Efficient Algorithms for Fuzzy Qualitative Temporal Reasoning

Steven Schockaert and Martine De Cock

Abstract— **Fuzzy qualitative temporal relations have been proposed to reason about events whose temporal boundaries are ill–defined. Although the corresponding reasoning tasks are in the same complexity class as their crisp counterparts, in practice, the scalability of fuzzy temporal reasoners may be insufficient for applications which require a high expressivity and deal with a large number of events. On the other hand, transitivity rules can be used to make sound, but incomplete inferences in polynomial time, utilizing a variant of Allen's path–consistency algorithm. The aim of this paper is to investigate how this polynomial time algorithm can be improved without altering its time complexity. To this end, we establish a characterization of 2–consistency of fuzzy temporal relations and provide transitivity rules which are significantly stronger than those resulting from straightforwardly generalizing transitivity rules for crisp temporal relations. We furthermore provide experimental evidence for the effectiveness of our improved algorithm.**

Index Terms— **Fuzzy Relations, Temporal Reasoning, Qualitative Reasoning**

I. INTRODUCTION

WHILE it is customary to talk about events and time periods like the Cold War, the Great Depression or the Age of Enlightenment, identifying appropriate beginning and ending dates for them is difficult, if not impossible. Many real–world events have an inherently gradual onset or ending, making it hard to pinpoint exact beginning and ending dates (e.g., the Dotcom Bubble), while other events are ill–defined, resulting in several possible beginning and ending dates (e.g., World War II). The vagueness of real–world events has been well–recognized in literature [1], [2], [3], [4], [5] and does not only pertain to large–scale events (e.g., when exactly does falling down a flight of stairs begin?). Finally, vague temporal markers are frequently used in everyday speech, e.g., late afternoon, some weeks ago, next summer, etc. Nevertheless, applications which employ temporal reasoning predominantly rely on the simplifying assumption that events and time periods can be represented by well–defined intervals of the real line.

Many applications can benefit from the use of qualitative temporal information acquired from natural language texts. Examples are multi–document summarization, where a chronological ordering of events occurring in different documents is needed to obtain a fluent narrative, and temporal question answering, where detailed temporal knowledge is used to find answers to questions which satisfy a temporal restriction imposed by the user (e.g., Which paintings did Salvator Dali create before his Surrealist period?). In such scenarios, however, assuming that all events have crisp temporal boundaries may quickly lead to inconsistencies. For example, while the Cold War is generally considered to have started shortly after the end of World War II, some texts mention the end of the Russian Revolution in 1917 as the real beginning of the Cold War.

In [6], a framework has been introduced to represent temporal relations between vague events as fuzzy relations. Reasoning in this framework is NP–complete in general [7], i.e., the time complexity of reasoning about fuzzy temporal relations is the same as that of reasoning in Allen's Interval Algebra (IA) [8], which is traditionally used for (crisp) reasoning about qualitative temporal relations (e.g., A happened before B, A happened during B, etc.). To support efficient reasoning in IA, a transitivity table was introduced in [9] conveying what temporal relations are possible between two intervals A and C, given that there exists some interval B such that a particular temporal relation r_1 holds between A and B and a particular temporal relation r_2 holds between B and C. If the initial knowledge base contains no disjunctive information (like *the beginning of A strictly precedes the beginning of B OR the ending of A equals the ending of B*), using this transitivity table yields a complete reasoning algorithm, requiring $O(n^3)$ time to complete, where n is the number of intervals (or events) involved [8]. Motivated by the importance of the transitivity table for crisp temporal reasoning, a generalized transitivity table for fuzzy temporal relations was provided in [6]. However, in contrast to crisp temporal reasoning frameworks, even in the absence of disjunctions, reasoning with fuzzy temporal relations is NP–complete in some cases [7], implying that any $O(n^3)$ time algorithm is necessarily incomplete. Moreover, as will become clear below, using only this generalized transitivity table for fuzzy temporal reasoning leads to rather poor performance, as in many cases, for example, inconsistencies cannot be detected by this method. Hence, on one hand we have a complete algorithm requiring exponential time, and on the other hand an efficient, but incomplete algorithm for fuzzy temporal reasoning.

