International Journal of General Systems, August 2003 Vol. 32 (4), pp. 343–360

Taylor & Francis
Taylor & Francis Group

OPENINGS AND CLOSURES OF FUZZY PREORDERINGS: THEORETICAL BASICS AND APPLICATIONS TO FUZZY RULE-BASED SYSTEMS

ULRICH BODENHOFER[a,*,†], MARTINE DE COCK[b,‡] and ETIENNE E. KERRE[b]

[a]*Software Competence Center Hagenberg, A-4232 Hagenberg, Austria;* [b]*Department of Applied Mathematics and Computer Science, Ghent University, B-9000 Gent, Belgium*

(Received 10 December 2002; In final form 4 March 2003)

The purpose of this paper is two-fold. Firstly, a general concept of closedness of fuzzy sets under fuzzy preorderings is proposed and investigated along with the corresponding opening and closure operators. Secondly, the practical impact of this notion is demonstrated by applying it to the analysis of ordering-based modifiers.

Keywords: Closedness; Fuzzy preordering; Fuzzy relation; Linguistic modifier

1. INTRODUCTION

Images of fuzzy sets under fuzzy relations have been investigated mainly in two contexts: on the one hand, mostly under the term "full image" (Gottwald, 1993), they can be regarded as very general tools for fuzzy inference, leading to the so-called "compositional rule of inference" (Gottwald, 1993; Bauer *et al.*, 1995). The theory of fuzzy relational equations makes direct use of this fundamental principle, too (Sanchez, 1984; Miyakoshi and Shimbo, 1985; di Nola *et al.*, 1991; Gottwald, 1993; De Baets, 2000). On the other hand, under the term "extensional hull", the image of a fuzzy set under a fuzzy equivalence relation yields the smallest fuzzy superset which is "closed" under the relation. This closedness property is usually called "extensionality" (Kruse *et al.*, 1994). The concepts of extensionality and extensional hulls have turned out to be extremely useful, in particular when the analysis and interpretation of fuzzy partitions and controllers is concerned (Klawonn, 1993; Klawonn and Kruse, 1993; Klawonn and Castro, 1995; Klawonn *et al.*, 1995).

*Corresponding author. Fax: +43-7236-3343-888. E-mail: ulrich.bodenhofer@scch.at

†This work was supported by the K_{plus} Competence Center Program which is funded by the Austrian Government, the Province of Upper Austria, and the Chamber of Commerce of Upper Austria.

‡This work was supported by the Fund for Scientific Research of Flanders (FWO). Parts of this work were done during a COST Action 15 short-term scientific mission of Martine De Cock at the Fuzzy Logic Laboratory Linz-Hagenberg in April 2000.

ISSN 0308-1079 print/ISSN 1563-5104 online
DOI: 10.1080/0308107031000135026

In the first part of this paper, we would like to generalize the concept of extensionality to arbitrary reflexive and T-transitive fuzzy relations—so-called fuzzy preorderings. Based on this general and powerful notion, smallest closed supersets and largest closed fuzzy subsets will be studied. It will turn out that again the two very common concepts of images under fuzzy relations are obtained.

The second part is devoted to a new view on these images of fuzzy sets under fuzzy relations—making use of the results on closedness and the corresponding closure operator, we are able to provide a new framework for defining the ordering-based modifiers "at least" and "at most".

2. PRELIMINARIES

Throughout the whole paper, we will not explicitly distinguish between fuzzy sets and their corresponding membership functions. Consequently, uppercase letters will be used for both synonymously. The set of all fuzzy sets on a domain X will be denoted with $\mathscr{F}(X)$.

For intersecting and unifying fuzzy sets, we will suffice with minimum and maximum:

$$(A \cap B)(x) = \min(A(x), B(x))$$
$$(A \cup B)(x) = \max(A(x), B(x)).$$

In general, aside from intersections and unions of fuzzy sets, triangular norms (Klement *et al.*, 2000) will be considered as our standard models of conjunction.

Definition 1 A *triangular norm* (*t-norm* for short) is an associative, commutative, and non-decreasing binary operation on the unit interval (i.e. a $[0,1]^2 \to [0,1]$ mapping) which has 1 as neutral element.

In this paper, unless stated otherwise, assume that T denotes a left-continuous triangular norm, i.e. a t-norm whose partial mappings $T(x,.)$ and $T(.,x)$ are left-continuous.

Correspondingly, so-called residual implications are used as the concepts of logical implication. In order to provide the reader with the basic properties of residual implications, we will now briefly recall them. For proofs, the reader is referred to the literature (Gottwald, 1993; Hájek, 1998).

Definition 2 A mapping $R : [0,1]^2 \to [0,1]$ is called *residual implication (residuum)* of T if and only if the following equivalence is fulfilled for all $x, y, z \in [0,1]$:

$$T(x,y) \leq z \Leftrightarrow x \leq R(y,z). \tag{1}$$

Lemma 3 For any left-continuous t-norm T, there exists a unique residuum $\vec{T}$ given as

$$\vec{T}(x,y) = \sup\{u \in [0,1] | T(u,x) \leq y\}.$$

Only briefly, we mention the concept of logical equivalence induced by a left-continuous t-norm.

Definition 4 The *biimplication* $\overleftrightarrow{T}$ of T is defined as

$$\overleftrightarrow{T}(x,y) = \min(\vec{T}(x,y), \vec{T}(y,x)).$$

For elementary properties of the fuzzy logical operations $\vec{T}$ and $\overleftrightarrow{T}$, the reader is referred to the relevant literature (Gottwald, 1993; Fodor and Roubens, 1994; Hájek, 1998; Klement *et al.*, 2000; Gottwald, 2001).

In this paper, we will solely consider *binary fuzzy relations*, i.e. fuzzy sets on a product space $X^2 = X \times X$, where X is an arbitrary crisp set. Let us recall some basics of binary fuzzy relations which will be important in the remaining paper.

DEFINITION 5 A binary fuzzy relation $R : X^2 \longrightarrow [0, 1]$ is called

1. *reflexive* if and only if $\forall x \in X : R(x, x) = 1$,
2. *symmetric* if and only if $\forall x, y \in X : R(x, y) = R(y, x)$,
3. *T-transitive* if and only if $\forall x, y, z \in X : T(R(x, y), R(y, z)) \leq R(x, z)$,
4. *strongly complete* if and only if $\forall x, y \in X : \max(R(x, y), R(y, x)) = 1$.

DEFINITION 6 A reflexive and T-transitive fuzzy relation is called *fuzzy preordering* with respect to a t-norm T, short *T-preordering*. A symmetric T-preordering is called *fuzzy equivalence relation* with respect to T, short *T-equivalence*.

DEFINITION 7 Consider an arbitrary fuzzy set $A \in \mathscr{F}(X)$. The *full image* of A under R, denoted $R \uparrow A$ and its dual $R \downarrow A$ are defined as

$$R \uparrow A(x) = \sup\{T(A(y), R(y, x)) | y \in X\},$$

$$R \downarrow A(x) = \inf\{\vec{T}(R(x, y), A(y)) | y \in X\}.$$

Note that $R \uparrow A$ has sometimes been called *direct image* (Kerre, 1993) or *conditioned fuzzy set* (Bellman and Zadeh, 1970), while the names *superdirect image* (Kerre, 1993) and *α-operation* (Sanchez, 1984) have already occurred for $R \downarrow A$.

