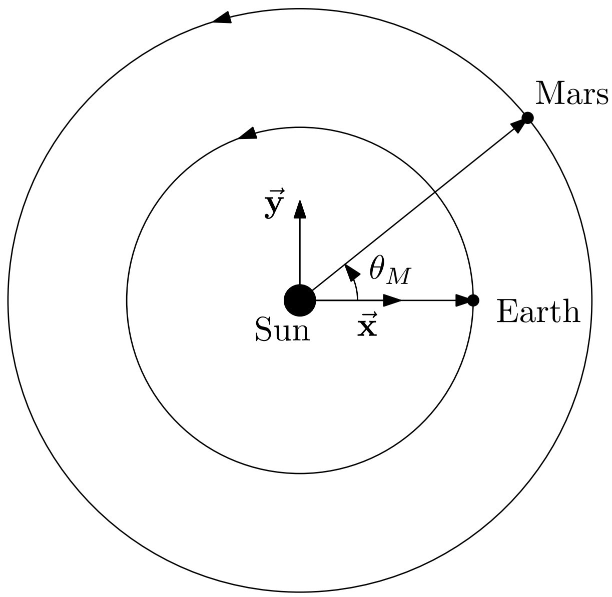

Figure 4.1 Earth to Mars transfer

5

Chapter 5 Exercises

For some of these questions, you will need the earth's gravitational constant

$$\mu_\oplus = 3.986 \times 10^5 \text{ km}^3/\text{s}^2.$$

1. It is desired to change an initially elliptical orbit of semimajor axis a_1 and eccentricity e_1 to a larger elliptical orbit with semimajor axis $a_2 > a_1$, with the same radius of perigee r_p, but different argument of perigee ($\omega_2 = \omega_1 + \Delta\omega$). Note that both orbits lie in the same plane. See Figure 5.1.

   (a) Describe a double tangential maneuver that can accomplish this.
   (b) Obtain an expression for the total Δv for the maneuver.
   (c) Obtain an expression for the total time taken to execute the maneuver.

2. Two spacecraft are in the same geocentric elliptical orbit with semi-major axis $a = 10,000$ km and eccentricity $e = 0.2$, as shown in Figure 5.2. At the current time, they have true anomalies

   $$\theta_1 = 45^o \text{ and } \theta_1 = 90^o,$$

   respectively. Determine the Δv spacecraft 1 must apply at periapsis if it is to catch spacecraft 2 with a single tangential maneuver.

3. A spacecraft is initially in a geocentric circular orbit of radius $r_c = 7,000$ km. It is desired to place the spacecraft in an elliptical orbit in the same plane, of semi-major axis $a = 20,000$ km and eccentricity $e = 0.665$.

   Suggest a double-impulse maneuver to accomplish the transfer. Compute the total Δv and the time of flight TOF.

4. A spacecraft is launched into a circular orbit of radius $r_1 = 8,000$ km with inclination, $i = 45^o$. Compute the total Δv required to transfer the spacecraft into a geostationary orbit (which has radius $r_2 = 42,221$ km), assuming the inclination change is performed at apoapsis of the transfer orbit.

5. A satellite leaves a circular parking orbit at inclination i and executes a Hohmann transfer to a larger circular orbit in the equatorial plane. Part of the required inclination change Δi_1 is performed during the first maneuver, and the remaining $\Delta i_2 = i - \Delta i_1$ is done during the second maneuver.

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(a) If the speeds in the circular orbits are v_{c1} and v_{c2} respectively, and the perigee and apogee speeds in the Hohmann transfer orbit are v_p and v_a respectively, show that the total Δv for both maneuvers is given by

$$\Delta v = \left[v_{c1}^2 + v_p^2 - 2v_{c1}v_p \cos\Delta i_1\right]^{\frac{1}{2}} + \left[v_{c2}^2 + v_a^2 - 2v_{c2}v_a \cos\left(i - \Delta i_1\right)\right]^{\frac{1}{2}}.$$

(b) Obtain an expression for

$$\frac{d\Delta v}{d\Delta i_1}.$$

(c) Using the result from part (b), show that performing the entire inclination change at apogee of the Hohmann transfer orbit (that is $\Delta i_1 = 0$) is not optimal.

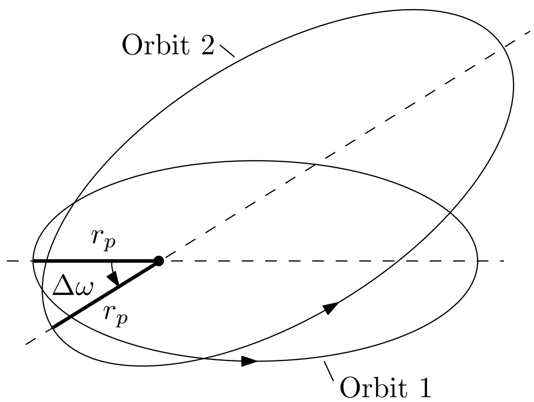

Figure 5.1 Desired orbit change

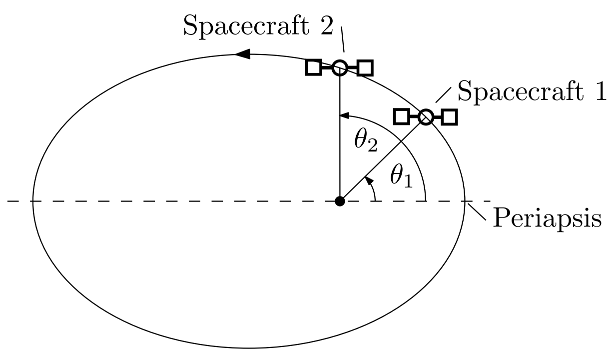

Figure 5.2 Question 2 Scenario

6

Chapter 6 Exercises

For the following exercises, you will need the Earth's gravitational constant

$$\mu_{earth} = 3.986 \times 10^5 \text{ km}^3/\text{s}^2,$$

the Sun's gravitational constant

$$\mu_{sun} = 1.3271244 \times 10^{11} \text{ km}^3/\text{s}^2,$$

Mars' gravitational constant

$$\mu_{mars} = 4.305 \times 10^4 \text{ km}^3/\text{s}^2,$$

Venus' gravitational constant

$$\mu_{venus} = 3.257 \times 10^{14} \text{ m}^3/\text{s}^2,$$

Jupiter's gravitational constant,

$$\mu_{Jup} = 1.268 \times 10^8 \text{ km}^3/\text{s}^2,$$

Earth's orbital radius about the sun

$$R_{earth} = 149.598023 \times 10^6 \text{ km},$$

Mars' orbital radius about the sun

$$R_{mars} = 227.939186 \times 10^6 \text{ km},$$

Venus' orbital radius about the sun

$$R_{venus} = 108.208601 \times 10^6 \text{ km},$$

Jupiter's orbital radius about the sun,

$$R_{Jup} = 777.8 \times 10^6 \text{ km},$$

and Saturn's orbital radius about the sun,

$$R_{Sat} = 1486 \times 10^6 \text{ km}.$$

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1. As part of a preliminary study for an exploration trip to Mars, it has been decided that a Hohmann transfer will be used to travel from the Earth to Mars. You may assume that the orbits of the Earth and Mars are circular and lie in the same plane.

   The spacecraft is initially in a circular parking orbit around the Earth of radius $r_{park} = 100,000$ km. It is desired to place the spacecraft in a circular orbit around Mars of radius $r_{capture} = 50,000$ km.

   (a) Compute the semi-major axis of the Hohmann transfer orbit.
   (b) Compute the time-of-flight for the Hohmann transfer.
   (c) Assuming that the Earth, Mars the Sun lie on the same line at $t = 0$, with Earth and Mars on opposite sides of the Sun, compute the time t in days of the required departure from Earth.
   (d) Compute the required hyperbolic excess speed $v_{\infty,dep}$ upon exiting the Earth's sphere of influence, and the hyperbolic excess speed $v_{\infty,arr}$ upon entering Mars' sphere of influence.
   (e) Determine the location and magnitude of the Δv_{dep} required for Earth departure.
   (f) Determine the required arrival hyperbola asymptote offset $-b$, and compute the magnitude of the Δv_{arr} required for Mars capture.
   (g) Compute the total Δv for the trip.

2. As part of a preliminary study for an exploration trip to Venus, it has been decided that a Hohmann transfer will be used to travel from the Earth to Venus. You may assume that the orbits of the Earth and Venus are circular and lie in the same plane.

   The spacecraft is initially in a circular parking orbit around the Earth of radius $r_{park} = 100,000$ km. It is desired to place the spacecraft in a circular orbit around Venus of radius $r_{capture} = 50,000$ km.

   (a) Compute the semi-major axis of the Hohmann transfer orbit.
   (b) Compute the time-of-flight for the Hohmann transfer.
   (c) Assuming that the Earth, Venus and the Sun lie along the same line at $t = 0$ (on the same side of the sun), compute the time t in days of the required departure from Earth.
   (d) Compute the required hyperbolic excess speed $v_{\infty,dep}$ upon exiting the Earth's sphere of influence, and the hyperbolic excess speed $v_{\infty,arr}$ upon entering Venus' sphere of influence.
   (e) Determine the location and magnitude of the Δv_{dep} required for Earth departure.
   (f) Determine the required arrival hyperbola asymptote offset $-b$, and compute the magnitude of the Δv_{arr} required for Venus capture.
   (g) Compute the total Δv for the trip.

3. Four incredibly lonely and homesick astronauts who got suckered into making a one-way trip to Mars, have found a resource (on Mars) that can be refined to create rocket fuel. However, this resource is limited, so they need to minimize the fuel required to get back to Earth. This necessitates a Hohmann transfer.

(a) The astronauts desperately want to return to Earth as soon as possible, so they do not want to miss the next launch window. Given that the current true anomalies of Mars and the Earth are $\theta_{Mars} = 45^o$ and $\theta_{Earth} = 90^o$, how long do the astronauts have to make preparations?

(b) Assuming that they can make the next launch window, how long will it be until the astronauts are reunited with their families?

(c) The astronauts will initially launch into a circular parking orbit around Mars of radius $r_{park} = 30,000$ km, where they will perform a final check-out of all their systems before embarking on the return journey to the Earth. What is the magnitude of the Δv they must apply to get on the required escape hyperbola, and at what location relative to the velocity vector of Mars must it be applied?

4. It is desired to perform an interplanetary transfer from Mars to Jupiter. Assume that Mars and Jupiter possess circular coplanar orbits and make other appropriate simplifying assumptions.

   (a) Calculate the required heliocentric velocities near Mars and near Jupiter.

   (b) What is the required hyperbolic excess speed, $v_{\infty,dep}$, upon leaving Mars' sphere of influence?

   (c) If the approach distance at Jupiter is $-b = 1,050,000$ km, calculate the perijovian distance.

   (d) Calculate the Δv to be applied at periapsis of the arrival hyperbola to capture the spacecraft into a circular orbit about Jupiter.

   (e) If Mars and Jupiter are currently aligned on the opposite sides of the Sun, how much time until the next launch window?

5. A spacecraft on a Hohmann transfer from the Earth to Saturn, flies unexpectedly through the sphere of influence of Jupiter. The spacecraft approaches Jupiter on an entry asymptote offset of $-b = 900,000$ km. Assume circular coplanar orbits for Earth, Jupiter and Saturn.

   (a) What is the perijovian distance?