The primary aim of this paper is to investigate how fuzzy temporal reasoning with transitivity rules can be improved such that more inconsistencies can be detected, and more interesting conclusions can be drawn, while maintaining the $O(n^3)$ time complexity. The paper is structured as follows. In the next section, we review related work on fuzzy temporal reasoning, while Section III introduces some preliminaries about fuzzy temporal relations. Next, in Section IV, we show how transitivity rules can be used to obtain an efficient,

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but incomplete algorithm for fuzzy temporal reasoning. In Section V, we present a first improvement, based on the notion of 2–consistency. Section VI discusses a second improvement, based on a notion of transitivity which is stronger than the transitivity rules from the generalized transitivity table. Subsequently, in Section VII, we demonstrate that both improvements have a significant impact on the performance of the fuzzy temporal reasoning algorithm. Appendix I contains the proofs of the propositions that are introduced in this paper. Finally, note that some of the results in Section VI appeared earlier in [10].

II. Related Work

The majority of existing approaches to fuzzy temporal reasoning are concerned with modelling vague temporal information about crisp events. For example, a possibilistic framework for modelling fuzzy dates (e.g., the beginning of next week) and fuzzy temporal constraints (e.g., A took place a few days after B) was introduced in [11]. It was later elaborated upon, resulting in the framework of fuzzy temporal constraint networks [12], [13], [14]. In [15], possibility theory is adopted to represent uncertain relations between time points. Specifically, a temporal relation between two time points t_1 and t_2 is represented as a triple of values between 0 and 1, expressing the degree of possibility that t_1 is before, equal to, or after t_2. Another reason why fuzzifications of temporal reasoning formalisms have been proposed is to cope with preferences [16], [17], [18]. The idea is to attach weights to each temporal relation revealing what information should be discarded if the available information turns out to be inconsistent.

In the approaches mentioned above, fuzzy temporal relations are mainly used to encode vague information about crisp events. In other words, while events are still assumed to begin and end at well–defined instants of time, available knowledge about these temporal boundaries is allowed to be imprecise. A more relevant line of research has focused on processing temporal information about vague events, i.e., events whose time span cannot accurately be represented as an interval. Such vague time periods can, in principle, be modelled in several ways. For example, [19] proposes definitions of temporal relations which do not explicitly refer to time, somewhat similar in spirit to the well–known region connection calculus for spatial reasoning [20]. In such an approach, however, temporal relations between vague events are crisp relations, which may be counterintuitive in many situations. In [21], rough sets are used to represent time spans of events. Temporal relations are then defined by specifying an upper bound of relations that possibly hold between two events, and a lower bound of relations that are guaranteed to hold. Most commonly, however, fuzzy sets are used to represent the time span of vague events, and temporal relations are defined as fuzzy relations [3], [4], [22]. The definitions of these fuzzy temporal relations are typically inspired by measures for comparing and ranking fuzzy numbers [23], [24]. In [7], a sound and complete algorithm for reasoning about fuzzy temporal relations has been introduced. Furthermore, the time complexity of fuzzy temporal reasoning was shown to be NP–complete.

III. Preliminaries

A. Fuzzy Time Intervals

In this paper, we represent time spans as fuzzy sets of real numbers, satisfying some additional, natural criteria:

Definition 1: A fuzzy (time) interval is a normalised, convex, upper semi–continuous fuzzy set in $\mathbb{R}$ with a bounded support.

Recall that a normalised fuzzy set A in $\mathbb{R}$ with a bounded support is convex and upper semi–continuous iff all α–level sets $A_\alpha = \{p | p \in \mathbb{R} \wedge A(p) \geq \alpha\}$ are closed intervals for $\alpha \in]0,1[$ and A_1 is either a closed interval or a singleton. A fuzzy set A in $\mathbb{R}$ is called normalised if $A(p) = 1$ for some p in $\mathbb{R}$; in this case, p is called a modal value of A.

Let T_W, I_W and S_W respectively denote the Łukasiewicz t–norm, implicator and t–conorm defined for a and b in $[0,1]$ by $T_W(a,b) = \max(0, a+b-1)$, $I_W(a,b) = \min(1, 1-a+b)$, $S_W(a,b) = \min(1, a+b)$. For convenience, we will use expressions like $T_W(a,b,c)$ as a shorthand for $T_W(a, T_W(b,c))$. The following well–known properties of the Łukasiewicz connectives will be useful below.