LEMMA 8 The following propositions hold for all $A, B \in \mathscr{F}(X)$ and all binary fuzzy relations $R, S \in \mathscr{F}(X^2)$:

1. $A \subseteq B \Rightarrow R \uparrow A \subseteq R \uparrow B$.
2. $A \subseteq B \Rightarrow R \downarrow A \subseteq R \downarrow B$.
3. $R \subseteq S \Rightarrow R \uparrow A \subseteq S \uparrow A$.
4. $R \subseteq S \Rightarrow R \downarrow A \supseteq S \downarrow A$.
5. $R \uparrow (A \cup B) = R \uparrow A \cup R \uparrow B$.
6. $R \downarrow (A \cap B) = R \downarrow A \cap R \downarrow B$.
7. $(R \cup S) \uparrow A = R \uparrow A \cup S \uparrow A$.
8. $(R \cup S) \downarrow A = R \downarrow A \cap S \downarrow A$.

Proof These propositions follow directly from the monotonicity properties of triangular norms and their residual implications (see Gottwald, 1993; Kerre, 1993 for more detailed proofs of 1–5). □

3. THE BASIC CONCEPT OF CLOSEDNESS AND ITS PROPERTIES

The closedness with respect to a fuzzy equivalence relation, often called "extensionality" (Klawonn and Kruse, 1993; Kruse *et al.*, 1994), and the induced closure operators are rather well-studied matters (Jacas, 1988; Kruse *et al.*, 1994; Jacas and Recasens, 1995; Klawonn and Castro, 1995; Boixader *et al.*, 2000; Bělohlávek, 2002). We will now define appropriate generalizations which do not assume symmetry.

Throughout this section, assume that R denotes a fuzzy preordering with respect to some left-continuous t-norm T.

DEFINITION 9 A fuzzy set $A \in \mathscr{F}(X)$ is called *closed* with respect to R, for brevity *R-closed*, if and only if, for all $x, y \in X$,

$$T(A(x), R(x, y)) \leq A(y).$$

In words, the meaning of closedness is that, for any element x of A, all y are also contained in A which are in relation to x.

Example 10 Let us briefly mention a few simple examples which demonstrate the variety of properties that can be expressed by means of closedness.

1. The universe X and the empty set $\emptyset$ are both closed with respect to any fuzzy preordering on X.
2. A crisp set is closed with respect to a crisp equivalence relation if and only if it can be represented as the union of equivalence classes.
3. A crisp set is closed with respect to a crisp ordering if and only if it is an up-set.
4. A fuzzy set is closed with respect to a crisp ordering $\preceq$ if and only if its membership function is non-decreasing with respect to $\preceq$.
5. If a fuzzy equivalence relation is considered, closedness is equivalent to extensionality (Klawonn and Kruse, 1993; Kruse *et al.*, 1994).

As immediate consequences of the residuation principle (1), we can derive equivalent formulations of R-closedness, which will be helpful later.

LEMMA 11 For any fuzzy set $A \in \mathscr{F}(X)$, R-closedness is equivalent to each of the following two propositions:

$$\forall x, y \in X : R(x, y) \leq \vec{T}(A(x), A(y)) \tag{2}$$

$$\forall x, y \in X : A(x) \leq \vec{T}(R(x, y), A(y)). \tag{3}$$

If R is, in addition, symmetric, A is R-closed if and only if the following inequality holds:

$$\forall x, y \in X : R(x, y) \leq \overleftrightarrow{T}(A(x), A(y)). \tag{4}$$

Proof The equivalence of R-closedness to formulae (2) and (3) follows directly from the definition of residual implications.

On the other hand, if we swap x and y in the definition of R-closedness, we obtain

$$T(A(y), R(y, x)) \leq A(x)$$

which is, due to Eq. (2), equivalent to

$$R(y, x) \leq \vec{T}(A(y), A(x)). \tag{5}$$

If we assume that R is symmetric and taking Eqs. (2) and (5) into account, we obtain

$$R(x, y) \leq \min(\vec{T}(A(x), A(y)), \vec{T}(A(y), A(x))) = \overleftrightarrow{T}(A(x), A(y)).$$

The opposite direction, i.e. that Eq. (4) implies R-closedness, is trivial if we consider Eq. (2) and the definition of the biimplication. $\square$

In particular, Eq. (2) has a trivial consequence we will need very often in the following.

COROLLARY 12 Let Q be another T-preordering. If a fuzzy set A is R-closed and $Q \subseteq R$, then A is also Q-closed.

The next result clarifies in which way closedness is preserved for finite and infinite unions and interactions (with respect to max and min, respectively).

LEMMA 13 For any family of R-closed fuzzy sets $(A_i)_{i\in I}$, the fuzzy sets defined by

$$\sup_{i\in I} A_i(x) \quad \text{and} \quad \inf_{i\in I} A_i(x)$$

are also R-closed. If the index set I is finite, the same holds even if T is not left-continuous.

Proof For arbitrary $x, y \in X$, we know that

$$T(A_i(x), R(x, y)) \leq A_i(y)$$

holds for all $i \in I$. Due to the monotonicity of t-norms, R-closedness is then preserved for finite intersections and unions (with respect to minimum and maximum, respectively).The same even holds for infinite intersections if we take the following into account (basic consequence of the monotonicity of t-norms):

$$T\left(\inf_{i\in I} u_i, v\right) \leq \inf_{i\in I} T(u_i, v).$$

For infinite unions, left-continuity has to be fulfilled:

$$T\left(\sup_{i\in I} A_i(x), R(x, y)\right) = \sup_{i\in I} T(A_i(x), R(x, y)) \leq \sup_{i\in I} A_i(y).$$

□

Nonchalantly speaking, Corollary 12 has shown that the smaller a fuzzy preordering R is, the easier fuzzy sets are R-closed. The next theorem gives a unique characterization of how large a relation R may be such that a given family of fuzzy sets is still R-closed.

THEOREM 14 Consider an arbitrary family of fuzzy sets $\tilde{A} = (A_i)_{i\in I}$. Then

$$R_{\tilde{A}}(x, y) = \inf\{\vec{T}(A_i(x), A_i(y)) | i \in I\}$$

is a T-preordering which is, in addition, the largest binary fuzzy relation R such that all A_i are R-closed. Furthermore,

$$R'_{\tilde{A}}(x, y) = \inf\{\overleftrightarrow{T}(A_i(x), A_i(y)) | i \in I\} \tag{6}$$

is a T-equivalence and the largest symmetric binary fuzzy relation R such that all A_i are R-closed.

Proof Reflexivity and T-transitivity of $R_{\tilde{A}}$ follow from basic properties of residual implications (see Valverde (1985) for more details). Analogously, reflexivity, symmetry, and T-transitivity of $R'_{\tilde{A}}$ follow from elementary properties of the biimplication $\overleftrightarrow{T}$. Closedness and maximality of both relations follow immediately from Lemma 11, formulae (2) and (4), respectively. □

Note that the construction (6) directly corresponds to the fundamental representation theorem for fuzzy preorderings by Valverde (1985). Theorem 14 shows that this representation naturally connects to the closedness property.