   (b) What is the angle between the entrance and exit velocity vectors relative to Jupiter?

   (c) What will the spacecraft's heliocentric energy gain be if the spacecraft passes behind Jupiter?

7

Chapter 7 Exercises

For the following questions, you will need the Earth's gravitational constant

$$\mu_\oplus = 3.986 \times 10^5 \text{ km}^3/\text{s}^2,$$

J_2 for the Earth

$$J_2 = 0.001082,$$

and the equatorial radius of the Earth

$$R_e = 6378.1363 \text{ km}.$$

1. The perturbing gravitational potential for the Earth may sometimes be approximated by

$$\phi_p = -\frac{\mu}{r}\left[J_2\left(\frac{R_e}{r}\right)^2 P_2(\sin\delta) + J_3\left(\frac{R_e}{r}\right)^3 P_3(\sin\delta)\right]$$

where R_e is the equatorial radius, J_2 and J_3 are zonal harmonic coefficients, $P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$ and $P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$ are Legendre polynomials.

(a) Find the perturbing force per unit mass due to the above perturbing potential in the spherical coordinate system (see Figure 7.1). The $\vec{\nabla}$ operator in spherical coordinates is given by

$$\vec{\nabla}(\cdot) = \frac{\partial}{\partial r}(\cdot)\vec{\mathbf{x}}_s + \frac{1}{r\cos\delta}\frac{\partial}{\partial\lambda}(\cdot)\vec{\mathbf{y}}_s + \frac{1}{r}\frac{\partial}{\partial\delta}(\cdot)\vec{\mathbf{z}}_s$$

Note that in Section 7.3.1 in the book, we obtained an expression for the acceleration due to the J_2 term directly in ECI coordinates. Strictly speaking, the Earth's gravitational potential is fixed in a frame attached to the Earth, namely the ECEF frame. The reason the acceleration due to J_2 could be evaluated directly in the ECI frame is because it depends only on latitude (δ), and not on longitude (λ). Under the assumption that the ECI and ECEF z-axes are equal ($\vec{\mathbf{z}}_G = \vec{\mathbf{z}}_F$), the latitude δ is the same in both ECI and ECEF frames, and the gravitational potential due to J_2 becomes identical in both frames. In reality, there is a slight difference between $\vec{\mathbf{z}}_G$ and $\vec{\mathbf{z}}_F$. However, by making the approximation that $\vec{\mathbf{z}}_G = \vec{\mathbf{z}}_F$, the analytical expressions for the effects of J_2

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on the orbital elements in Section 7.3.4 in the book could be obtained. Strictly speaking, in the final equation for the acceleration due to J_2 *(equation (7.40)),* $\vec{\mathbf{z}}_G$ *should be replaced by* $\vec{\mathbf{z}}_F$.

(b) Noting that the spherical coordinate frame is obtained from the ECEF frame by a rotation λ about the $\vec{\mathbf{z}}_F$ axis, followed by a rotation $-\delta$ about the $\vec{\mathbf{y}}_s$ axis, find the perturbing force per unit mass due to the J_2 term in ECEF coordinates, and verify that this is the same as that presented in equation (7.40) in the book.

(c) By transforming the perturbing force per unit mass due to the J_3 term to ECEF coordinates, show that the force per unit mass in terms of physical vectors is

$$\vec{\mathbf{f}}_{p,J_3} = \frac{\mu J_3 R_e^3}{2r^5}\left[\frac{5(\vec{\mathbf{r}}\cdot\vec{\mathbf{z}}_F)}{r^2}\left(7\frac{(\vec{\mathbf{r}}\cdot\vec{\mathbf{z}}_F)^2}{r^2} - 3\right)\vec{\mathbf{r}} + 3\left(1 - 5\frac{(\vec{\mathbf{r}}\cdot\vec{\mathbf{z}}_F)^2}{r^2}\right)\vec{\mathbf{z}}_F\right]$$

2. Consider the perturbing potential for a non-spherical primary due to zonal terms only

$$\phi_p(\vec{\mathbf{r}}) = -\frac{\mu}{r}\sum_{n=2}^{\infty} J_n\left(\frac{R_e}{r}\right)^n P_n(\sin\delta).$$

Show that the associated perturbing force/unit mass is given by

$$\vec{\mathbf{f}}_p = \frac{\mu}{r^3}\sum_{n=2}^{\infty} J_n\left(\frac{R_e}{r}\right)^n\left[\left((n+1)P_n(\sin\delta) + \sin\delta P_n'(\sin\delta)\right)\vec{\mathbf{r}} - rP_n'(\sin\delta)\vec{\mathbf{z}}_F\right],$$

where $P_n'(x) = dP_n(x)/dx$ and $\vec{\mathbf{z}}_F$ is the z-axis of the ECEF frame.

3. A satellite is initially in a close-to-circular Earth orbit (very small eccentricity), as shown in Figure 7.2. However, a small disturbing force due to solar radiation pressure acts continuously on the spacecraft in an inertially fixed direction, as shown. Assume the solar radiation pressure force per unit mass is in the plane of the orbit, and has magnitude f.

(a) Express the tangential force component f_θ and the radial force component f_r in terms of f and the true anomaly, θ.

(b) Show that for the initially close-to-circular orbit, the evolutionary equations for the semi-major axis a and the eccentricity e are (approximate by setting $e = 0$)

$$\frac{da}{dt} = \frac{2a^2}{\sqrt{\mu a}} f\cos\theta$$

$$\frac{de}{dt} = \sqrt{\frac{a}{\mu}} f\left[1 + \cos^2\theta\right]$$

(c) For a circular orbit, the angular rate is approximately constant, with $\dot{\theta} = \sqrt{\frac{\mu}{a^3}}$. Show that the evolutionary equations for a and e with respect to true anomaly, θ are

$$\frac{da}{d\theta} = \frac{2f}{n^2}\cos\theta$$

$$\frac{de}{d\theta} = \frac{f}{an^2}\left[1 + \cos^2\theta\right]$$

where $n = \sqrt{\frac{\mu}{a^3}}$ is the orbital mean motion.

(d) As shown in Figure 7.3, only the lit portion of the orbit is affected by the solar pressure force. The portion of the orbit shadowed by the Earth has a range of true anomalies $\theta_v \leq \theta \leq 180^o - \theta_v$ as shown in the figure. Using the result in part (c), show that the changes in a and e over one orbit are given by

$$\Delta a = 0$$

$$\Delta e = \frac{f}{an^2}\left[3\left(\theta_v + \frac{\pi}{2}\right) + \frac{\sin(2\theta_v)}{2}\right]$$

4. (a) Show that for a sun-synchronous frozen orbit with semi-major axis, a, the eccentricity e is given by

$$e = \left[1 - \left(\frac{3J_2R_e^2}{2\langle\dot{\Omega}\rangle_{ss}}\sqrt{\frac{\mu}{5a^7}}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}},$$

where

$$\langle\dot{\Omega}\rangle_{ss} = 360^o/\text{year}.$$

(b) Compute the eccentricities and radii of perigee for a geocentric sun-synchronous frozen orbits with semi-major axes $a = 10,000$ km, $a = 15,000$ km and $a = 20,000$ km. Conclude that there is a range of semi-major axes for which a geocentric sun-synchronous frozen orbit is possible.

(c) Sketch a plot of semi-major axis vs. radius of perigee for a geocentric sun-synchronous frozen orbit.

(d) Find the minimum value of a for which a geocentric sun-synchronous frozen orbit is possible.

(e) Using an iterative procedure, determine the maximum value of a for which a geocentric frozen orbit is possible, given that the orbit should stay at least 200 km above the earth.

5. For this question, make use of the impulsive form of Gauss' variational equations.

(a) Consider a circular orbit. Suppose that it is desired to simultaneously change the inclination i, and the right ascension of the ascending node Ω, by a small amount δi and $\delta\Omega$ respectively.

i. What should be the magnitude of the impulsive velocity change?

ii. Where in the orbit should it be applied (at what value of θ)? (You may take $\omega = 0$)

(b) Consider an elliptical orbit ($0 < e < 1$). Suppose that it is desired to change the right ascension of the ascending node Ω by a small amount $\delta\Omega$, while keeping all other elements unaffected.

Describe a double-impulse maneuver that accomplishes this. That is, specify $\delta\vec{\mathbf{v}}_1$ and $\delta\vec{\mathbf{v}}_2$, and their locations of application in the orbit (θ). Hint: Consider the $\delta\Omega$ change first.

(c) Given a spacecraft in a sun-synchronous orbit of semi-major axis $a = 7000$ km, and eccentricity $e = 0.05$.

i. Compute the secular rate of change of the argument of perigee $\langle\dot{\omega}\rangle$ due to J_2 effects.
ii. It is desired to keep the secular part of ω within a range $\omega_{min} \leq \omega \leq \omega_{max}$, where $\omega_{max} - \omega_{min} = 2^o$. How often does the orbit need to be corrected?
iii. Assuming that $\delta\omega = 2^o$, what is the required $\delta\vec{\mathbf{v}}$, and where should it be applied in the orbit such that the other elements are not affected?

6. Atmospheric drag has a significant impact upon the lifetime of a space mission. The force per unit mass due to atmospheric drag is given by

$$\vec{\mathbf{f}}_p = -C\vec{\mathbf{v}},$$

where $C = \frac{1}{2}\frac{c_d A}{m}\rho v$, and c_d is the drag coefficient, A the cross-sectional area of the spacecraft, m the spacecraft mass, ρ the atmospheric density, and $v = |\vec{\mathbf{v}}|$ the magnitude of the spacecraft velocity vector.

(a) Starting from the energy equation for an orbit, show that the effect of atmospheric drag on the semi-major axis is given by

$$\dot{a} = -\frac{2Ca^2v^2}{\mu}.$$

(b) Given that the atmospheric drag does not affect the eccentricity for circular orbits, what does the result in part (a) mean for a spacecraft in a circular orbit?
(c) The atmospheric density decreases exponentially with radial distance from the earth surface (altitude). As such, highly elliptical orbits can be considered under the influence of atmospheric drag only near perigee. That is, the effect of atmospheric drag on highly elliptical orbits may be approximated by a tangential Δv near perigee of every orbit.

Based upon this, what is the long-term effect of atmospheric drag on highly elliptical orbits?
(d) Starting from the definition of the semi-latus rectum, show that the effect of atmospheric drag on the semi-major semi-latus rectum is given by

$$\dot{p} = -2Cp.$$

(e) Show that the effect of atmospheric drag on the eccentricity is given by

$$\dot{e} = \frac{Cp}{e}\left(\frac{2}{a} - \frac{2}{r}\right).$$

Hint: You will need the vis-viva equation.
(f) Using the result from part (e), what happens to the eccentricity at apogee? What happens at perigee? Hint: Substitute the expression for the radii at apogee and perigee into the result from part (e).
(g) Can you provide a physical explanation for the phenomena observed in part (f)?

7. The perturbation force (per unit mass) acting on a spacecraft in a geocentric close-to-circular orbit is given by

$$\vec{\mathbf{f}}_p = 0.005 \sin\theta \vec{\mathbf{y}}_o \text{ N/kg},$$

where $\vec{\mathbf{x}}_o$, $\vec{\mathbf{y}}_o$ and $\vec{\mathbf{z}}_o$ are the unit vectors of the orbital cylindrical coordinate frame, and θ is the true anomaly. If the orbit has a period of 10 hours, calculate the secular changes in the semi-major axis and the eccentricity after one orbit.