$$T_W(a,b) \leq c \Leftrightarrow a \leq I_W(b,c) \tag{1}$$

$$\max(I_W(a,c), I_W(b,c)) = I_W(\min(a,b), c) \tag{2}$$

$$I_W(a,b) = S_W(1-a, b) \tag{3}$$

$$1 - T_W(a,b) = S_W(1-a, 1-b) \tag{4}$$

$$T_W(a, I_W(a,b)) = \min(a,b) \tag{5}$$

$$I_W(a,b) = I_W(1-b, 1-a) \tag{6}$$

B. Fuzzy Temporal Relations

For crisp intervals, qualitative temporal relations are usually defined as constraints on the boundary points of these intervals. For example, it holds that $[a^-, a^+]$ is during $[b^-, b^+]$ iff $b^- < a^-$ and $a^+ < b^+$. Because beginnings and endings of fuzzy time intervals are gradual, a different approach is required when defining fuzzy temporal relations. Our definitions are inspired by the fact that the constraints on the boundary points can equivalently be expressed using a first–order formulation which does not explicitly refer to these boundary points. For example, let $A = [a^-, a^+]$ and $B = [b^-, b^+]$. It holds that

$$a^- < b^- \Leftrightarrow (\exists p)(p \in A \wedge (\forall q)(q \in B \Rightarrow p < q)) \tag{7}$$

The right–hand side of (7) can straightforwardly be generalized using the Łukasiewicz connectives, i.e., we define the degree $bb^{\ll}(A,B)$ to which the beginning of a fuzzy time interval A is strictly before the beginning of a fuzzy time interval B as

$$bb^{\ll}(A,B) = \sup_{p \in \mathbb{R}} T_W(A(p), \inf_{q \in \mathbb{R}} I_W(B(q), L^{\ll}(p,q)))$$

where $L^{\ll}(p,q) = 1$ if $p < q$ and $L^{\ll}(p,q) = 0$ otherwise. In the same way, we define the degree $ee^{\ll}(A,B)$ to which the ending of A is strictly before the ending of B, the degree $be^{\ll}(A,B)$ to which the beginning of A is strictly before the ending of B, the degree $eb^{\ll}(A,B)$ to which the ending of A is strictly before the beginning of B, the degree $bb^{\preccurlyeq}(A,B)$ to which the beginning of A is before or equal to the beginning of B, the degree $ee^{\preccurlyeq}(A,B)$ to which the ending of A is before or equal to the ending of B, the degree $be^{\preccurlyeq}(A,B)$ to which the beginning of A is before or equal to the ending of B and the degree $eb^{\preccurlyeq}(A,B)$ to which the ending of A is before or equal to the beginning of B as shown in Table I; $L^{\preccurlyeq}$ is defined as $L^{\preccurlyeq}(p,q) = 1 - L^{\ll}(q,p)$ for all p and q in $\mathbb{R}$. Note that the definitions of our fuzzy temporal relations coincide with the corresponding classical definitions when A and B are crisp intervals. This stands in contrast to the definitions in [4], for example, where temporal relations between two crisp intervals can be still be satisfied to a degree between 0 and 1, e.g., to model the degree to which most of interval A is before the beginning of interval B.

Our commitment to the Łukasiewicz connectives is motivated by various reasons. One example is the following lemma, which is of paramount importance for the discussions throughout this paper and which would not hold if the minimum or product, together with their residual implicator, were used instead of the Łukasiewicz t–norm. Intuitively, the lemma states that the degree to which the beginning of A is strictly before the beginning of B is equal to the degree to which the beginning of B is not before or equal to the beginning of A, and similar for $ee^{\ll}(A,B)$, $be^{\ll}(A,B)$ and $eb^{\ll}(A,B)$.

Lemma 1: [6] Let A and B be fuzzy time intervals. It holds that

$$bb^{\ll}(A,B) = 1 - bb^{\preccurlyeq}(B,A) \quad ee^{\ll}(A,B) = 1 - ee^{\preccurlyeq}(B,A)$$
$$be^{\ll}(A,B) = 1 - eb^{\preccurlyeq}(B,A) \quad eb^{\ll}(A,B) = 1 - be^{\preccurlyeq}(B,A)$$

Not all possible constraints between the boundary points of crisp intervals $A = [a^-, a^+]$ and $B = [b^-, b^+]$ are independent of each other. For example, if $a^- < b^-$ then also $a^- < b^+$, $a^- \leq b^-$ and $a^- \leq b^+$. The following lemma presents a generalization of this.

Lemma 2: [6] Let A and B be fuzzy time intervals. It holds that

$$bb^{\ll}(A,B) \leq bb^{\preccurlyeq}(A,B)$$
$$ee^{\ll}(A,B) \leq ee^{\preccurlyeq}(A,B)$$
$$be^{\ll}(A,B) \leq be^{\preccurlyeq}(A,B)$$
$$eb^{\ll}(A,B) \leq eb^{\preccurlyeq}(A,B)$$
$$eb^{\preccurlyeq}(A,B) \leq bb^{\preccurlyeq}(A,B) \leq be^{\preccurlyeq}(A,B)$$
$$eb^{\preccurlyeq}(A,B) \leq ee^{\preccurlyeq}(A,B) \leq be^{\preccurlyeq}(A,B)$$
$$eb^{\ll}(A,B) \leq bb^{\ll}(A,B) \leq be^{\ll}(A,B)$$
$$eb^{\ll}(A,B) \leq ee^{\ll}(A,B) \leq be^{\ll}(A,B)$$