4. OPENING AND CLOSURE OPERATORS

Now we can turn to our actual objects of study—opening and closure operators induced by fuzzy preorderings. We will soon see that the two image operators $R\uparrow$ and $R\downarrow$ play a central role; so, let us start to investigate their properties in terms of closedness. Again, we make the convention that R denotes a T-preordering on some fixed domain X.

PROPOSITION 15 All images $R\uparrow A$ and $R\downarrow A$ are R-closed.

Proof For proving that $R\uparrow A$ is R-closed, consider the left-continuity of T and the T-transitivity of R:

$$\begin{aligned} T(R\uparrow A(x), R(x,y)) &= T(\sup\{T(A(z), R(z,x)) \mid z \in X\}, R(x,y)) \\ &= \sup\{T(T(A(z), R(z,x)), R(x,y)) \mid z \in X\} \\ &= \sup\{T(A(z), T(R(z,x), R(x,y))) \mid z \in X\} \\ &\leq \sup\{T(A(z), R(z,y)) \mid z \in X\} \\ &= R\uparrow A(y). \end{aligned}$$

If we take T-transitivity of R, the monotonicity properties of $\vec{T}$ and the well-known inequality $\vec{T}(T(x,y),z) \leq \vec{T}(x, \vec{T}(y,z))$ into account, we obtain

$$\begin{aligned} R\downarrow A(x) &= \inf\{\vec{T}(R(x,z), A(z)) \mid z \in X\} \\ &\leq \inf\{\vec{T}(T(R(x,y), R(y,z)), A(z)) \mid z \in X\} \\ &\leq \inf\{\vec{T}(R(x,y), \vec{T}(R(y,z), A(z))) \mid z \in X\} \\ &\leq \vec{T}(R(x,y), \inf\{\vec{T}(R(y,z), A(z)) \mid z \in X\}) \\ &= \vec{T}(R(x,y), R\downarrow A(y)) \end{aligned}$$

which is, by Lemma 11, Eq. (3), a sufficient condition for R-closedness. □

THEOREM 16 For any $A \in \mathscr{F}(X)$, $R\uparrow A$ is the smallest R-closed fuzzy superset of A and $R\downarrow A$ is the largest R-closed fuzzy subset.

Proof From Proposition 15, we know that $R\uparrow A$ and $R\downarrow A$ are R-closed.
The inclusion properties can be proved as follows:

$$\begin{aligned} R\downarrow A(x) &= \inf\{\vec{T}(R(x,y), A(y)) \mid y \in X\} \\ &\leq \vec{T}(R(x,x), A(x)) = \vec{T}(1, A(x)) = A(x) \\ &= T(A(x), 1) = T(A(x), R(x,x)) \\ &\leq \sup\{T(A(y), R(y,x)) \mid y \in X\} \\ &= R\uparrow A(x). \end{aligned}$$

It remains to show minimality/maximality. Suppose B is an arbitrary R-closed fuzzy superset of A. Then we obtain, for all $x, y \in X$,

$$B(x) \geq T(B(y), R(y,x)) \geq T(A(y), R(y,x)).$$

Hence, we can even take the supremum over all y on the right-hand side, i.e.

$$B(x) \geq \sup\{T(A(y), R(y,x)) | y \in X\} = R\uparrow A(x),$$

which shows that B must be a fuzzy superset of $R\uparrow A$. Since B was chosen arbitrarily, $R\uparrow A$ must be the smallest R-closed fuzzy superset.

Now let us consider an arbitrary R-closed fuzzy set C, such that $C \subseteq A$. In a similar way as above, we obtain the following for each $x, y \in X$:

$$C(x) \leq \vec{T}(R(x,y), C(y)) \leq \vec{T}(R(x,y), A(y).$$

Since this holds for any $x, y \in X$, we can also take the infimum over all y on the right-hand side and the proof of maximality is finished:

$$C(x) \leq \inf\{\vec{T}(R(x,y), A(y)) | y \in X\} = R\downarrow A(x). \qquad \square$$

According to Theorem 16, it is therefore justified to call $R\uparrow$ the *closure operator* of R and to call $R\downarrow$ the *opening operator* of R.

COROLLARY 17 The closure and the opening operator of a T-preordering R can also be represented in the following way:

$$R\uparrow A(x) = \inf\{B(x) | B \quad \text{is an} \quad R\text{-closed fuzzy superset of } A\}$$

$$R\downarrow A(x) = \sup\{C(x) | C \quad \text{is an } R\text{-closed fuzzy subset of } A\}.$$

Proof From Theorem 16, we know that any R-closed fuzzy superset of A is a fuzzy superset of $R\uparrow A$. Since $R\uparrow A$ is an R-closed fuzzy superset of A itself, the representation must hold. The representation of $R\downarrow A$ can be proved analogously. $\square$

Theorem 16 provides us with the mathematical apparatus for proving several basic properties of closures and openings.

COROLLARY 18 The following propositions hold for any $A \in \mathscr{F}(X)$:

1. A is R-closed if and only if $A = R\uparrow A$.
2. A is R-closed if and only if $A = R\downarrow A$.
3. $R\uparrow(R\uparrow A) = R\uparrow A$.
4. $R\downarrow(R\downarrow A) = R\downarrow A$.
5. $R\uparrow(R\downarrow A) = R\downarrow A$.
6. $R\downarrow(R\uparrow A) = R\uparrow A$.

Proof The first two propositions follow directly from Theorem 16. The others are immediate consequences of the first one. $\square$

Items 3 and 4 in Corollary 18 refer to idempotency with respect to composition, i.e. that $R\uparrow \circ R\uparrow \equiv R\uparrow$ and $R\downarrow \circ R\downarrow \equiv R\downarrow$. In order to investigate such algebraic properties a little further, we now formulate a sufficient condition under which the applications of closure and opening operators commute.

THEOREM 19 Given two T-preorderings R_1 and R_2 such that $R_1 \cup R_2$ is T-transitive, the following propositions hold for any $A \in \mathscr{F}(X)$:

$$(R_1 \cup R_2) \uparrow A = R_1 \uparrow (R_2 \uparrow A) = R_2 \uparrow (R_1 \uparrow A) = R_1 \uparrow A \cup R_2 \uparrow A$$

$$(R_1 \cup R_2) \downarrow A = R_1 \downarrow (R_2 \downarrow A) = R_2 \downarrow (R_1 \downarrow A) = R_1 \downarrow A \cap R_2 \downarrow A.$$

Proof Let $R_1 \cup R_2$ be T-transitive. Then the reflexivity of R_1 and R_2 implies that $R_1 \cup R_2$ is a T-preordering, and all the results achieved so far are applicable to $R_1 \cup R_2$ as well.