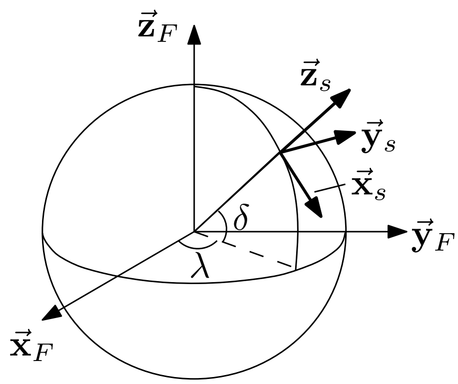

Figure 7.1 Spherical coordinate frame

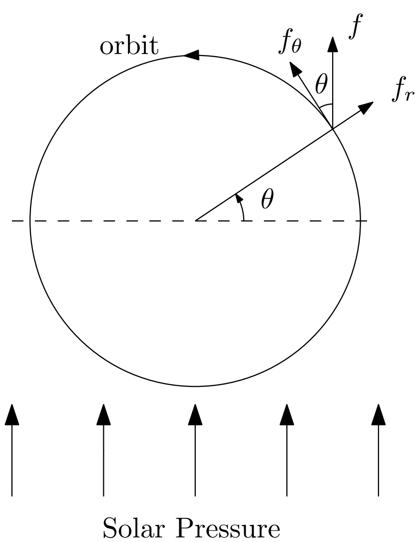

Figure 7.2 Solar radiation pressure

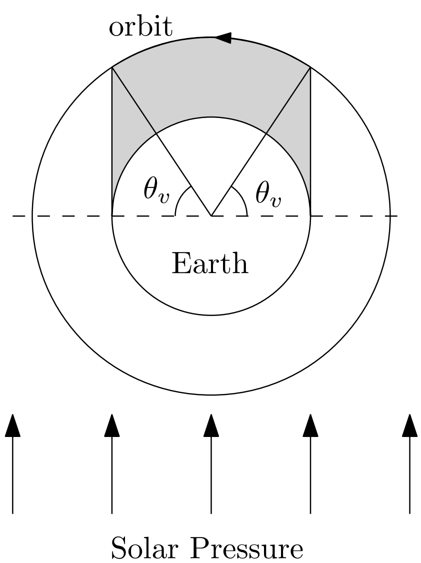

Figure 7.3 Shadowing by Earth

8

Chapter 9 Exercises

For some of the following questions, you will need the Earth's gravitational constant

$$\mu = 3.986 \times 10^5 \text{ km}^3/\text{s}^2.$$

1. Prove the expressions in (9.29) in the book.
2. Prove the expression in (9.32) in the book.
3. Consider a geocentric leader-follower spacecraft formation, with the leader in a circular orbit of radius $r_l = 7000$ km. Determine the initial conditions for the follower spacecraft (in the Hill frame), if it is to be in a Projected Circular Orbit about the leader of radius $R = 100$ m, with initial phase angle $\phi_0 = 45^o$. Numerically simulate the relative orbit using Hill's equations to validate the initial conditions.
4. Repeat the development in Section 9.3.2 to obtain the initial conditions for a translated Projected Elliptical Orbit, where everything is the same as in Section 9.3.2, except that the Projected Elliptical Orbit is to be centered at a point $\bar{x} = x_d$, $\bar{z} = 0$, where x_d is non-zero.
5. Specialize the results from Question 4 to the case of a translated Projected Circular Orbit of radius R.
6. Repeat Question 3 for a translated Projected Circular Orbit of the same dimension, but with center at $x_d = 200$ m.
7. Consider a geocentric leader-follower spacecraft formation, with the leader in a circular orbit of radius $r_l = 7200$ km. At the current time, the follower has position and velocity relative to the leader given by (in Hill frame coordinates)

$$x = 234.2020 \text{ m}, \quad y = 70.7107 \text{ m}, \quad z = 66.9846 \text{ m},$$

$$\dot{x} = -0.1281 \text{ m/s}, \quad \dot{y} = 0.0731 \text{ m/s}, \quad \dot{z} = 0.0177 \text{ m/s}.$$

   Determine $\bar{x}$, $\bar{z}$, P, ϕ, Q and α (using the notation from Sections 9.2.4 and 9.2.5 in the book). Is the relative motion bounded?

8. In Sections 9.2.4 and 9.2.5 in the book, the following transformations were provided from $x, y, z, \dot{x}, \dot{y}, \dot{z}$ to $\bar{x}, \bar{z}, P, \phi, Q, \alpha$:

$$\bar{x} = x - \frac{2\dot{z}}{\omega_o},$$

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$$\bar{z} = 4z + \frac{2\dot{x}}{\omega_o},$$

$$P = \left(\left(3z + \frac{2\dot{x}}{\omega_o} \right)^2 + \left(\frac{\dot{z}}{\omega_0} \right)^2 \right)^{\frac{1}{2}},$$

$$\sin\phi = \frac{-\left(3z + \frac{2\dot{x}}{\omega_o}\right)}{P},$$

$$\cos\phi = \frac{\dot{z}}{\omega_o P},$$

$$Q = \left(y^2 + \left(\frac{\dot{y}}{\omega_0} \right)^2 \right)^{\frac{1}{2}},$$

$$\sin\alpha = \frac{y}{Q},$$

$$\cos\alpha = \frac{\dot{y}}{\omega_o Q}.$$

Using only the expressions given above, prove the inverse transformations given below, from $\bar{x}, \bar{z}, P, \phi, Q, \alpha$ to $x, y, z, \dot{x}, \dot{y}, \dot{z}$:

$$x = \bar{x} + 2P\cos\phi,$$

$$z = \bar{z} + P\sin\phi,$$

$$\dot{x} = -\frac{\omega_o}{2}\left(3\bar{z} + 4P\sin\phi\right),$$

$$\dot{z} = \omega_o P\cos\phi,$$

$$y = Q\sin\alpha,$$

$$\dot{y} = \omega_o Q\cos\alpha.$$

9. In Chapter 3 in the book, it was shown that the classical orbital elements provide much greater physical insight into an orbit than do the inertial position and velocity vectors. Likewise, for a leader-follower formation, the quantities $\bar{x}, \bar{z}, P, \phi, Q, \alpha$ provide much greater physical insight into the relative motion of a leader-follower formation than do $x, y, z, \dot{x}, \dot{y}, \dot{z}$. The physical meanings of $\bar{x}, \bar{z}, P, \phi, Q, \alpha$, were investigated in Sections 9.2.3 to 9.2.5 in the book.

   However, $\bar{x}, \bar{z}, P, \phi, Q, \alpha$ were defined on the basis of natural formation motion, that is, without any disturbances or spacecraft control forces. It was shown that under these conditions, $\bar{z}, P, Q$ are constant, and $\dot{\bar{x}} = -3\omega_0\bar{z}/2$ and $\dot{\phi} = \dot{\alpha} = \omega_o$. However, in practise, just as for a geocentric orbit, there will be disturbances or intentional control forces which will cause $\bar{z}, P, Q$ to vary with time, and the rates of $\dot{\bar{x}}, \dot{\phi}, \dot{\alpha}$ to also vary. Therefore, similar to the Gauss variational equations for the orbital elements, it will be useful to obtain dynamic equations for $\bar{x}, \bar{z}, P, \phi, Q, \alpha$, when the follower spacecraft is under the influence of external forces.

As shown in Question 8, by simply taking as definitions the transformations from $x, y, z, \dot{x}, \dot{y}, \dot{z}$ to $\bar{x}, \bar{z}, P, \phi, Q, \alpha$ (without any consideration of physical meaning, or whether or not there are disturbance forces), the inverse transformation from $\bar{x}, \bar{z}, P, \phi, Q, \alpha$ to $x, y, z, \dot{x}, \dot{y}, \dot{z}$ is also well-defined, and takes the same form regardless of whether or not there are disturbance forces.

Now, considering the forced equations of relative motion (equations (9.11) and (9.12) in the book)

$$\begin{aligned}
\ddot{x} &= -2\omega_o \dot{z} + f_x, \\
\ddot{z} &= 2\omega_o \dot{x} + 3\omega_o^2 z + f_z, \\
\ddot{y} &= -\omega_o^2 y + f_y,
\end{aligned}$$

show that the dynamics for $\bar{x}, \bar{z}, P, \phi, Q, \alpha$ are given by

$$\begin{aligned}
\dot{\bar{x}} &= -\frac{3\omega_o}{2}\bar{z} - \frac{2}{\omega_o} f_z, \\
\dot{\bar{z}} &= \frac{2}{\omega_o} f_x, \\
\dot{P} &= -\frac{2\sin\phi}{\omega_o} f_x + \frac{\cos\phi}{\omega_o} f_z, \\
\dot{\phi} &= \omega_o - \frac{2\cos\phi}{\omega_o P} f_x - \frac{\sin\phi}{\omega_o P} f_z, \\
\dot{Q} &= \frac{\cos\alpha}{\omega_o} f_y, \\
\dot{\alpha} &= \omega_o - \frac{\sin\alpha}{\omega_o Q} f_y.
\end{aligned}$$

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Chapter 10 Exercises

1. Determine the extension of Equations (10.4) and (10.5) when m_3 is no longer confined to the plane of the orbit of m_1 and m_2.
2. Using numerical root-finding software, validate the locations of L_1, L_2, and L_3 for the Earth-Moon system.
3. Consider Equations (10.13) and (10.14) and adopt the nondimensionalizations

$$\delta\hat{x} = \delta x / r_{12}$$
$$\delta\hat{y} = \delta y / r_{12}$$
$$(\dot{\ }) = \frac{d(\ )}{d\tau}, \quad \tau = \omega t$$

(We have redefined the symbol $(\dot{\ })$). Show that the equations for the triangle equilibrium points L_4 and L_5 become

$$\delta\ddot{\hat{x}} - 2\delta\dot{\hat{y}} - \frac{3}{4}\delta\hat{x} - \frac{3\sqrt{3}}{2}\left(\rho - \frac{1}{2}\right)\delta\hat{y} = 0$$
$$\delta\ddot{\hat{y}} + 2\delta\dot{\hat{x}} - \frac{3\sqrt{3}}{2}\left(\rho - \frac{1}{2}\right)\delta\hat{x} - \frac{9}{4}\delta\hat{y} = 0$$

for L_4 and

$$\delta\ddot{\hat{x}} - 2\delta\dot{\hat{y}} - \frac{3}{4}\delta\hat{x} + \frac{3\sqrt{3}}{2}\left(\rho - \frac{1}{2}\right)\delta\hat{y} = 0$$
$$\delta\ddot{\hat{y}} + 2\delta\dot{\hat{x}} + \frac{3\sqrt{3}}{2}\left(\rho - \frac{1}{2}\right)\delta\hat{x} - \frac{9}{4}\delta\hat{y} = 0$$

for L_5, where $\rho = m_2/(m_1 + m_2)$. Determine the range of mass ratios ρ leading to stability of the triangle points. In particular, verify that they are stable for the Earth-Moon system where $\rho = 0.01215$.