Crisp temporal relations exhibit a lot of interesting transitivity properties, which form the basis for temporal reasoning algorithms. For example, if the beginning of A is strictly before the ending of B and the ending of B is before or equal to the beginning of C, we know that the beginning of A is strictly before the beginning of C. The fuzzy temporal relations $bb^{\ll}, bb^{\preccurlyeq}, \ldots, eb^{\preccurlyeq}$ exhibit a similar transitive behaviour.

Lemma 3: [6] The fuzzy temporal relations satisfy the transitivity rules that are summarized in Table II. Specifically, if $R(A,C)$ is the entry in this table on the row corresponding to $S(A,B)$ and the column corresponding to $Q(B,C)$, it holds that $T_W(S(A,B), Q(B,C)) \leq R(A,C)$.

IV. Fuzzy Temporal Reasoning

Let X be a finite set of variables. The reasoning task which we consider in this paper consists of deciding whether a set of formulas Θ of the form $bb^{\ll}(x,y) \leq 0.6$, $eb^{\preccurlyeq}(x,y) \geq 0.9, \ldots$ is consistent, i.e., whether we can assign a fuzzy time interval to each of the variables in X such that all lower and upper bounds on fuzzy temporal relations in Θ are satisfied.

Example 1: Let $X = \{x,y,z\}$ and let $\Theta = \{bb^{\ll}(x,y) \geq 0.7, be^{\preccurlyeq}(y,z) \geq 0.6, eb^{\preccurlyeq}(z,x) \geq 0.8\}$ then Θ is not consistent. Indeed, by Lemma 3, we know that

$$be^{\ll}(x,z) \geq T_W(bb^{\ll}(x,y), be^{\preccurlyeq}(y,z)) \geq T_W(0.7, 0.6) = 0.3$$

and by Lemma 1 we know that $eb^{\preccurlyeq}(z,x) = 1 - be^{\ll}(x,z) \leq 1 - 0.3 = 0.7$. Hence $eb^{\preccurlyeq}(z,x) \geq 0.8$ cannot be satisfied when the other two formulas from Θ are satisfied.

Most other interesting reasoning tasks for fuzzy time intervals can be reduced to this problem [7], hence it is of prime importance to have an efficient algorithm for consistency checking at our disposal. Unfortunately this problem is NP–complete in general [7]. However, it is possible to derive polynomial–time algorithms which are sound, but incomplete, i.e., which will detect some, but not all inconsistencies. Despite the incompleteness of such an algorithm, we can still hope that it would only fail to detect inconsistencies in some pathological cases. In many applications, such as multi–document summarization and temporal question answering, this would be sufficient, as long as *most* inconsistencies can be detected.

In the following, let $C(x,y)$ denote the set of formulas from Θ involving the variables x and y, and let $X = \{x_1, x_2, \ldots, x_n\}$. Due to Lemma 1, we can assume that Θ only contains lower bounds; e.g., an upper bound like $eb^{\preccurlyeq}(x_i,x_j) \leq 0.7$ can be replaced by the equivalent formula $be^{\ll}(x_j,x_i) \geq 0.3$. Moreover, without loss of generality, we can assume that Θ contains exactly one lower bound for each of the fuzzy temporal relations $bb^{\ll}, bb^{\preccurlyeq}, ee^{\ll}, ee^{\preccurlyeq}, be^{\ll}, be^{\preccurlyeq}, eb^{\ll}, eb^{\preccurlyeq}$ and each pair of variables (x,y) from X^2. Typically, many of these lower bounds will be 0, which means that we have no information at all about the corresponding fuzzy temporal relation for the corresponding pair of variables. Note that $C(x,y)$ is completely specified by 16 values from $[0,1]$, corresponding to 8 lower bounds for the fuzzy temporal relations applied to (x,y) and 8 lower bounds for the fuzzy temporal relations applied to (y,x). For the ease of presentation, we will therefore represent $C(x,y)$ as two lists of 8 values. Specifically, we write

$$C(x,y) = \langle[\alpha_1, \beta_1, \gamma_1, \delta_1, \alpha'_1, \beta'_1, \gamma'_1, \delta'_1], \tag{8}$$
$$[\alpha_2, \beta_2, \gamma_2, \delta_2, \alpha'_2, \beta'_2, \gamma'_2, \delta'_2]\rangle \tag{9}$$

TABLE I

DEFINITION OF THE QUALITATIVE TEMPORAL RELATIONS BETWEEN FUZZY TIME INTERVALS A AND B, AND THEIR CORRESPONDENCE WITH THE CLASSICAL DEFINITIONS WHEN $A = [a^-, a^+]$ AND $B = [b^-, b^+]$ ARE CRISP INTERVALS.