First of all, $(R_1 \cup R_2) \uparrow A$ is $R_1 \cup R_2$-closed. Therefore, by Corollary 12, $(R_1 \cup R_2) \uparrow A$ is R_1-closed and, due to Corollary 18,

$$R_1 \uparrow ((R_1 \cup R_2) \uparrow A) = (R_1 \cup R_2) \uparrow A.$$

Since $R_2 \subseteq R_1 \cup R_2$, monotonicity (cf. Eqs. (1) and (3) of Lemma 8) entails

$$R_1 \uparrow (R_2 \uparrow A) \subseteq R_1 \uparrow ((R_1 \cup R_2) \uparrow A) = (R_1 \cup R_2) \uparrow A. \tag{7}$$

The inclusion property (see Theorem 16) and monotonicity (cf. Lemma 8) yield

$$R_1 \uparrow A \subseteq R_1 \uparrow (R_2 \uparrow A), \tag{8}$$

$$R_2 \uparrow A \subseteq R_1 \uparrow (R_2 \uparrow A). \tag{9}$$

Putting Eqs. (8) and (9) together, we obtain

$$R_1 \uparrow A \cup R_2 \uparrow A \subseteq R_1 \uparrow (R_2 \uparrow A). \tag{10}$$

Since

$$(R_1 \cup R_2) \uparrow A = R_1 \uparrow A \cup R_2 \uparrow A$$

holds anyway due to Lemma 8, Eq. (10) is equivalent to

$$(R_1 \cup R_2) \uparrow A \subseteq R_1 \uparrow (R_2 \uparrow A). \tag{11}$$

Then Eqs. (7) and (11) together prove that

$$(R_1 \cup R_2) \uparrow A = R_1 \uparrow (R_2 \uparrow A).$$

The second equality

$$(R_1 \cup R_2) \uparrow A = R_2 \uparrow (R_1 \uparrow A)$$

follows immediately if we swap R_1 and R_2.

Now let us turn to the second line of equalities. Again, trivially, $(R_1 \cup R_2) \downarrow A$ is $R_1 \cup R_2$-closed. Hence, due to Corollary 12, $(R_1 \cup R_2) \downarrow A$ is R_1-closed and, again by Corollary 18,

$$R_1 \downarrow ((R_1 \cup R_2) \downarrow A) = (R_1 \cup R_2) \downarrow A.$$

Since $R_2 \subseteq R_1 \cup R_2$, monotonicity (see Eqs. (2) and (4) of Lemma 8) implies

$$R_1 \downarrow (R_2 \downarrow A) \supseteq R_1 \downarrow ((R_1 \cup R_2) \downarrow A) = (R_1 \cup R_2) \downarrow A. \tag{12}$$

On the other hand, the inclusion property (see Theorem 16) and monotonicity imply

$$R_1 \downarrow A \supseteq R_1 \downarrow (R_2 \downarrow A),$$

$$R_2 \downarrow A \supseteq R_1 \downarrow (R_2 \downarrow A).$$

Joining these two inclusions yields

$$R_1 \downarrow A \cap R_2 \downarrow A \supseteq R_1 \downarrow (R_2 \downarrow A). \tag{13}$$

Since we know from Prop. (8) of Lemma 8 that

$$(R_1 \cup R_2) \downarrow A = R_1 \downarrow A \cap R_2 \downarrow A,$$

the inequalities (12) and (13) imply

$$(R_1 \cup R_2) \downarrow A = R_1 \downarrow (R_2 \downarrow A).$$

The second equality

$$(R_1 \cup R_2) \downarrow A = R_2 \downarrow (R_1 \downarrow A)$$

follows again immediately if we swap R_1 and R_2. □

5. AN APPLICATION: ORDERING-BASED MODIFIERS

Already in their beginning, fuzzy systems were considered as appropriate tools for controlling complex systems and for carrying out complicated decision processes (Zadeh, 1973). It is well known and easy to see that, if rule bases are represented as complete tables, the number of rules grows exponentially with the number of variables—a fact which can be regarded as a serious limitation in terms of surveyability and interpretability.

Almost all fuzzy systems make implicit use of orderings. More specifically, it is quite common to decompose the universe of a system variable into a certain number of fuzzy sets by means of the ordering of the universe—an approach which is often reflected in labels like "small", "medium", or "large". We will now demonstrate by means of a simple example how such ordering information can be used to reduce the size of a rule base while improving expressiveness and interpretability. Consider a typical PD-style fuzzy controller with two inputs e, Δe and one output variable f, where the universes of all these variables are covered by five fuzzy sets labeled *NB*, *NS*, *Z*, *PS*, and *PB*:

	Δe				
e	*NB*	*NS*	*Z*	*PS*	*PB*
NB	*NB*	*NB*	*NB*	*NS*	*Z*
NS	*NB*	*NB*	*NS*	*Z*	*PS*
Z	*NB*	*NS*	*Z*	*PS*	*PB*
PS	*NS*	*Z*	*PS*	*PB*	*PB*
PB	*Z*	*PS*	*PB*	*PB*	*PB*

One possibility to reduce the size of this rule base is to take neighboring rules with the same consequents, such as,

IF e is *NB* AND Δe is *NB* THEN f is *NB*.
IF e is *NS* AND Δe is *NB* THEN f is *NB*.
IF e is *Z* AND Δe is *NB* THEN f is *NB*

and to replace them by a single rule like the following one[1]:

IF e is at most *Z* AND Δe is *NB* THEN f is *NB*.

Of course, there is actually no need to do so in such a simple case. Anyway, grouping neighboring rules by means of expressions, such as, "at least", "at most", or "between", could help to reduce the size of larger rule bases considerably.

[1]It depends on the underlying inference scheme whether the result is actually the same. We leave this aspect aside for the present paper, since this is not its major concern.

In addition, such elements can be useful in rule interpolation. Sometimes, when experts or automatic tuning procedures only provide an incomplete description of a fuzzy rule base, it can still be necessary to obtain a conclusion even if an observation does not match any antecedent in the rule base (Kóczy and Hirota, 1993). Moreover, it is considered as another opportunity for reducing the size of a rule base to store only some representative rules and to interpolate between them (Kóczy and Hirota, 1997). In any case, it is indispensable to have criteria for determining between which rules the interpolation should take place. Beside distance, orderings play a fundamental role in this selection. As an alternative to distance-based methods (Kóczy and Hirota, 1997), it is possible to fill the gap between the antecedents of two rules using a fuzzy concept of "between", which leads us to the ordering-based modifiers mentioned above.

The fact remains that we are still lacking a way to represent such expressions under the presence of fuzziness. In order to have a universal approach which is applicable in a wide variety of practical problems, at least the following two properties should be satisfied:

1. If there is a kind of inherent context of gradual equality in the given environment, ordering-based modifiers should take it into account. Stressing the well-known example of the height of men, this means that a fuzzy set "at least 180 cm" should not exclude 179.9 cm completely, since both values are almost indistinguishable.
2. Of course, the operators should be applicable to fuzzy sets, too, in order to be able to model expressions like "at least medium".

Usually, an expression like "at least" deeply relies on an underlying concept of ordering. Taking the first of the two above requirements into account, it is, however, not sufficient to consider only crisp concepts of ordering. With the aim to have a vague model of ordering based on an underlying vague concept of equality/equivalence, a generalization of fuzzy orderings has been proposed (Höhle and Blanchard, 1985; Bodenhofer, 2000).