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Chapter 12 Exercises

1. A spacecraft with a principal axes body-fixed frame $\mathcal{F}_b$, has corresponding principal moments of inertia $I_x = 100$ kg·m^2, $I_y = 120$ kg·m^2, $I_z = 80$ kg·m^2. The spacecraft attitude relative to the Earth centered inertial frame $\mathcal{F}_G$ is described by a yaw-pitch-roll (3-2-1) Euler sequence, represented by the rotation matrix

$$\mathbf{C}_{bG}(\phi,\theta,\psi) = \begin{bmatrix} c_\theta c_\psi & c_\theta s_\psi & -s_\theta \\ s_\phi s_\theta c_\psi - c_\phi s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & s_\phi c_\theta \\ c_\phi s_\theta c_\psi + s_\phi s_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi & c_\phi c_\theta \end{bmatrix}$$

where $s_b = \sin b$ and $c_b = \cos b$. Where ϕ, θ and ψ are the roll, pitch and yaw angles, respectively. Currently, the attitude is represented by $\phi = \theta = \psi = \frac{\pi}{4}$ rad, and the spacecraft orbital position (in ECI coordinates) is

$$\vec{\mathbf{R}}_o = \vec{\mathcal{F}}_G^T \begin{bmatrix} 0 \\ 0 \\ R_o \end{bmatrix} \text{ km.}$$

Determine the gravity-gradient torque acting on the spacecraft. Express the result in spacecraft body coordinates. Note that $\mu = 3.986 \times 10^5$ km^3/s^2 is the Earth's gravitational constant.

2. The International Geomagnetic Reference Field (IGRF) is a global model of the Earth's magnetic field. The IGRF model gives the Earth magnetic field vector at the spacecraft position $\vec{\mathbf{r}}$ in spherical coordinates, where the spherical coordinate frame $\mathcal{F}_s$ is defined as shown in Figure 10.1. That is, the IGRF model provides B_r, B_θ and B_λ such that

$$\vec{\mathbf{B}} = \vec{\mathcal{F}}_s^T \mathbf{B}_s = B_\theta \vec{\mathbf{x}}_s + B_\lambda \vec{\mathbf{y}}_s + B_r \vec{\mathbf{z}}_s.$$

The magnetic field components B_r, B_θ and B_λ are functions of the spacecraft orbital radius r, the spacecraft geocentric longitude λ and the spacecraft geocentric co-latitude $\theta = 90^o - \delta$ (δ is the geocentric latitude), as shown in Figure 10.2. For example, the first set of terms of the IGRF model are given by the dipole approximation

$$\begin{aligned} B_r &= 2\left(\frac{R_e}{r}\right)^3 \left[g_1^0 \cos\theta + \left(g_1^1 \cos\lambda + h_1^1 \sin\lambda\right)\sin\theta\right], \\ B_\theta &= \left(\frac{R_e}{r}\right)^3 \left[g_1^0 \sin\theta - \left(g_1^1 \cos\lambda + h_1^1 \sin\lambda\right)\cos\theta\right], \\ B_\lambda &= \left(\frac{R_e}{r}\right)^3 \left[g_1^1 \sin\lambda - h_1^1 \cos\lambda\right], \end{aligned}$$

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where R_e is the Earth's equatorial radius, and g_1^0, g_1^1 and h_1^1 are given IGRF coefficients.

Let the spacecraft have residual magnetic dipole moment $\vec{\mathbf{m}} = \vec{\mathcal{F}}_b^T \mathbf{m}_b$, where the components $\mathbf{m}_b$ in the spacecraft body frame are given. Also given are the spacecraft inertial attitude $\mathbf{C}_{bG}$, and the spacecraft orbital position vector in ECI coordinates $\vec{\mathbf{r}} = \vec{\mathcal{F}}_G^T \mathbf{r}$.

Write down the equations required to compute the residual magnetic disturbance torque in spacecraft body coordinates, using only the given information $\mathbf{m}_b$, $\mathbf{C}_{bG}$ *and* $\mathbf{r}$ *(as well as the magnetic field parameters* g_1^0, g_1^1 *and* h_1^1 *and the Earth's equatorial radius,* R_e*, and rate of rotation* ω_{earth}*).*

Note: As shown in Figure 10.2, the Earth Centered Earth Fixed (ECEF) frame $\mathcal{F}_F$ is obtained from the Earth Centered Inertial (ECI) frame $\mathcal{F}_G$ by a rotation about the z-axis through an angle $\alpha_G = \omega_{earth}(t - t_0)$, which is known as the right ascension of the Greenwich meridian, and ω_{earth} is the rate of rotation of the earth. It is clear that the spacecraft geocentric longitude is given by $\lambda = \alpha - \alpha_G$, where α is the spacecraft right ascension.

3. Consider a flat surface S illuminated by the sun as shown in Figure 10.3. The surface has normal vector $\vec{\mathbf{n}}$, and the unit vector pointing from the surface to the sun is $\vec{\mathbf{s}}$.

   (a) Show that the torque about the point c due to the solar pressure on the side of S with outward normal $\vec{\mathbf{n}}$ is given by

$$\vec{\mathbf{T}}_S = \begin{cases} \vec{\boldsymbol{\rho}}_A \times \vec{\mathbf{F}}_S, & \vec{\mathbf{n}} \cdot \vec{\mathbf{s}} \geq 0, \\ \vec{\mathbf{0}}, & \vec{\mathbf{n}} \cdot \vec{\mathbf{s}} < 0. \end{cases}$$

   where $\vec{\boldsymbol{\rho}}_A = \frac{\int_S \vec{\boldsymbol{\rho}} dS}{A}$ is the center of area of S, $A = \int_S dS$ is the area of S, $\vec{\mathbf{F}}_S = -pA(\vec{\mathbf{n}} \cdot \vec{\mathbf{s}})\vec{\mathbf{s}}$ is the total solar pressure force on S and p is the solar pressure magnitude.

   (b) The surface of a spacecraft may be approximated by a number of flat surfaces S_1, S_2,....S_n. Each surface has area A_i, outward pointing normal vector given in body coordinates as $\vec{\mathbf{n}}_i = \vec{\mathcal{F}}_b^T \mathbf{n}_i$, and center of area located from the spacecraft center of mass also given on body coordinates as $\vec{\boldsymbol{\rho}}_{A,i} = \vec{\mathcal{F}}_b^T \boldsymbol{\rho}_{A,i}$ for $i = 1, ..., n$. If the sun pointing vector is given in inertial coordinates as $\vec{\mathbf{s}} = \vec{\mathcal{F}}_I^T \mathbf{s}_I$, obtain the expression required to compute the total solar pressure torque about the spacecraft center of mass in body coordinates.

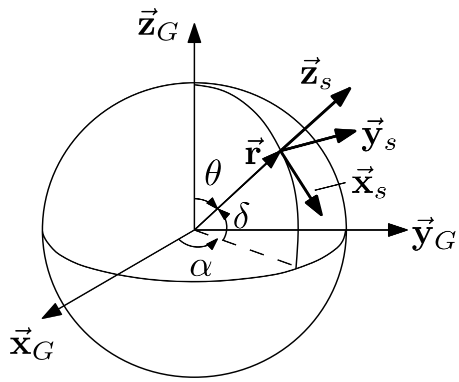

Figure 10.1 Spherical coordinate frame definition

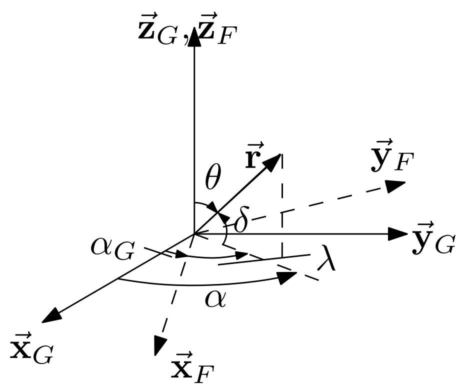

Figure 10.2 ECI and ECEF frames

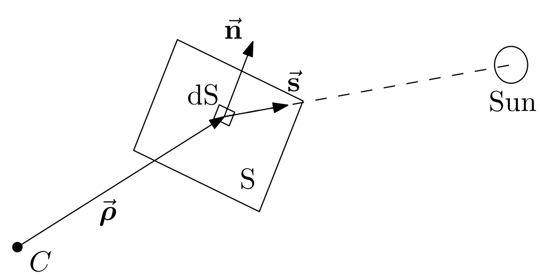

Figure 10.3 Flat surface under solar pressure

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Chapter 13 Exercises

1. A spacecraft is launched into a low Earth orbit. The spacecraft principal moments of inertia are $I_x = 98$ kg·m^2, $I_y = 102$ kg·m^2, $I_z = 150$ kg·m^2. For stability, the launch vehicle deploys the spacecraft such that it is in a major axis spin when released, with $\omega_z = 0.5$ rad/s. Because no deployment is perfect, the spacecraft also has some angular velocity about the other two principal axes, given by $\omega_x = 0.1$ rad/s, $\omega_y = 0.02$ rad/s. Making appropriate approximations:

   (a) Describe the resulting spacecraft attitude motion if there are no disturbance torques.
   (b) Determine the nutation angle.
   (c) Determine the precession rate.

2. A uniform thin disk is thrown into the air, and is observed to wobble such that its axis of symmetry traces out a cone with half angle 60^o, once per second.

   (a) Show that the transverse and axial moments of inertia of an uniform infinitesimally thin disk satisfy

   $$I_t = \frac{1}{2} I_a.$$

   Assume that the disk has uniform mass per unit area σ.
   (b) Determine the relative spin-rate of the disk.
   (c) Determine the disk angular velocity vector in inertial coordinates.

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Chapter 14 Exercises

1. Consider a spacecraft with principal inertias satisfying $I_x > I_z > I_y$. Sometimes it may be necessary to spin the spacecraft about the intermediate axis (in this case the principal z-axis). One method to accomplish this could be to apply control torques $T_{cx} = -k\omega_x$, $T_{cy} = -k\omega_y$ and $T_{cz} = -k_s\,(\omega_z - \nu)$, where ν is the desired spin-rate about the body z-axis. The equations of motion become

$$\begin{aligned} I_x\dot{\omega}_x + (I_z - I_y)\omega_y\omega_z &= T_{cx}, \\ I_y\dot{\omega}_y + (I_x - I_z)\omega_x\omega_z &= T_{cy}, \\ I_z\dot{\omega}_z + (I_y - I_x)\omega_x\omega_y &= T_{cz}. \end{aligned}$$

   (a) Find the values of k and k_s that make a spin about the intermediate axis asymptotically stable.
   (b) Is the feedback $T_{cz} = -k_s\,(\omega_z - \nu)$ necessary to stabilize the intermediate axis spin?

2. Consider a rigid axisymmetric spinning body with principal moments of inertia $I_x = I_y = I_t$ and $I_a = I_z$.

   (a) Show that the rotational kinetic energy is given by

$$T = \frac{h^2}{2}\left(\frac{\sin^2\gamma}{I_t} + \frac{\cos^2\gamma}{I_a}\right),$$

   where h is the magnitude of the total angular momentum and γ is the angle between the axis of symmetry ($\vec{\mathbf{z}}_b$) and the angular momentum vector $\vec{\mathbf{h}}$.
   Note: for a rigid body, the rotational kinetic energy is given by $T = \frac{1}{2}I_x\omega_x^2 + \frac{1}{2}I_y\omega_y^2 + \frac{1}{2}I_z\omega_z^2$.