Crisp intervals			Fuzzy time intervals		
$a^- < b^-$	$\Leftrightarrow$	$(\exists p)(p \in A \wedge (\forall q)(q \in B \Rightarrow p < q))$	$bb^{\ll}(A,B)$	$=$	$\sup_p T_W(A(p), \inf_q I_W(B(q), L^{\ll}(p,q)))$
$a^- \leq b^-$	$\Leftrightarrow$	$(\forall q)(q \in B \Rightarrow (\exists x)(p \in A \wedge p \leq q))$	$bb^{\preccurlyeq}(A,B)$	$=$	$\inf_q I_W(B(q), \sup_p T_W(A(p), L^{\preccurlyeq}(p,q)))$
$a^+ < b^+$	$\Leftrightarrow$	$(\exists q)(q \in B \wedge (\forall p)(p \in A \Rightarrow p < q))$	$ee^{\ll}(A,B)$	$=$	$\sup_q T_W(B(q), \inf_p I_W(A(p), L^{\ll}(p,q)))$
$a^+ \leq b^+$	$\Leftrightarrow$	$(\forall p)(p \in A \Rightarrow (\exists q)(q \in B \wedge p \leq q))$	$ee^{\preccurlyeq}(A,B)$	$=$	$\inf_p I_W(A(p), \sup_q T_W(B(q), L^{\preccurlyeq}(p,q)))$
$a^- < b^+$	$\Leftrightarrow$	$(\exists p)(\exists q)(p \in A \wedge q \in B \wedge p < q)$	$be^{\ll}(A,B)$	$=$	$\sup_p T_W(A(p), \sup_q T_W(B(q), L^{\ll}(p,q)))$
$a^- \leq b^+$	$\Leftrightarrow$	$(\exists x)(\exists q)(p \in A \wedge q \in B \wedge p \leq q)$	$be^{\preccurlyeq}(A,B)$	$=$	$\sup_p T_W(A(p), \sup_q T_W(B(q), L^{\preccurlyeq}(p,q)))$
$a^+ < b^-$	$\Leftrightarrow$	$(\forall p)(\forall q)(p \in A \wedge q \in B \Rightarrow p < q)$	$eb^{\ll}(A,B)$	$=$	$\inf_p I_W(A(p), \inf_q I_W(B(q), L^{\ll}(p,q)))$
$a^+ \leq b^-$	$\Leftrightarrow$	$(\forall p)(\forall q)(p \in A \wedge q \in B \Rightarrow p \leq q)$	$eb^{\preccurlyeq}(A,B)$	$=$	$\inf_p I_W(A(p), \inf_q I_W(B(q), L^{\preccurlyeq}(p,q)))$

TABLE II

TRANSITIVITY TABLE FOR FUZZY TEMPORAL RELATIONS.