DEFINITION 20 A T-transitive binary fuzzy relation $R \in \mathscr{F}(X^2)$ is called a *fuzzy ordering* on X with respect to a t-norm T and a T-equivalence E, for brevity *T–E-ordering*, if and only if it additionally satisfies the following two axioms:

1. *E-reflexivity*: $\forall x, y \in X : R(x, y) \geq E(x, y)$.
2. *T–E-antisymmetry*: $\forall x, y \in X : T(R(x, y), R(y, x)) \leq E(x, y)$.

For more details on this concept of fuzzy orderings, its properties and applications, the reader is referred to Bodenhofer (2000; 2003). We just mention that, by replacing the fuzzy equivalence relation E by the crisp equality, the well-known definition of fuzzy partial orderings (Zadeh, 1971) is obtained. Moreover, one easily verifies that this still includes crisp orderings.

Now let us start with the problem of how to define an operator "at least". If we restrict ourselves to crisp sets and crisp orderings, the following definition seems intuitively correct:

$$x \in \text{“at least } M\text{”} \Leftrightarrow (\exists y \in X : y \in M \wedge y \preceq x).$$

For generalizing this formula to a fuzzy set A and a given T–E-ordering R, two logical concepts have to be fuzzified as well—the conjunction and the existential quantifier. For conjunction, the underlying t-norm T seems to be the ready-made choice. If we take, as usual in t-norm-based predicate logic (Hájek, 1998), the supremum as fuzzy substitute for the existential quantifier, the following generalization is obtained:

$$\text{“at least } A\text{”}(x) = \sup\{T(A(y), R(y, x)) \mid y \in X\}.$$

Actually, this is nothing other than the full image or closure of A with respect to R:

$$\text{“at least } A\text{''} = R \uparrow A.$$

In order to make our formulas a little shorter and easier to read, we will denote this operator with ATL in the following.

Analogously, it is possible to define an operator “at most” just by taking the inverse ordering $R^{-1}(x, y) = R(y, x)$

$$\text{“at most } A\text{''}(x) = \sup\{T(A(y), R(x, y)) | y \in X\}$$

which will be denoted ATM in the following. To make notation consistent, let us denote the closure of E—the so-called *extensional hull*—as $\mathrm{EXT}(A) = E \uparrow A$.

The question arises what the benefits of the results from Section 3, as promised earlier, are. First of all, and this is neither surprising nor really spectacular, ATL(A) is R-closed and ATM(A) is R^{-1}-closed. As an immediate consequence of Corollary 12, ATL(A) and ATM(A) are both *extensional*, i.e. E-closed. Moreover, we know from Corollary 18 that both operators are idempotent in the sense that

$$\mathrm{ATL}(\mathrm{ATL}(A)) = \mathrm{ATL}(A),$$
$$\mathrm{ATM}(\mathrm{ATM}(A)) = \mathrm{ATM}(A).$$

We have mentioned above that the T-equivalence-based approach to fuzzy orderings is very much inspired by the typical practical situation that there is a given crisp concept of (mostly linear) ordering, however, with an additional context of gradual equality (like in the height example). We will now study this case in more detail. It will turn out that the results from Section 3, enable us to represent ATL and ATM by closures with respect to the crisp ordering and the fuzzy concept of equality. Before that, let us formalize this typical case in a mathematically exact way.

DEFINITION 21 A T–E-ordering R is called a *direct fuzzification* of a crisp ordering $\preceq$ if and only if it admits the following resolution:

$$R(x, y) = \begin{cases} 1 & \text{if } x \preceq y \\ E(x, y) & \text{otherwise.} \end{cases} \tag{14}$$

It is important to mention that strongly complete fuzzy orderings are uniquely characterized as direct fuzzifications of linear orderings (Bodenhofer, 2000). An alternative, slightly more restrictive definition of antisymmetry using the minimum t-norm $T_{\mathbf{M}}(x, a) = \min(x, y)$ has been proposed by Bělohlávek (2002). As the minimum t-norm is the largest t-norm, any fuzzy relation fulfilling $T_{\mathbf{M}}$–E-antisymmetry (for some fuzzy equivalence relation E) also fulfills T–E-antisymmetry (for any t-norm T). Note that strongly complete T–E-orderings—no matter which underlying t-norm T we consider—also fulfill the stronger condition of $T_{\mathbf{M}}$–E-antisymmetry.

As is easy to see from Eq. (14) a direct fuzzification of a crisp ordering is the max-union of a crisp ordering and a T-equivalence, which allows us to apply Theorem 19.

THEOREM 22 Let R be a T–E-ordering which is a direct fuzzification of a crisp ordering $\preceq$. Then the following equalities hold

$$\mathrm{ATL}(A) = \mathrm{EXT}(\mathrm{LTR}(A)) = \mathrm{LTR}(\mathrm{EXT}(A)) = \mathrm{EXT}(A) \cup \mathrm{LTR}(A), \tag{15}$$
$$\mathrm{ATM}(A) = \mathrm{EXT}(\mathrm{RTL}(A)) = \mathrm{RTL}(\mathrm{EXT}(A)) = \mathrm{EXT}(A) \cup \mathrm{RTL}(A), \tag{16}$$

where the operator LTR denotes the closure with respect to $\preceq$ while RTL stands for the closure with respect to the inverse relation of $\preceq$:

$$\mathrm{LTR}(A)(x) = \sup\{A(y) \mid y \preceq x\}$$
$$\mathrm{RTL}(A)(x) = \sup\{A(y) \mid x \preceq y\}.$$

Moreover, ATL(A) is the smallest fuzzy superset of A which is extensional and has a non-decreasing membership function. Analogously, ATM(A) is the smallest fuzzy superset of A which is extensional and has a non-increasing membership function.

Proof Let us start with the closures induced by the relations $\preceq$ and $\succeq$. For representing $\preceq$ as a fuzzy relation, we consider its *characteristic function*

$$\chi_{\preceq}(x, y) = \begin{cases} 1 & \text{if } x \preceq y, \\ 0 & \text{otherwise.} \end{cases}$$

Taking into account that $x \preceq y \Leftrightarrow \chi_{\preceq}(x, y) = 1$, Prop. (2) in Lemma 11 implies that a fuzzy set A is $\preceq$-closed if and only if, for all $x, y \in X$,

$$\chi_{\preceq}(x, y) \leq \vec{T}(A(x), A(y)).$$

In particular this means that, if $x \preceq y$, the equality $\vec{T}(A(x), A(y)) = 1$ must hold. Since $\vec{T}(x, y) = 1$ if and only if $x \leq y$, we obtain that $\preceq$-closedness is equivalent to non-decreasingness of the membership function:

$$x \preceq y \Rightarrow A(x) \leq A(y).$$

Analogously, we can show that $\succeq$-closedness corresponds to the non-increasingness of the membership function. Since $\preceq$ is a crisp relation, the following holds:

$$\mathrm{LTR}(A)(x) = \sup\{T(A(y), \chi_{\preceq}(y, x)) \mid y \in X\} = \sup\{A(y) \mid y \preceq x\}.$$

The analogous argument applies to prove the corresponding representation of RTL.