   (b) Under the Energy Sink Hypothesis, internal energy dissipation results in a reduction in rotational kinetic energy, while the angular momentum is conserved. That is, $\dot{T} < 0$ and $h = constant$. Show that under the energy-sink hypothesis for an axisymmetric quasi-rigid body,

$$\dot{T} = \frac{h^2 \sin 2\gamma}{2I_aI_t}\,(I_a - I_t)\,\dot{\gamma} < 0,$$

   for $\gamma \neq 0, 90^o$.

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(c) Noting that we can always choose the $\vec{\mathbf{z}}_b$-axis such that $0 \leq \gamma \leq 90^o$, explain how the result obtained in part (b) is consistent with the major axis rule.

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Chapter 15 Exercises

1. Find all equilibrium solutions for a dual-spin satellite with dynamics given by

$$\begin{aligned} I_x\dot{\omega}_x + (I_z - I_y)\,\omega_y\omega_z + h_s\omega_y &= 0, \\ I_y\dot{\omega}_y + (I_x - I_z)\,\omega_x\omega_z - h_s\omega_x &= 0, \\ I_z\dot{\omega}_z + (I_y - I_x)\,\omega_x\omega_y &= 0. \end{aligned}$$

2. Consider an axisymmetric dual-spin satellite, with principal inertias $I_x = I_y = I_t$ and $I_z = I_a$. The wheel spin axis coincides with the satellite axis of symmetry, namely the principal z-axis. The wheel has a positive constant spin angular momentum relative to the spacecraft $h_s > 0$. The spacecraft is nominally non-spinning, that is $\omega_z(0) = 0$.

   (a) Obtain the solution for the resulting torque-free attitude motion.

   (b) Provide a physical interpretation of the resulting torque-free attitude motion, analogous to that obtained in Section 13.2 in the book.

   (c) Suppose $h_s = 1$ Nms, $I_t = 10$ kg·m^2 and $I_a = 12$ kg·m^2, and the spacecraft z-axis is observed to trace out a cone in inertial space with half-angle 30^o. How long does it take to trace out a single cone?

3. Consider a nominally non-spinning dual-spin spacecraft with principal inertias I_x, I_y and I_z. The wheel axis coincides with the $\vec{\mathbf{z}}_b$ axis of a body-fixed principal axes frame. The wheel relative angular momentum is given by $h_s > 0$.

   (a) Show that this situation corresponds to an equilibrium for torque-free motion.

   (b) Show that small perturbations to the spacecraft angular velocity lead to purely oscillatory behavior in ω_x and ω_y with frequency

$$\Omega_p = \frac{h_s}{\sqrt{I_x I_y}}.$$

4. A spacecraft with principal axes body frame $\mathcal{F}_b$, has corresponding principal moments of inertia $I_x = 8$ kg·m^2, $I_y = 12$ kg·m^2, $I_z = 10$ kg·m^2.

   It is desired to spin the spacecraft about the principal z-axis with angular velocity $\omega_z = 0.1$ rad/s. Assuming that the spacecraft has a momentum wheel with spin-axis aligned with the principal z-axis, determine the required relative wheel angular

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momentum h_s to make the desired attitude motion passively stable under torque-free conditions.

5. Chapter 2, Question 3.
6. Chapter 2, Question 4.

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Chapter 16 Exercises

1. Chapter 2, Question 2
2. Consider an arbitrary spacecraft body frame $\mathcal{F}_b$ (not necessarily a principal axes frame). That is, the spacecraft inertia matrix evaluated in $\mathcal{F}_b$ has the form

$$\mathbf{I} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix}.$$

   Let the spacecraft body-frame $\mathcal{F}_b$ coincide with the orbiting reference frame $\mathcal{F}_o$ for a circular orbit about earth. That is, the spacecraft has angular velocity

$$\vec{\boldsymbol{\omega}}_{bI} = \vec{\mathcal{F}}_b^T \boldsymbol{\omega}_{bI}, \;\; \boldsymbol{\omega}_{bI} = \begin{bmatrix} 0 \\ -\omega_o \\ 0 \end{bmatrix},$$

   where $\omega_o = \sqrt{\mu/r^3}$ is the orbital angular velocity, μ is Earth's gravitational constant and r is the spacecraft orbital radius.

   (a) Show that the gravity-gradient torque vector in spacecraft body coordinates is given by

$$\mathbf{T}_g = 3\omega_o^2 \begin{bmatrix} -I_{yz} \\ I_{xz} \\ 0 \end{bmatrix}.$$

   (b) Evaluate $\boldsymbol{\omega}_{bI}^{\times} \mathbf{I} \boldsymbol{\omega}_{bI}$.

   (c) Making use of parts (a) and (b) in the equations of motion, what are the requirements on the inertia matrix $\mathbf{I}$ for an earth-pointing equilibrium ($\mathcal{F}_b = \mathcal{F}_o$) in the presence of gravity-gradient torque? What does this say about $\mathcal{F}_b$?

3. Consider the spacecraft shown in Figure 14.1. The spacecraft hub is a solid cubic block of mass 100 kg and side 1 m. To provide gravity-gradient stability, the spacecraft inertia is augmented by the addition of four lump masses of equal mass m, located from the spacecraft center of mass by massless rods of length l.

   (a) Determine the moment of inertia matrix for the spacecraft in terms of m and l.

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(b) Based on the result of part (a), is the spacecraft gravity-gradient stabilized?

(c) In addition to the gravity-gradient torque, there is a constant solar pressure torque about the pitch axis given by $T_{d,y} = 10^{-5}$ Nm. Assuming that the spacecraft is in a circular orbit, and $\theta(0) = \dot{\theta}(0) = 0$, find the solution for the pitch angle $\theta(t)$ in terms of the orbital rate ω_o, the mass m and the rod length l.

(d) Assume now that the spacecraft is in a circular orbit with a period of 6 hours. How long must the rods be to ensure that the maximum excursion in pitch is limited to 5^o? Assume that $m = 2$ kg.

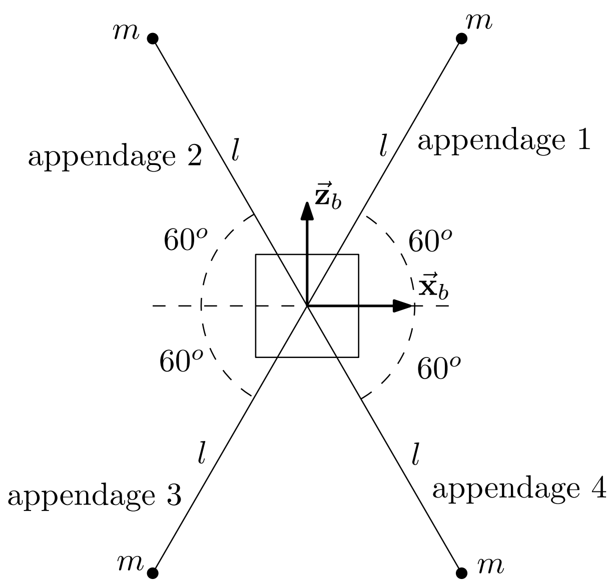

Figure 14.1 Satellite with inertia augmentation for Question 3

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Chapter 17 Exercises

1. Consider a spin-stabilized spacecraft nominally spinning about the principal z-axis with spin-rate ν. The desired angular velocities are therefore $\omega_x = 0$, $\omega_y = 0$ and $\omega_z = \nu$. It may be desirable to control the spin-rate. The equations of motion are (as usual)

$$\begin{aligned} I_x\dot{\omega}_x + (I_z - I_y)\omega_y\omega_z &= T_{cx} + T_{dx}, \\ I_y\dot{\omega}_y + (I_x - I_z)\omega_x\omega_z &= T_{cy} + T_{dy}, \\ I_z\dot{\omega}_z + (I_y - I_x)\omega_x\omega_y &= T_{cz} + T_{dz}. \end{aligned}$$

where $T_{c.}$ are control torques, and $T_{d.}$ are disturbance torques. The principal moments of inertia are $I_x = 10$ kg·m^2, $I_y = 12$ kg·m^2, $I_z = 8$ kg·m^2.

   (a) Assuming small angular velocities $\omega_x = \epsilon_x$, $\omega_y = \epsilon_y$, obtain the linearized equation for the spin-rate ω_z.

   (b) Given that the output of interest is $y = \omega_z$, and the control input is $u = T_{cz}$, find the plant transfer function $G_p(s)$ such that

$$Y(s) = G_p(s)\left[U(s) + \hat{T}_{dz}(s)\right].$$

   (c) The reference signal is the desired spin-rate $r = \nu$. Therefore, the spin-rate error is $e = \nu - \omega = r - y$. Assuming proportional control

$$u(t) = K_p e(t),$$

   Draw a block-diagram for the closed-loop system.

   (d) Find the closed-loop transfer function relationships from the reference signal $R(s)$ to the error $E(s)$ and from the disturbance $\hat{T}_{dz}(s)$ to the error $E(s)$. What restriction must be placed on the proportional gain K_p to ensure asymptotic stability?

   (e) Find the response $e(t)$ to a step disturbance $T_{dz}(t) = \bar{T}_{dz}$. What restriction must be placed on the proportional gain K_p if the steady-state error to a disturbance of magnitude $\bar{T}_{dz} = 10^{-5} Nm$ is to be kept below 1 deg/s?

   (f) Find the response $e(t)$ to a step reference signal $r(t) = \bar{\nu}$. What restriction must be placed on the proportional gain K_p if the spin-rate $y(t)$ is to be within 2% of the desired spin-rate $\bar{\nu}$ within 10 seconds?

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2. Consider the spacecraft attitude control problem for a single axis. The attitude dynamics are given by

$$I\ddot{\theta} = u + T_d,$$

where I is the related moment of inertia, θ is the related attitude angle, u is the control torque applied about the axis by an actuator, and T_d is a disturbance torque. The actuator has dynamics

$$\dot{u} = -\frac{1}{T_a}(u - u_c),$$

where u is the control torque applied by the actuator, T_a is the actuator time constant, and u_c is the control torque commanded by the control law.

   (a) Find the actuator transfer function.
   (b) The control law is chosen to be a modified PD control law

   $$u_c = K_p e(t) - K_d \dot{y}(t),$$

   where the plant output is the attitude angle ($y(t) = \theta(t)$), the reference signal is the desired attitude angle ($r(t) = \theta_d$), and $e(t) = r(t) - y(t)$ is the attitude error. Draw a block diagram representation of the closed-loop system.
   (c) Find the transfer function from reference signal $R(s)$ to output $Y(s)$, and the transfer function from disturbance $T_d(s)$ to output $Y(s)$.

3. Consider the spacecraft attitude control problem for a single axis. Neglecting the disturbance torque, the attitude dynamics are given by

$$I\ddot{\theta} = u,$$

where u is the control torque. The moment of inertia is $I = 1$ kg·m^2.

The following transient specifications are given for the closed-loop step response:

- Maximum overshoot requirement, $M_p = 20\%$
- Settling time requirement, $t_s = 60$ seconds

Design a modified PD control law such that the transient specifications are satisfied.

4. Consider the modified proportional-derivative attitude control about a single spacecraft axis as shown in Figure 15.1, where $I = 10\ kg \cdot m^2$ is the corresponding moment of inertia, $y = \theta$ is the corresponding attitude angle, $r = \theta_d$ is the desired attitude angle, $K_p = 0.1$ is the proportional gain and $K_d = 0.5$ is the derivative gain. With regards to a unit step input:

   (a) Determine the settling time.
   (b) Determine the percent overshoot.
   (c) Determine the rise time.
   (d) Is it possible to reduce both the settling time and percent overshoot without changing the proportional gain? Illustrate your reasoning with a diagram.