	$be^{\preccurlyeq}(B,C)$	$bb^{\preccurlyeq}(B,C)$	$ee^{\preccurlyeq}(B,C)$	$eb^{\preccurlyeq}(B,C)$	$be^{\ll}(B,C)$	$bb^{\ll}(B,C)$	$ee^{\ll}(B,C)$	$eb^{\ll}(B,C)$
$be^{\preccurlyeq}(A,B)$	1	1	$be^{\preccurlyeq}(A,C)$	$bb^{\preccurlyeq}(A,C)$	1	1	$be^{\ll}(A,C)$	$bb^{\ll}(A,C)$
$bb^{\preccurlyeq}(A,B)$	$be^{\preccurlyeq}(A,C)$	$bb^{\preccurlyeq}(A,C)$	$be^{\preccurlyeq}(A,C)$	$bb^{\preccurlyeq}(A,C)$	$be^{\ll}(A,C)$	$bb^{\ll}(A,C)$	$be^{\ll}(A,C)$	$bb^{\ll}(A,C)$
$ee^{\preccurlyeq}(A,B)$	1	1	$ee^{\preccurlyeq}(A,C)$	$eb^{\preccurlyeq}(A,C)$	1	1	$ee^{\ll}(A,C)$	$eb^{\ll}(A,C)$
$eb^{\preccurlyeq}(A,B)$	$ee^{\preccurlyeq}(A,C)$	$eb^{\preccurlyeq}(A,C)$	$ee^{\preccurlyeq}(A,C)$	$eb^{\preccurlyeq}(A,C)$	$ee^{\ll}(A,C)$	$eb^{\ll}(A,C)$	$ee^{\ll}(A,C)$	$eb^{\ll}(A,C)$
$be^{\ll}(A,B)$	1	1	$be^{\ll}(A,C)$	$bb^{\ll}(A,C)$	1	1	$be^{\ll}(A,C)$	$bb^{\ll}(A,C)$
$bb^{\ll}(A,B)$	$be^{\ll}(A,C)$	$bb^{\ll}(A,C)$	$be^{\ll}(A,C)$	$bb^{\ll}(A,C)$	$be^{\ll}(A,C)$	$bb^{\ll}(A,C)$	$be^{\ll}(A,C)$	$bb^{\ll}(A,C)$
$ee^{\ll}(A,B)$	1	1	$ee^{\ll}(A,C)$	$eb^{\ll}(A,C)$	1	1	$ee^{\ll}(A,C)$	$eb^{\ll}(A,C)$
$eb^{\ll}(A,B)$	$ee^{\ll}(A,C)$	$eb^{\ll}(A,C)$	$ee^{\ll}(A,C)$	$eb^{\ll}(A,C)$	$ee^{\ll}(A,C)$	$eb^{\ll}(A,C)$	$ee^{\ll}(A,C)$	$eb^{\ll}(A,C)$

to denote the following set of lower bounds

$$
\begin{array}{ll}
be^{\preccurlyeq}(x,y) \geq \alpha_1 & be^{\ll}(x,y) \geq \alpha'_1 \\
bb^{\preccurlyeq}(x,y) \geq \beta_1 & bb^{\ll}(x,y) \geq \beta'_1 \\
ee^{\preccurlyeq}(x,y) \geq \gamma_1 & ee^{\ll}(x,y) \geq \gamma'_1 \\
eb^{\preccurlyeq}(x,y) \geq \delta_1 & eb^{\ll}(x,y) \geq \delta'_1 \\
be^{\preccurlyeq}(y,x) \geq \alpha_2 & be^{\ll}(y,x) \geq \alpha'_2 \\
bb^{\preccurlyeq}(y,x) \geq \beta_2 & bb^{\ll}(y,x) \geq \beta'_2 \\
ee^{\preccurlyeq}(y,x) \geq \gamma_2 & ee^{\ll}(y,x) \geq \gamma'_2 \\
eb^{\preccurlyeq}(y,x) \geq \delta_2 & eb^{\ll}(y,x) \geq \delta'_2
\end{array}
$$

We will furthermore write $C_1(x,y)$ (resp. $C_2(x,y)$) to denote the subset of $C(x,y)$ containing the lower bounds for the fuzzy temporal relations applied to (x,y) (resp. (y,x)). Both $C_1(x,y)$ and $C_2(x,y)$ can be represented by a list of 8 values; for the set $C(x,y)$ defined in (8), we write

$$
\begin{aligned}
C_1(x,y) &= [\alpha_1, \beta_1, \gamma_1, \delta_1, \alpha'_1, \beta'_1, \gamma'_1, \delta'_1] \\
C_2(x,y) &= [\alpha_2, \beta_2, \gamma_2, \delta_2, \alpha'_2, \beta'_2, \gamma'_2, \delta'_2]
\end{aligned}
$$

Note that $C_1(x,y) = C_2(y,x)$ and $C_2(x,y) = C_1(y,x)$.

For (x,y) in X^2, $C(x,y)$ acts as a constraint on the possible values of x and y. The idea of our algorithm is to incrementally refine these constraints, i.e., increase some of the corresponding lower bounds, based on known properties of the fuzzy temporal relations. A first way to do this is by using Lemma 2. For example, if $\alpha_1 = 0.4$ and $\beta_1 = 0.6$, we could change the value of α_1 to 0.6 since whenever $bb^{\preccurlyeq}(x,y) \geq 0.6$, we also have that $be^{\preccurlyeq}(x,y) \geq 0.6$. Procedure `Normalise` shows how the dependencies from Lemma 2 can be used for updating the various lower bounds conveyed by $C(x,y)$.