Equality (15) follows directly from Theorem 19 if we consider $R_1 = E$ and $R_2 = \chi_{\preceq}$, while equality (16) follows in the same way with $R_1 = E$ and $R_2 = \chi_{\succeq}$.

Of course, ATL(A) is extensional and has a non-decreasing membership function (by Corollary 12, since E and $\preceq$ are both subrelations of R). For proving that ATL(A) is the smallest extensional fuzzy superset of A with a non-decreasing membership function, suppose that a superset $B \supseteq A$ is extensional and has a non-decreasing membership function. Hence, B is a superset of both EXT(A) and LTR(A). Then

$$B \supseteq \mathrm{EXT}(A) \cup \mathrm{LTR}(A) = \mathrm{ATL}(A)$$

must hold, which proves the minimality of ATL(A). Analogous arguments can be applied to prove that ATM(A) is the smallest extensional fuzzy superset of A with a non-increasing membership function. □

The representations (15) and (16) can be interpreted as commutative diagrams, one of which is shown in Fig. 1.

Finally, let us clarify the interplay between the operators ATL, ATM, and EXT and convex hulls, where we base our understanding of convexity of fuzzy sets on an underlying crisp notion of ordering. Note that, more traditionally (Zadeh, 1965; Lowen, 1980), the convexity of a fuzzy set is defined as the convexity (in the usual sense common in linear algebra) of all its α-cuts. For the real numbers with their canonical linear ordering, these two ways of defining convexity coincide.

$$\begin{tikzcd} A \arrow[r, "\mathrm{EXT}"] \arrow[d, "\mathrm{LTR}"'] & \mathrm{EXT}(A) \arrow[d, "\mathrm{LTR}"'] \\ \mathrm{LTR}(A) \arrow[r, "\mathrm{EXT}"] & \mathrm{ATL}(A) \end{tikzcd}$$

FIGURE 1 A commutative diagram depicting the relationships (15) for a given fuzzy set A.

DEFINITION 23 Provided that the domain X is equipped with some crisp ordering $\preceq$ (not necessarily linear), a fuzzy set $A \in \mathscr{F}(X)$ is called *convex* if and only if, for all $x, y, z \in X$,

$$x \preceq y \preceq z \Rightarrow A(y) \geq \min(A(x), A(z)).$$

LEMMA 24 Any fuzzy set with non-decreasing or non-increasing membership function is convex.

Proof Consider a fuzzy set $A \in \mathscr{F}(X)$ with a non-decreasing membership function. Then the following holds for all $x, y, z \in X$:

$$x \preceq y \preceq z \Rightarrow A(x) \leq A(y) \leq A(z).$$

Therefore, $A(y) \geq A(x) = \min(A(x), A(z))$ must always be fulfilled for an ascending sequence $x \preceq y \preceq z$, and A is guaranteed to be convex. Analogously, the same can be proved for a fuzzy set with non-increasing membership function. □

Therefore, we can conclude, under the assumption that R is a direct fuzzification of some crisp ordering $\preceq$, that ATL(A), ATM(A), LTR(A), and RTL(A) are convex fuzzy sets for any $A \in \mathscr{F}(X)$.

LEMMA 25 The min-intersection of any two convex fuzzy sets is again convex.

Proof Assume that A and B are two convex fuzzy sets, i.e.

$$x \preceq y \preceq z \Rightarrow A(y) \geq \min(A(x), A(z)),$$
$$x \preceq y \preceq z \Rightarrow B(y) \geq \min(B(x), B(z)).$$

Taking an arbitrary ascending sequence $x \preceq y \preceq z$, we obtain

$$\begin{aligned}\min(A(y), B(y)) &\geq \min(\min(A(x), A(z)), \min(B(x), B(z)))\\ &= \min(A(x), A(z), B(x), B(z))\\ &= \min(A(x), B(x), A(z), B(z))\\ &= \min(\min(A(x), B(x)), \min(A(z), B(z))).\end{aligned}$$ □

LEMMA 26 Assume that $\preceq$ is an arbitrary, not necessarily linear ordering on a domain X. Then the fuzzy set

$$\mathrm{CVX}(A) = \mathrm{LTR}(A) \cap \mathrm{RTL}(A)$$

is the smallest convex fuzzy superset of A.

Proof First of all, CVX(A) is a convex fuzzy superset of A, since it is the intersection of two convex fuzzy sets both of which are supersets of A.

Now assume that B is a convex fuzzy superset of A, i.e. for all $x, y, z \in X$,

$$x \preceq y \preceq z \Rightarrow B(y) \geq \min(B(x), B(z)).$$

Since this holds for all chains $x \preceq y \preceq z$, we can even, for a fixed y, take the suprema over all $x \preceq y$ and $z \succeq y$ and the following is obtained:

$$\begin{aligned} B(y) &\geq \min(\sup\{B(x) | x \preceq y\}, \sup\{B(z) | y \preceq z\}) \\ &= \min(\mathrm{LTR}(B)(y), \mathrm{RTL}(B)(y)) \\ &\geq \min(\mathrm{LTR}(A)(y), \mathrm{RTL}(A)(y)) \\ &= \mathrm{CVX}(A)(y). \end{aligned}$$

The fuzzy set B was supposed to be an arbitrary convex fuzzy superset of A; therefore, CVX(A) must be the smallest convex fuzzy superset of A. □

THEOREM 27 With the assumptions of Theorem 22 and the definition

$$\mathrm{ECX}(A) = \mathrm{ATL}(A) \cap \mathrm{ATM}(A),$$

the following representation holds:

$$\mathrm{ECX}(A) = \mathrm{EXT}(\mathrm{CVX}(A)) = \mathrm{CVX}(\mathrm{EXT}(A)) = \mathrm{EXT}(A) \cup \mathrm{CVX}(A). \tag{17}$$

Furthermore, ECX(A) is the smallest fuzzy superset of A which is extensional and convex.

Proof Taking into account that, for min and max, the laws of distributivity hold, we obtain the following from Theorem 22:

$$\begin{aligned} \mathrm{ECX}(A)(x) &= \min(\mathrm{ATL}(A)(x), \mathrm{ATM}(A)(x)) \\ &= \min(\max(\mathrm{EXT}(A)(x), \mathrm{LTR}(A)(x)), \max(\mathrm{EXT}(A)(x), \mathrm{RTL}(A)(x))) \\ &= \max(\mathrm{EXT}(A)(x), \min(\mathrm{LTR}(A)(x), \mathrm{RTL}(A)(x))) \\ &= \max(\mathrm{EXT}(A)(x), \mathrm{CVX}(A)(x)). \end{aligned}$$

Using Eqs. (15) and (16), we immediately obtain from the definition of CVX(A) that

$$\begin{aligned} \mathrm{ECX}(A) &= \mathrm{ATL}(A) \cap \mathrm{ATM}(A) \\ &= \mathrm{LTR}(\mathrm{EXT}(A)) \cap \mathrm{RTL}(\mathrm{EXT}(A)) \\ &= \mathrm{CVX}(\mathrm{EXT}(A)). \end{aligned}$$