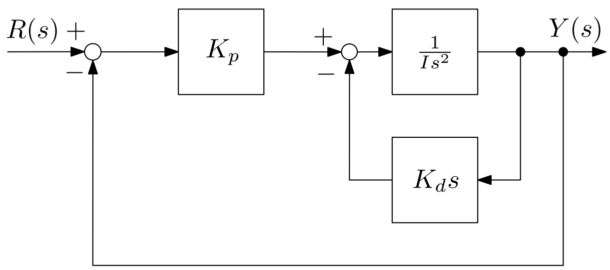

Figure 15.1 Modified PD control law for Question 4

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Chapter 18 Exercises

1. A plant has transfer function

$$G_p(s) = \frac{Y(s)}{U(s)} = \frac{1}{s\,(s^3 + s^2 + 2s + 1)}.$$

   The plant is to be controlled using a controller $U(s) = G_c(s)E(s)$, as shown in Figure 16.1.

   (a) A proportional control $G_c(s) = K_p$ is proposed. Determine the range of proportional gain, K_p over which the closed-loop system is asymptotically stable.
   (b) What is the system type with proportional control?
   (c) What are the steady-state errors of the closed-loop system to unit step and ramp inputs when $K_p = \frac{1}{2}$? What are they when $K_p = 2$?

2. A plant has transfer function

$$G_p(s) = \frac{Y(s)}{U(s)} = \frac{s+1}{s^2\,(s^2 + 2s + 2)}.$$

   The plant is to be controlled using a controller $U(s) = G_c(s)E(s)$, as shown in Figure 16.1.

   (a) Determine the closed-loop transfer function $\frac{Y(s)}{R(s)}$, when the control is proportional, that is when $G_c(s) = K_p$.
   (b) Using a Routh analysis, determine if it is possible to asymptotically stabilize the system using proportional control only. If it can, determine the range of K_p that makes it asymptotically stable.
   (c) Determine the closed-loop transfer function $\frac{Y(s)}{R(s)}$ when the control is proportional-derivative, that is when $G_c(s) = K_p + sK_d$.
   (d) Using a Routh analysis, determine conditions on the proportional gain K_p to asymptotically stabilize the system if the derivative gain is $K_d = 1$.

3. Consider the small-angle roll-yaw equations for a nominally non-spinning dual-spin satellite,

$$I_x\ddot{\phi} - h_s\dot{\psi} = T_x,$$
$$I_z\ddot{\psi} + h_s\dot{\phi} = T_z.$$

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with proportional-derivative control for each axis

$$T_x = -I_x\left(k_p\phi + k_d\dot{\phi}\right),$$
$$T_z = -I_z\left(k_p\psi + k_d\dot{\psi}\right).$$

(a) Show that the closed-loop roll and yaw poles satisfy the characteristic equation

$$s^4 + 2k_d s^3 + \left(2k_p + k_d^2 + \frac{h_s^2}{I_x I_z}\right)s^2 + 2k_p k_d s + k_p^2 = 0.$$

(b) Using a Routh analysis, show that if $k_p > 0$ and $k_d > 0$, the closed-loop roll and yaw equations are asymptotically stable.

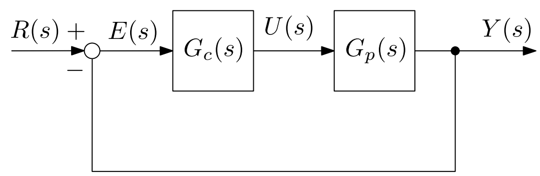

Figure 16.1 Feedback control system

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Chapter 19 Exercises

1. Consider the feedback control system in Figure 17.1. Assuming proportional control $G_c(s) = K$ with $K \geq 0$, sketch the root locus plots when the plant transfer functions are:

   (a)
   $$G_p(s) = \frac{10}{(s+1)(s+2)(s+3)}$$

   (b)
   $$G_p(s) = \frac{1}{(s+1)(s^2+2s+2)}$$

   (c)
   $$G_p(s) = \frac{s+2}{(s+1)(s+3)(s+4)}$$

   (d)
   $$G_p(s) = \frac{s+1}{s^2+2s+2}$$

   (e)
   $$G_p(s) = \frac{s+3}{(s+1)(s+2)}.$$

   (f)
   $$G_p(s) = \frac{1}{(s+1)(s+2)(s^2+2s+2)}$$

   (g)
   $$G_p(s) = \frac{s^2+4s+5}{(s+1)(s+2)(s^2+2s+2)}$$

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Note: If there are asymptotes, compute the angles and center. Do not compute breakaway and break-in points or imaginary axis crossings.

2. Consider the small-angle roll-yaw equations for a nominally non-spinning dual-spin satellite,

$$I_x\ddot{\phi} - h_s\dot{\psi} = T_x,$$
$$I_z\ddot{\psi} + h_s\dot{\phi} = T_z.$$

with proportional-derivative control for each axis

$$T_x = -I_x\left(k_p\phi + k_d\dot{\phi}\right),$$
$$T_z = -I_z\left(k_p\psi + k_d\dot{\psi}\right).$$

As found in Question 3, Chapter 18, the closed-loop roll and yaw poles satisfy the characteristic equation

$$s^4 + 2k_ds^3 + \left(2k_p + k_d^2 + \frac{h_s^2}{I_xI_z}\right)s^2 + 2k_pk_ds + k_p^2 = 0.$$

(a) Show that the characteristic equation can be rewritten equivalently as

$$1 + \frac{h_s^2}{I_xI_z}\frac{s^2}{\left(s^2 + k_ds + k_p\right)^2} = 0.$$

(b) Assume that the gains k_p and k_d are chosen according to $k_d = 2\zeta\omega_n$ and $k_p = \omega_n^2$, where $0 < \zeta < 1$ is the desired damping ratio and ω_n is the desired undamped natural frequency. Sketch a root-locus plot for the closed-loop poles as the wheel angular momentum changes, that is, for $\frac{h_s^2}{I_xI_z} \geq 0$. Compute the asymptote angles and center, but do not compute any other details.

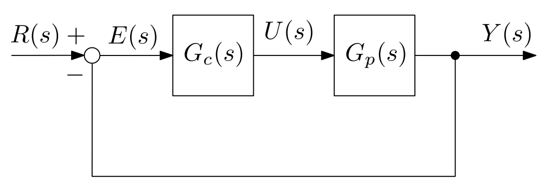

Figure 17.1 Feedback control system

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Chapter 20 Exercises

Consider the feedback control system in Figure 18.1.

1. The plant has transfer function

$$G_p(s) = \frac{a}{(s+1)(s+2)(s+3)},$$

   where $a > 0$. Suggest a controller $G_c(s)$ if the closed-loop system is to exhibit no oscillatory behavior (no complex closed-loop poles) regardless of the value of a.

2. Consider again the plant with transfer function

$$G_p(s) = \frac{a}{(s+1)(s+2)(s+3)},$$

   where $a > 0$. What are the minimum number of zeros that must be contained in the controller $G_c(s)$ such that the closed-loop system will never go unstable, regardless of the value of a? Where must it be placed?

3. The plant has transfer function

$$G_p(s) = \frac{1}{s+1}.$$

   The control law $G_c(s)$ is to be designed such that the closed-loop system has dominant poles at $s = -3 \pm j3$, and a zero steady-state error to a step input $r(t) = 1$.

   (a) Can the control objective be met with pure integral control, $G_c(s) = \frac{K}{s}$? Hint: do the desired dominant poles lie on the root locus?

   (b) Design a combined lead-integral control

$$G_c(s) = K\frac{\left(s + \frac{1}{T}\right)}{s\left(s + \frac{1}{\alpha T}\right)},$$

   with $T > 0$ and $0 < \alpha < 1$. such that the closed-loop requirements are met. You may take $T = \frac{1}{3}$. Hint: you may augment the plant with the integrator.

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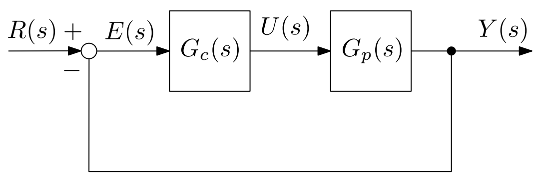

Figure 18.1 Feedback control system

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Bias Momentum Control Design Exercises - Chapters 17 to 20

The following exercises will involve the design of an attitude control system for a bias momentum satellite, and ties together the subjects of Chapters 17 to 20.

Background
A bias momentum satellite is an earth-pointing dual-spin satellite, with attitude controlled relative to an orbiting reference frame $\mathcal{F}_o$ as shown in Figure 19.1. The momentum wheel spin axis is aligned with the orbit normal (an inertially fixed direction for a two-body orbit), making it a pitch wheel. We describe the spacecraft attitude relative to the orbiting frame by a yaw-pitch-roll (3-2-1) Euler sequence, where ϕ, θ and ψ denote the roll, pitch and yaw angles respectively. For small rotations, ϕ constitutes a rotation about $\vec{\mathbf{x}}_o$, θ is a rotation about $\vec{\mathbf{y}}_o$ and ψ is a rotation about $\vec{\mathbf{z}}_o$ (as shown in Figure 19.1).

For many bias momentum satellites, the only attitude sensor is an Earth sensor. An Earth sensor gives the direction of the center of earth from the spacecraft ($-\vec{\mathbf{r}}$) in body coordinates. The roll and pitch angles ϕ and θ can be determined from this measurement. However, as can be seen from Figure 19.1, if the spacecraft undergoes a pure yaw rotation, the rotation is about $-\vec{\mathbf{r}}$, so it does not change in body coordinates. Therefore, the yaw angle cannot be measured by an earth sensor.

Bias momentum stabilization for an earth pointing satellite has the following benefits:

- Coupling between the roll and yaw axes due to the bias momentum. This means that the yaw angle can be stabilized without using a yaw sensor.
- Gyroscopic stability of the roll and yaw angles due to the bias momentum.
- The pitch wheel can be used to control the pitch angle (by changing it's speed)
- The pitch wheel can be used to control roll and yaw angles (by making small changes to the spin axis (aka control moment gyro))

The equations of motion are given by

$$\mathbf{I}\dot{\boldsymbol{\omega}} + \dot{h}_s\mathbf{a} + \boldsymbol{\omega}^\times\left[\mathbf{I}\boldsymbol{\omega} + h_s\mathbf{a}\right] = \mathbf{T}_c + \mathbf{T}_d + \mathbf{T}_g,$$

where $\mathbf{T}_c$ is the control torque, $\mathbf{T}_d$ is the external disturbance torque, $\mathbf{T}_g$ is the gravity-gradient torque, and all other symbols have the same meaning as in the book. We take the

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wheel spin-axis to be

$$\mathbf{a} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}.$$

Assuming that the body frame is a principal axes frame, the spacecraft is in a circular orbit, and that the angles and rates are small, the equations of motion relative to the orbiting frame become

$$\begin{aligned} I_x\ddot{\phi} - [(I_x + I_z - I_y)\omega_o - h_s]\,\dot{\psi} + \left[4(I_y - I_z)\omega_o^2 + \omega_o h_s\right]\phi &= T_{cx} + T_{dx}, \\ I_y\ddot{\theta} + 3(I_x - I_z)\omega_o^2\theta &= T_{cy} + T_{dy}, \\ I_z\ddot{\psi} + [(I_x + I_z - I_y)\omega_o - h_s]\,\dot{\phi} + \left[(I_y - I_x)\omega_o^2 + \omega_o h_s\right]\psi &= T_{cz} + T_{dz}, \end{aligned}$$

where $T_{cy} = -\dot{h}_s$ if the momentum wheel is used to control the pitch, and $\omega_o = \sqrt{\mu/r^3}$ is the orbital angular rate. These equations may be compared to equations (16.14) to (16.16) in the book, for a gravity-gradient stabilized spacecraft.