Procedure `Normalise`

Data: $C(x,y) =$
$\langle [\alpha_1, \beta_1, \gamma_1, \delta_1, \alpha'_1, \beta'_1, \gamma'_1, \delta'_1], [\alpha_2, \beta_2, \gamma_2, \delta_2, \alpha'_2, \beta'_2, \gamma'_2, \delta'_2] \rangle$

Result: If possible, the lower bounds in $C(x,y)$ are increased by using the dependencies from Lemma 2.

for i *in* $\{1,2\}$ **do**

$\beta'_i \leftarrow \max(\beta'_i, \delta'_i)$
$\gamma'_i \leftarrow \max(\gamma'_i, \delta'_i)$
$\alpha'_i \leftarrow \max(\alpha'_i, \beta'_i, \gamma'_i)$
$\delta_i \leftarrow \max(\delta'_i, \delta_i)$
$\beta_i \leftarrow \max(\beta'_i, \beta_i, \delta_i)$
$\gamma_i \leftarrow \max(\gamma'_i, \gamma_i, \delta_i)$
$\alpha_i \leftarrow \max(\alpha'_i, \alpha_i, \beta_i, \gamma_i)$

Example 2: Let $C(x,y)$ be given by

$$
\begin{aligned}
C(x,y) = \langle & [0.3, 0.6, 0.2, 0.3, 0.6, 0.3, 0.5, 0.4], \\
& [0.7, 0.5, 0.4, 0.3, 0.5, 0.1, 0.6, 0.2] \rangle
\end{aligned}
$$

Applying `Normalise` to $C(x,y)$ yields

$$
\begin{aligned}
C(x,y) = \langle & [0.6, 0.6, 0.5, 0.4, 0.6, 0.4, 0.5, 0.4], \\
& [0.7, 0.5, 0.6, 0.3, 0.6, 0.2, 0.6, 0.2] \rangle
\end{aligned}
$$

If $C(x,y)$ does not change by applying `Normalise`, $C(x,y)$ is called normalised. Note that after applying `Normalise` once, $C(x,y)$ is always normalised.

Another way of deriving stronger lower bounds is by using the transitivity rules from Table II, i.e., given $C_1(x,y)$ and $C_1(y,z)$, we can draw some conclusions concerning the lower bounds in $C_1(x,z)$. Function `Compose` takes as input the lists of lower bounds $C_1(x,y)$ and $C_1(y,z)$ and returns a list S of lower bounds for $be^{\preccurlyeq}(x,z)$, $bb^{\preccurlyeq}(x,z)$, $\ldots$, $eb^{\ll}(x,z)$. We can then refine the lower bounds in $C_1(x,z)$ by including all constraints from S. Let $C_1(x,y)$ be defined as before and let S be defined as

$$
S = [\alpha, \beta, \gamma, \delta, \alpha', \beta', \gamma', \delta']
$$

We write $C_1(x,y) \cup S$ to denote the union of the lower bounds in $C_1(x,y)$ and S, i.e.,

$$\begin{aligned} &C_1(x,y) \cup S \\ &= [\max(\alpha_1,\alpha), \max(\beta_1,\beta), \max(\gamma_1,\gamma), \max(\delta_1,\delta), \\ &\quad \max(\alpha'_1,\alpha'), \max(\beta'_1,\beta'), \max(\gamma'_1,\gamma'), \max(\delta'_1,\delta')] \end{aligned} \tag{10}$$

Finally, we need a way to detect inconsistent constraints.

Function `Compose`

Input: $C_1(x,y) = [\alpha_1,\beta_1,\gamma_1,\delta_1,\alpha'_1,\beta'_1,\gamma'_1,\delta'_1]$,
$C_1(y,z) = [\alpha_2,\beta_2,\gamma_2,\delta_2,\alpha'_2,\beta'_2,\gamma'_2,\delta'_2]$
Output: A set S of lower bounds for $be^{\preccurlyeq}(x,z)$, $bb^{\preccurlyeq}(x,z)$, ..., $eb^{\ll}(x,z)$; $S = [\alpha,\beta,\gamma,\delta,\alpha',\beta',\gamma',\delta']$
$\alpha \leftarrow \max(T_W(\alpha_1,\gamma_2), T_W(\beta_1,\alpha_2))$
$\beta \leftarrow \max(T_W(\alpha_1,\delta_2), T_W(\beta_1,\beta_2))$
$\gamma \leftarrow \max(T_W(\gamma_1,\gamma_2), T_W(\delta_1,\alpha_2))$
$\delta \leftarrow \max(T_W(\gamma_1,\delta_2), T_W(\delta_1,\beta_2))$
$\alpha' \leftarrow \max(T_W(\alpha'_1,\gamma_2), T_W(\beta'_1,\alpha_2), T_W(\alpha_1,\gamma'_2), T_W(\beta_1,\alpha'_2))$
$\beta' \leftarrow \max(T_W(\alpha'_1,\delta_2), T_W(\beta'_1,\beta_2), T_W(\alpha_1,\delta'_2), T_W(\beta_1,\beta'_2))$
$\gamma' \leftarrow \max(T_W(\gamma'_1,\gamma_2), T_W(\delta'_1,\alpha_2), T_W(\gamma_1,\gamma'_2), T_W(\delta_1,\alpha'_2))$
$\delta' \leftarrow \max(T_W(\gamma'_1,\delta_2), T_W(\delta'_1,\beta_2), T_W(\gamma_1,\delta'_2), T_W(\delta_1,\beta'_2))$