On the other hand, ECX(A) is an intersection of two convex fuzzy sets and, therefore, convex. Thus, by Lemma 26, ECX(A) is a fuzzy superset of CVX(A). Moreover, ECX(A) is extensional, since it is the intersection of two extensional fuzzy sets (cf. Lemma 13). All together, ECX(A) is an extensional fuzzy superset of CVX(A), which implies (cf. Theorem 16)

$$\mathrm{ECX}(A) \supseteq \mathrm{EXT}(\mathrm{CVX}(A)). \tag{18}$$

Since $A \subseteq \mathrm{CVX}(A)$ always holds, the following is obtained (see Eq. (1) of Lemma 8 and Theorem 16):

$$\mathrm{EXT}(A) \subseteq \mathrm{EXT}(\mathrm{CVX}(A))$$
$$\mathrm{CVX}(A) \subseteq \mathrm{EXT}(\mathrm{CVX}(A)).$$

This immediately implies

$$\mathrm{ECX}(A) = \mathrm{EXT}(A) \cup \mathrm{CVX}(A) \subseteq \mathrm{EXT}(\mathrm{CVX}(A))$$

which, together with Eq. (18), completes the proof of Eq. (17).

Now assume that B is an extensional and convex fuzzy superset of A. Since extensionality implies $B \supseteq \mathrm{EXT}(A)$ while convexity implies $B \supseteq \mathrm{CVX}(A)$, we see that

$$B \supseteq \mathrm{CVX}(A) \cup \mathrm{EXT}(A) = \mathrm{ECX}(A)$$

and the minimality of $\mathrm{ECX}(A)$ is proved as well. □

Example 28 Figure 2 shows a simple example of a non-trivial fuzzy set $A \in \mathscr{F}(\mathbb{R})$ and the results which are obtained by applying various operators we have discussed so far.

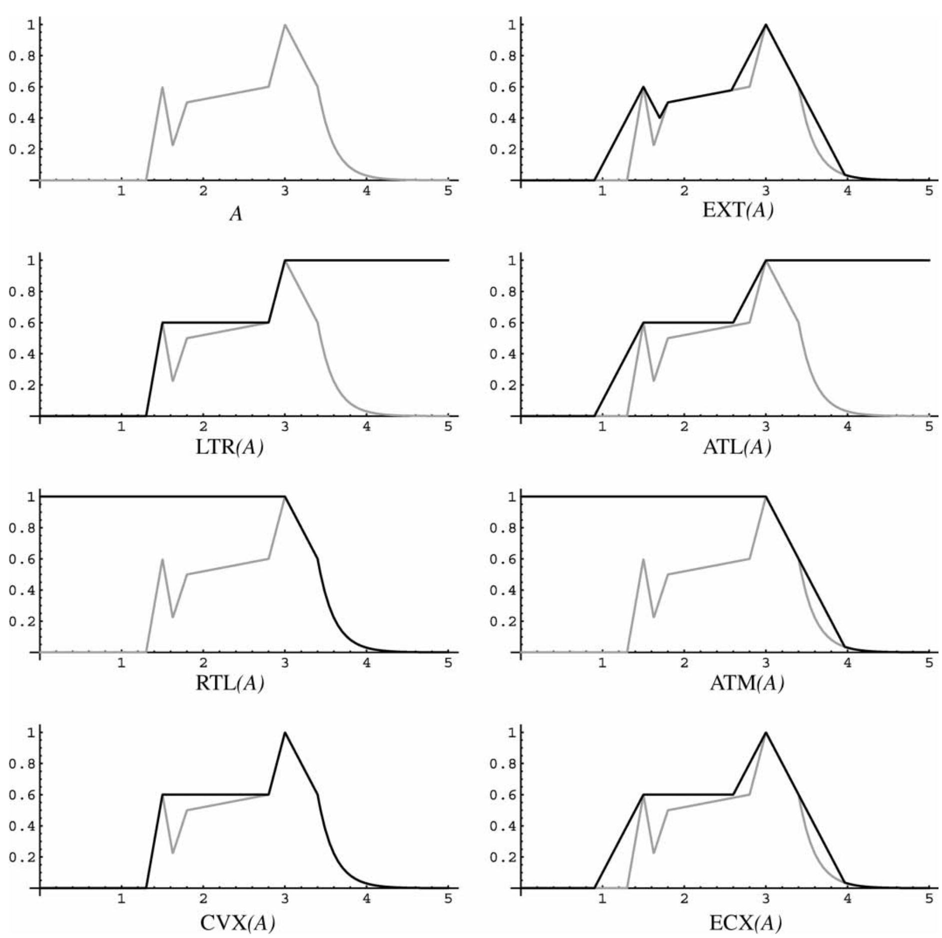

FIGURE 2 A fuzzy set $A \in \mathscr{F}(\mathbb{R})$ and the results which are obtained when applying various ordering-based operators.

The relations used for representing these operators are the natural linear ordering of real numbers $\leq$ and the following two fuzzy relations:

$$E(x,y) = \max(1 - |x-y|, 0)$$

$$R(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ \max(1 - x + y, 0) & \text{otherwise.} \end{cases}$$

One easily verifies that E is indeed a $T_{\mathbf{L}}$-equivalence on the real numbers and that R is a $T_{\mathbf{L}}$–E-ordering, which directly fuzzifies the linear ordering of real numbers, where $T_{\mathbf{L}}$ stands for the so-called *Łukasiewicz t-norm*

$$T_{\mathbf{L}}(x,y) = \max(x + y - 1, 0).$$

In particular, Fig. 2 demonstrates the commutative diagram shown in Fig. 1 and all the other equalities of Eqs. (15)–(17).

The unary ordering-based modifiers ATL and ATM can be used as the basis for defining more sophisticated binary ordering-based modifiers, such as, "between". A starting point is provided in Bodenhofer (2002).

6. CONCLUDING REMARKS

This paper provides a theoretical framework for studying opening and closure operators of fuzzy preorderings. While many results transfer directly from the symmetric case, i.e. from the theory of fuzzy equivalence relations, fundamental additional insight has been gained about the way the successive application of opening and closure operators commutes (cf. Theorem 19). Based on these considerations, we have seen that the results on closure operators have fruitful applications in the construction and analysis of ordering-based modifiers.

References

Bauer, P., Klement, E.P., Moser, B. and Leikermoser, A. (1995) "Modeling of control functions by fuzzy controllers", In: Nguyen, H.T., Sugeno, M., Tong, R.M. and Yager, R.R., eds, *Theoretical Aspects of Fuzzy Control* (Wiley, New York), Chapter 5, pp 91–116.

Bellman, R.E. and Zadeh, L.A. (1970) "Decision making in a fuzzy environment", *Man. Sci.* **17**(4), 141–164.

Bělohlávek, R. (2002) *Fuzzy Relational Systems. Foundations and Principles*, *IFSR Int. Series on Systems Science and Engineering* (Kluwer Academic, New York).