Finally, we make the approximation that $|I_x\omega_o|, |I_y\omega_o|, |I_z\omega_o| << |h_s|$, greatly simplifying the equations of motion to

$$\begin{aligned} I_x\ddot{\phi} + h_s\dot{\psi} + \omega_o h_s\phi &= T_{cx} + T_{dx}, \\ I_y\ddot{\theta} + k_g\theta &= T_{cy} + T_{dy}, \\ I_z\ddot{\psi} - h_s\dot{\phi} + \omega_o h_s\psi &= T_{cz} + T_{dz}, \end{aligned}$$

where $k_g = 3(I_x - I_z)\omega_o^2$. It can be seen that the pitch equation is decoupled from the roll and yaw equations.

Spacecraft and Orbital Parameters

The spacecraft principal inertias are $I_x = 0.4$ kg·m^2, $I_y = 0.5$ kg·m^2, $I_z = 0.6$ kg·m^2. The spacecraft is in a circular orbit with altitude 600 km, such that the orbital angular rate is $\omega_o = 0.001083$ rad/s. The expected maximum disturbance torques are $T_{dx,max} = T_{dy,max} = T_{dz,max} = 5 \times 10^{-6}$ Nm.

Control System Requirements

- The maximum allowable steady-state errors in response to constant disturbances are $\phi_{ss} \leq 0.1$ deg, $\theta_{ss} \leq 0.1$ deg and $\psi_{ss} \leq 4$ deg.
- The closed-loop poles for the roll/yaw loop are to have damping ratios $\zeta = 0.7$.

The spacecraft is expected to be capable of making pitch maneuvers, with transient specification for a pitch step response:

- Settling-time, $t_s \leq 200$ seconds.
- Percent overshoot, $M_p \leq 30$ %.

You will need the Earth's gravitational constant

$$\mu = 3.986 \times 10^5 \text{ km}^3/\text{s}^2,$$

and the equatorial radius of the Earth

$$R_e = 6378.1363\,\text{km}.$$

Exercises

1. The pitch controller is to be of the modified PD type

$$T_{cy}(t) = k_p^p(\theta_d - \theta) - k_d^p\dot{\theta},$$

where θ_d is the desired pitch angle.

(a) Find the plant transfer function for the pitch angle.

(b) What is the system type for the pitch loop with the above control law? What are the implications on the steady-state pitch error for a pitch step command in the absence of a disturbance torque?

(c) Obtain an expression for the steady-state pitch error due to a pitch disturbance torque $T_{dy}(s) = \frac{\bar{T}_{dy}}{s}$.

(d) Find the closed-loop transfer function from desired pitch angle θ_d to actual pitch angle θ.

(e) Design the pitch control gains such that the closed-loop pitch specifications are met. Note: keep the proportional gain k_p^p small.

2. The control law for the roll/yaw loop only makes use of roll measurements, and takes the form

$$\begin{aligned} T_{cx} &= -\left(k_p^{ry}\phi + k_d^{ry}\dot{\phi}\right), \\ T_{cz} &= a\left(k_p^{ry}\phi + k_d^{ry}\dot{\phi}\right), \end{aligned}$$

where $a > 0$, $k_p^{ry} > 0$ and $k_d^{ry} > 0$ are parameters to be determined.

(a) Show that the closed-loop roll/yaw equations after taking Laplace transforms are given by

$$\begin{bmatrix} s^2I_x + sk_d^{ry} + \omega_o h_s + k_p^{ry} & sh_s \\ -\left(s(h_s + ak_d^{ry}) + ak_p^{ry}\right) & s^2I_z + \omega_o h_s \end{bmatrix} \begin{bmatrix} \hat{\phi}(s) \\ \hat{\psi}(s) \end{bmatrix} = \begin{bmatrix} \hat{T}_{dx}(s) \\ \hat{T}_{dz}(s) \end{bmatrix}.$$

(b) Show that the solution for the roll and yaw angles satisfies

$$\begin{bmatrix} \hat{\phi}(s) \\ \hat{\psi}(s) \end{bmatrix} = \frac{1}{\Delta(s)} \begin{bmatrix} s^2I_z + \omega_o h_s & -sh_s \\ s(h_s + ak_d^{ry}) + ak_p^{ry} & s^2I_x + sk_d^{ry} + \omega_o h_s + k_p^{ry} \end{bmatrix} \begin{bmatrix} \hat{T}_{dx}(s) \\ \hat{T}_{dz}(s) \end{bmatrix},$$

where

$$\begin{aligned} \Delta(s) = {} & I_xI_zs^4 + I_zk_d^{ry}s^3 + \left[(I_x + I_z)\omega_o h_s + I_zk_p^{ry} + h_s^2 + ak_d^{ry}h_s\right]s^2 \\ & + \left[\omega_o h_s k_d^{ry} + ah_s k_p^{ry}\right]s + \omega_o h_s(\omega_o h_s + k_p^{ry}). \end{aligned}$$

(c) Assuming that the closed-loop system is asymptotically stable ($\Delta(s) = 0$ has roots with negative real parts), show that the steady-state roll and yaw errors for step disturbances $T_{dx}(s) = \frac{\bar{T}_{dx}}{s}$ and $T_{dz}(s) = \frac{\bar{T}_{dz}}{s}$ are

$$\phi_{ss} = \frac{\bar{T}_{dx}}{\omega_o h_s + k_p^{ry}}, \quad \psi_{ss} = \frac{ak_p^{ry}\bar{T}_{dx}}{\omega_o h_s(\omega_o h_s + k_p^{ry})} + \frac{\bar{T}_{dz}}{\omega_o h_s}.$$

(d) Typically, the control system shall be designed such that $a < 1$ and $k_p^{ry} >> \omega_o h_s$. Therefore, the yaw error may be upper-bounded by

$$|\psi_{ss}| \leq \frac{|\bar{T}_{dx}| + |\bar{T}_{dz}|}{\omega_o |h_s|}.$$

Using this approximation, and the expression for ϕ_{ss} from part (c), determine the required bias momentum h_s and the proportional gain k_p^{ry} such that the roll and yaw steady-state specifications are satisfied. Note: take $h_s, k_p^{ry} > 0$.

(e) Having selected h_s and k_p^{ry}, all that remains is to select k_d^{ry} and a such that the closed-loop poles have the required damping ratio. This is done as follows. From part (b), the closed-loop poles satisfy the characteristic equation

$$\Delta(s) = 0.$$

We would like the closed-loop poles to be $s = -\zeta\omega_{n1} \pm j\omega_{n1}\sqrt{1-\zeta^2}, -\zeta\omega_{n2} \pm j\omega_{n2}\sqrt{1-\zeta^2}$, where ζ is the required damping ratio of the poles, and ω_{n1} and ω_{n2} are the undamped natural frequencies of the poles. Therefore, the characteristic equation should have the form

$$(s^2 + 2\zeta\omega_{n1}s + \omega_{n1}^2)(s^2 + 2\zeta\omega_{n2}s + \omega_{n2}^2) = 0.$$

To find the required k_d^{ry} and a, we equate

$$\Delta(s) = I_x I_z (s^2 + 2\zeta\omega_{n1}s + \omega_{n1}^2)(s^2 + 2\zeta\omega_{n2}s + \omega_{n2}^2).$$

Equating the coefficients of powers of s, leads to four equations in four unknowns, namely k_d^{ry}, a, ω_{n1} and ω_{n2} (the damping ration ζ is given). We are only interested in the first two.

i. Show that the solutions for k_d^{ry} and a are

$$k_d^{ry} = \sqrt{\frac{E\,(B - (4\zeta^2 - 2)F)}{EA^2 - C(AF - D)}},$$

$$a = \frac{(AF - D)}{E} k_d^{ry},$$

where

$$A = \frac{1}{2\zeta I_x}, \quad B = \frac{(I_x + I_z)h_s\omega_o}{I_x I_z} + \frac{k_p^{ry}}{I_x} + \frac{h_s^2}{I_x I_z},$$

$$C = \frac{h_s}{I_x I_z}, \quad D = \frac{\omega_o h_s}{2\zeta I_x I_z}, \quad E = \frac{h_s k_p^{ry}}{2\zeta I_x I_z},$$

$$F = \sqrt{\frac{\omega_o h_s(\omega_o h_s + k_p^{ry})}{I_x I_z}}.$$

ii. Find k_d^{ry} and a.

3. The pitch control command obtained in question 1 is to be applied by an actuator with dynamics

$$T_{cy}(s) = \frac{1}{Ts+1}U_c(s),$$

where $T > 0$ is the actuator time constant, $U_c(s)$ is the control torque commanded by the control law given in Question 1, and $T_{cy}(s)$ is the pitch torque applied to the spacecraft by the actuator.

   (a) Show that the characteristic equation for the closed-loop system is

$$I_yTs^3 + I_ys^2 + (k_gT + k_d^p)s + k_g + k_p^p = 0.$$

   (b) Determine the condition that the actuator time-constant T must satisfy for the closed-loop system to be asymptotically stable.

   (c) Sketch a root locus for the closed-loop poles for $\frac{1}{T} \geq 0$. It does not need to be to scale. Do not determine details like breakaway points and imaginary axis crossings. Note: Be mindful of the sign of k_g. What can you conclude from the root locus?

4. The pitch actuator is much slower than you had originally specified as the control system designer, with a time constant $T = 5$ seconds. Is it possible to redesign the PD controller obtained in Question 1 so that the closed-loop system (when actuator dynamics are included) has the dominant poles you obtained in Question 1? If it is possible, redesign the PD controller.

   To simplify the problem, you may set $k_g = 0$. To save some effort in finding the open-loop transfer function, note that the characteristic equation in part (a) of Question 3 may be rewritten as (with $k_g = 0$)

$$1 + \frac{k_d^p}{I_yT}\frac{(s + \frac{k_p^p}{k_d^p})}{s^2(s + \frac{1}{T})} = 0.$$

   From this, we can identify

$$G_o(s) = \frac{k_d^p}{I_yT}\frac{(s + \frac{k_p^p}{k_d^p})}{s^2(s + \frac{1}{T})},$$

   as the open-loop transfer function, $\frac{k_d^p}{I_yT}$ as the open-loop gain, and $s = -\frac{k_p^p}{k_d^p}$ as the PD zero.

   Hint for solving the problem: Check the maximum possible angle $\angle\left(s + \frac{k_p^p}{k_d^p}\right)$ that the PD zero can provide, and compare it to what is required.

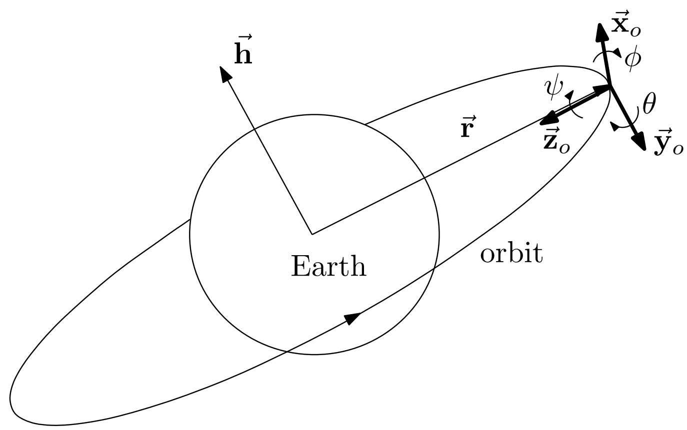

Figure 19.1 Orbiting Reference Frame

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Chapter 21 Exercises

1. Sketch the frequency responses (Bode plots) for the systems with transfer functions:

   (a)
   $$G(s) = \frac{10}{(s+0.1)(s+1)(s+10)}$$

   (b)
   $$G(s) = \frac{s+1}{(s+0.1)(s+10)}$$

   (c)
   $$G(s) = \frac{(s+1)(s+10)}{(s+0.1)(s+100)}.$$

2. If a filter $H(s)$ is to be designed such that low frequency inputs are passed, but high frequency inputs are blocked, what is the requirement on the number of zeros and poles of $H(s)$?

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Chapter 22 Exercises

1. Sketch the polar plots for the systems with transfer functions:

   (a)

$$G(s) = \frac{5}{(s+1)(s+2)(s+0.5)}$$

   (b)

$$G(s) = \frac{s+2}{(s+1)(s+3)}$$

   (c)

$$G(s) = \frac{s+2}{s(s+1)(s+3)}$$

2. Consider the feedback system with open-loop transfer function

$$G_o(s) = \frac{K}{(s+3)(s+2)(s-0.5)}.$$

   Using the Nyquist criterion, determine whether or not the closed-loop system is stable or unstable in each of the following cases.

   (a) $K = 1$ (see Figure 21.1)
   (b) $K = 10$ (see Figure 21.2)
   (c) $K = 100$ (see Figure 21.3)
   (d) Sketch a root-locus for the above system. Can you explain the results in parts (a), (b) and (c) using the root locus?

3. Consider the spacecraft attitude control with output filtering as shown in Figure 21.4. Sketch the Nyquist plot in each of the following two cases. Using the Nyquist criterion, verify that the closed-loop system is stable, and find the gain margins.

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(a) PD control

$$G_c(s) = K_p + K_d s.$$

Low pass filter

$$H(s) = \frac{1}{Ts+1}$$

System parameters:

$$I = 1, \;\; T = 1, \;\; K_p = 0.01, \;\; K_d = 0.1.$$

(b) PID control

$$G_c(s) = K_p + K_d s + \frac{K_i}{s}.$$

Double low pass filter

$$H(s) = \frac{1}{(Ts+1)^2}$$

System parameters:

$$I = 1, \;\; T = 1, \;\; K_p = 0.01, \;\; K_d = 0.1, \;\; K_i = 10^{-4}.$$

4. Sketch the Bode plot corresponding to part (a) of Question 3. Estimate the phase margins and the allowable time-delay in the feedback loop.
5. Figure 21.5 shows the Bode plot corresponding to part (b) of Question 3. Estimate all stability margins.

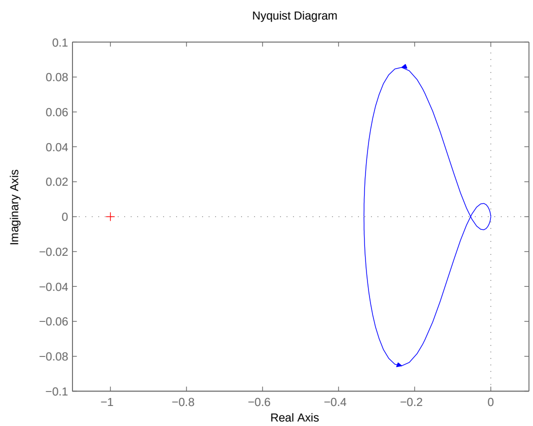

Figure 21.1 Nyquist plot for Question 2(a), $K = 1$

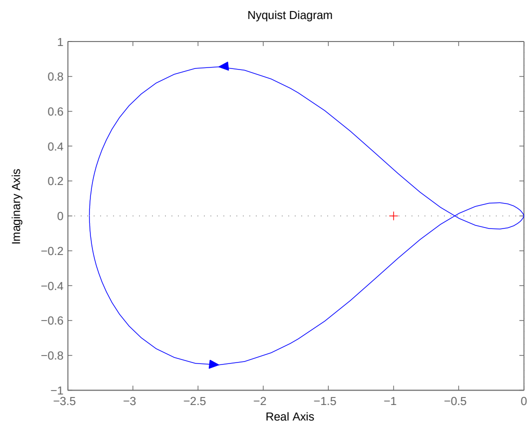

Figure 21.2 Nyquist plot for Question 2(b), $K = 10$

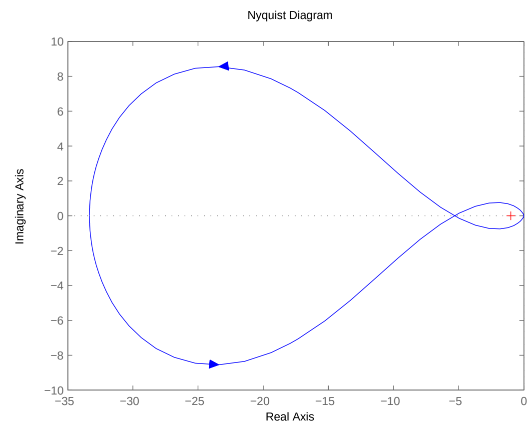

Figure 21.3 Nyquist plot for Question 2(c), $K = 100$

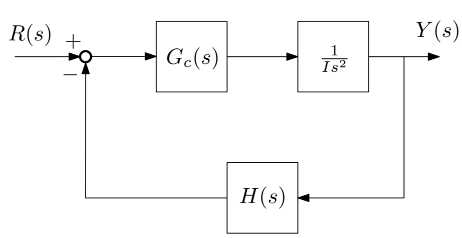

Figure 21.4 Spacecraft feedback attitude control with output filtering

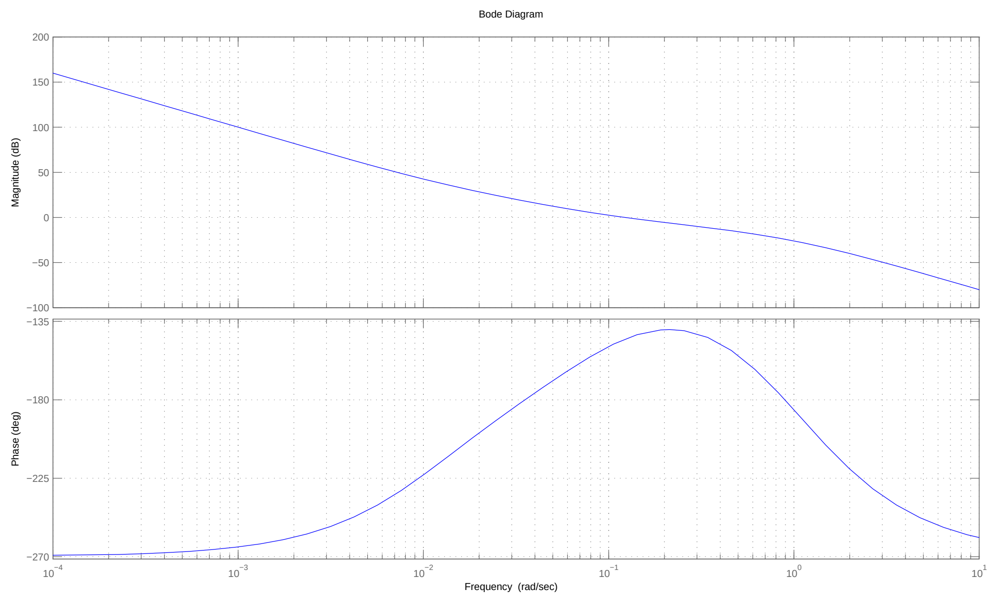

Figure 21.5 Bode plot for Question 3(b)

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Chapter 25 Exercises

1. A spacecraft is orbiting the Earth, as shown in Figure 22.1. As shown in the figure, at this particular location in the orbit, the earth-pointing and sun-pointing vectors are given in the ECI frame as

$$\vec{\mathbf{n}}_e = \vec{\mathcal{F}}_G^T \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{\mathbf{n}}_s = \vec{\mathcal{F}}_G^T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Also, as shown in Figure 22.1, the spacecraft attitude is obtained by a rotation of 45^o about $\vec{\mathbf{z}}_G$.

(a) Determine $\mathbf{C}_{bG}$.

(b) Determine the coordinates of the earth and sun vectors $\vec{\mathbf{n}}_e$ and $\vec{\mathbf{n}}_s$ respectively, in the spacecraft body frame, $\mathcal{F}_b$.

(c) Determine the coordinates of the unit vectors defining the intermediate frame $\mathcal{F}_t$ (see Section 25.2.4 in the book), in the spacecraft body frame, and in the ECI frame (as in the TRIAD method).

(d) Obtain the rotation matrices $\mathbf{C}_{bt}$ and $\mathbf{C}_{Gt}$ as in Section 25.2.4 in the book.

(e) Using your solution to part (c) above, compute $\mathbf{C}_{bG}$ using the TRIAD method. Compare this with the result in part (a).

(f) Using the measured vectors obtained in part (b), compute $\mathbf{C}_{bG}$ using the Davenport q-method and QUEST. Verify that you obtain the same result as in part (d). Note that it should be exactly the same, since no measurement noise has been added.

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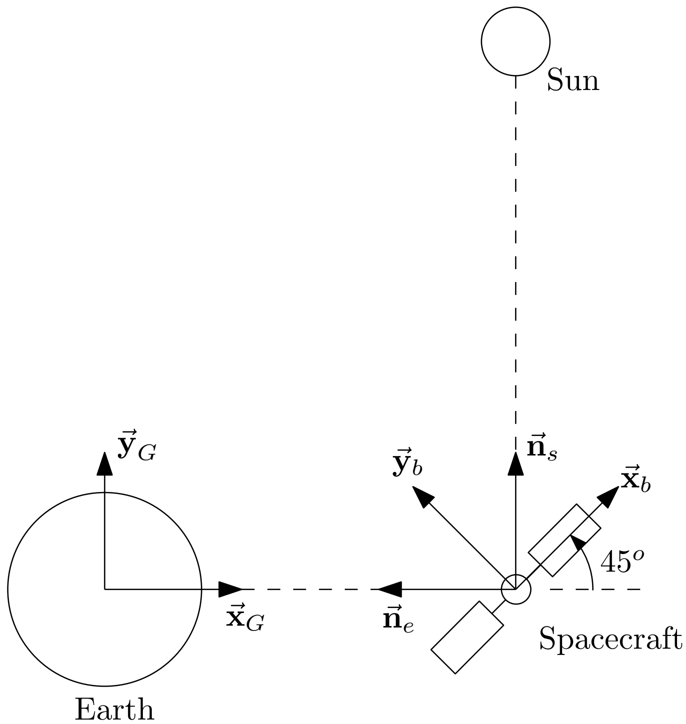

Figure 22.1 Attitude determination question