Bodenhofer, U. (2000) "A similarity-based generalization of fuzzy orderings preserving the classical axioms", *Int. J. Uncertainty Fuzziness Knowledge-based Syst.* **8**(5), 593–610.

Bodenhofer, U. (2002) "Binary ordering-based modifiers," *in Proc. 9th Int. Conf. on Information Processing and Management of Uncertainty in Knowledge-based Systems*, Vol. 3, Annecy, pp 1953–1959.

Bodenhofer, U. (2003) "Representations and constructions of similarity-based fuzzy orderings", *Fuzzy Sets Syst.* **137**(1), 113–116.

Boixader, D., Jacas, J. and Recasens, J. (2000) "Fuzzy equivalence relations: advanced material", In: Dubois, D. and Prade, H., eds, *Fundamentals of Fuzzy Sets*, *The Handbooks of Fuzzy Sets* (Kluwer Academic Publishers, Boston) Vol. 7, pp 261–290.

De Baets, B. (2000) "Analytical solution methods for fuzzy relational equations", In: Dubois, D. and Prade, H., eds, *Fundamentals of Fuzzy Sets*, *The Handbooks of Fuzzy Sets* (Kluwer Academic Publishers, Boston) Vol. 7, pp 291–340.

di Nola, A., Pedrycz, W., Sessa, S. and Sanchez, E. (1991) "Fuzzy relational equations theory as a basis for fuzzy modelling: An overview", *Fuzzy Sets Syst.* **40**, 415–429.

Fodor, J. and Roubens, M. (1994) *Fuzzy Preference Modelling and Multicriteria Decision Support* (Kluwer Academic Publishers, Dordrecht).

Gottwald, S. (1993) *Fuzzy Sets and Fuzzy Logic* (Vieweg, Braunschweig).

Gottwald, S. (2001) *A Treatise on Many-Valued Logics, Studies in Logic and Computation* (Research Studies Press, Baldock).
Hájek, P. (1998) *Metamathematics of Fuzzy Logic*, *Trends in Logic* (Kluwer Academic Publishers, Dordrecht) Vol. 4.
Höhle, U. and Blanchard, N. (1985) "Partial ordering in L-underdeterminate sets", *Inf. Sci.* **35**, 133–144.
Jacas, J. (1988) "On the generators of T-indistinguishability operators", *Stochastica* **12**, 49–63.
Jacas, J. and Recasens, J. (1995) "Fuzzy T-transitive relations: eigenvectors and generators", *Fuzzy Sets Syst.* **72**, 147–154.
Kerre, E.E., ed. (1993) *Introduction to the Basic Principles of Fuzzy Set Theory and Some of its Applications*, Communication and Cognition, Gent.
Klawonn, F. (1993) "Mamdani's model in the view of equality relations", *Proc. EUFIT* **93**(1), 364–369.
Klawonn, F. and Castro, J.L. (1995) "Similarity in fuzzy reasoning", *Mathware Soft Comput.* **3**(2), 197–228.
Klawonn, F. and Kruse, R. (1993) "Equality relations as a basis for fuzzy control", *Fuzzy Sets Syst.* **54**(2), 147–156.
Klawonn, F., Gebhardt, J. and Kruse, R. (1995) "Fuzzy control on the basis of equality relations—with an example from idle speed control", *IEEE Trans. Fuzzy Syst.* **3**, 336–356.
Klement, E.P., Mesiar, R. and Pap, E. (2000) *Triangular Norms*, *Trends in Logic* (Kluwer Academic Publishers, Dordrecht) Vol. 8.
Kóczy, L.T. and Hirota, K. (1993) "Ordering, distance and closeness of fuzzy sets", *Fuzzy Sets Syst.* **59**(3), 281–293.
Kóczy, L.T. and Hirota, K. (1997) "Size reduction by interpolation in fuzzy rule bases", *IEEE Trans. Syst. Man Cybern.* **27**(1), 14–25.
Kruse, R., Gebhardt, J. and Klawonn, F. (1994) *Foundations of Fuzzy Systems* (Wiley, New York).
Lowen, R. (1980) "Convex fuzzy sets", *Fuzzy Sets Syst.* **3**, 291–310.
Miyakoshi, M. and Shimbo, M. (1985) "Solutions of composite fuzzy relational equations with triangular norms", *Fuzzy Sets Syst.* **16**, 53–63.
Sanchez, E. (1984) "Solution of fuzzy equations with extended operations", *Fuzzy Sets Syst.* **12**, 237–248.
Valverde, L. (1985) "On the structure of F-indistinguishability operators", *Fuzzy Sets Syst.* **17**(3), 313–328.
Zadeh, L.A. (1965) "Fuzzy sets", *Inf. Control* **8**, 338–353.
Zadeh, L.A. (1971) "Similarity relations and fuzzy orderings", *Inf. Sci.* **3**, 177–200.
Zadeh, L.A. (1973) "Outline of a new approach to the analysis of complex systems and decision processes", *IEEE Trans. Syst. Man Cybern.* **3**(1), 28–44.

Ulrich Bodenhofer was born in Upper Austria in 1972. He received an M.Sc. degree in applied mathematics in 1996 and a PhD in technical sciences in 1998, both from the Johannes Kepler University, Linz, Austria. Dr Bodenhofer is currently scientific director of the Software Competence Center Hagenberg (SCCH). He is also working as a lecturer at the Johannes Kepler University, Linz. He is a member of IEEE, the IEEE Communications Society, the IEEE Systems, Man and Cybernetics Society, the IEEE Neural Networks Society, the North-American Fuzzy Information Processing Society (NAFIPS), and a board member of the European Society for Fuzzy Logic and Technology (EUSFLAT). His research interests include fuzzy and many-valued logics, fuzzy relations, decision analysis, genetic algorithms, machine learning, and image processing.

Martine De Cock was born on September 9, 1976. After obtaining an M.Sc. degree in Computer Science and a teaching degree at Ghent University in 1998, she became a member of the Fuzziness and Uncertainty Modelling Research Unit. In 2002, she obtained a PhD in Computer Science. Currently she is working as a postdoctoral researcher of the Fund for Scientific Research—Flanders (FWO).

Etienne E. Kerre was born in Zele, Belgium on May 8, 1945. He obtained his M.Sc. Degree in Mathematics in 1967 and his PhD in Mathematics in 1970 from Ghent University. Since 1984, he has been a lecturer, and since 1991, a full professor at Ghent University. He is a referee for more than 30 international scientific journals, and also a member of the editorial board of international journals and conferences on fuzzy set theory. He was an honorary chairman at various international conferences. In 1976, he founded the Fuzziness and Uncertainty Modelling Research Unit (FUM) and since then his research has been focused on the modelling of fuzziness and uncertainty, and has resulted in a great number of contributions in fuzzy set theory and its various generalizations, and in evidence theory. Especially the theories of fuzzy relational calculus and of fuzzy mathematical structures owe a very great deal to him. Over the years he has also been a promotor of 16 PhDs on fuzzy set theory. His current research interests include fuzzy and intuitionistic fuzzy relations, fuzzy topology, and fuzzy image processing. He has authored or co-authored 11 books, and more than 100 papers that have appeared in international refereed journals.