Function `Consistent` finds inconsistencies by checking whether the dependencies from Lemma 1 are violated. For example, regardless of the fuzzy time interval that is assigned to x and y, it holds that $be^{\preccurlyeq}(x,y) = 1 - eb^{\ll}(y,x)$. Hence, for $C(x,y)$ defined in (8), if $\alpha_1 > 1 - \delta'_2$, this constraint can never be satisfied. Procedure `Closure` is the resulting procedure for finding inconsistencies, similar in spirit to Allen's path consistency algorithm [9]. Lines 1–5 ensure that all constraints are initially normalised, and that no inconsistencies can be detected. Subsequently, constraints are composed using the function `Compose` until no lower bounds can be strengthened anymore. Each time a lower bound is increased, the consistency of the corresponding constraint is checked. Note that $C_1(x_{i_0},x_{k_0}) \subset S$ iff $C_1(x_{i_0},x_{k_0}) \cup S \neq C_1(x_{i_0},x_{k_0})$, where $C_1(x_{i_0},x_{k_0}) \cup S$ is defined as in (10). If the constraint $C_1(x_{i_0},x_{k_0})$ is changed, some triplets need to be reconsidered. Therefore, on line 14 the set *todo* is updated.

Function `Consistent`

Input: $C(x,y) = \langle[\alpha_1,\beta_1,\gamma_1,\delta_1,\alpha'_1,\beta'_1,\gamma'_1,\delta'_1], [\alpha_2,\beta_2,\gamma_2,\delta_2,\alpha'_2,\beta'_2,\gamma'_2,\delta'_2]\rangle$
Output: *false* if it is known that the formulas in $C(x,y)$ cannot be satisfied by assigning a fuzzy time interval to x and y; *true* otherwise.
if $\alpha_1 > 1-\delta'_2 \vee \beta_1 > 1-\beta'_2 \vee \gamma_1 > 1-\gamma'_2 \vee \delta_1 > 1-\alpha'_2$ $\vee \alpha'_1 > 1-\delta_2 \vee \beta'_1 > 1-\beta_2 \vee \gamma'_1 > 1-\gamma_2 \vee \delta'_1 > 1-\alpha_2$ **then**
  return *false*
else
  return *false*

Procedure `Closure`

1 **for** $i \leftarrow 1$ **to** n **do**
2   **for** $j \leftarrow i+1$ **to** n **do**
3   *Normalise*$(C(x_i,x_j))$
4   **if** $\neg$*Consistent*$(C(x_i,x_j))$ **then**
5   **return** inconsistency found
6 $todo \leftarrow \{(i,j,k) | 1 \leq i,j,k \leq n \wedge i \neq j \neq k\}$
7 **while** $todo \neq \emptyset$ **do**
8   Select and remove a triplet (i_0,j_0,k_0) from *todo*
9   $S \leftarrow C_1(x_{i_0},x_{k_0}) \cup$ *Compose*$(C_1(x_{i_0},x_{j_0}), C_1(x_{j_0},x_{k_0}))$
10   **if** $C_1(x_{i_0},x_{k_0}) \subset S$ **then**
11     $C_1(x_{i_0},x_{k_0}) \leftarrow S$
12   *Normalise*$(C(x_{i_0},x_{k_0}))$
13   **if** *Consistent*(S) **then**
14       $todo \leftarrow todo$
15         $\cup \{(i_0,k_0,l) | 1 \leq l \leq n \wedge l \neq i_0 \neq k_0\}$
16         $\cup \{(l,i_0,k_0) | 1 \leq l \leq n \wedge l \neq i_0 \neq k_0\}$
17   **else**
18   **return** inconsistency found

Note that in the discussion above, for simplicity and ease of presentation, we have assumed that Θ does not contain disjunctive formulas such as $bb^{\ll}(x,y) \geq 0.7 \vee eb^{\preccurlyeq}(y,x) \geq 0.8$. As a consequence, some (generalizations of) relations from the Interval Algebra cannot be represented, in particular non-convex relations like "either A is completely before B or B is completely before A". However, if needed, the present framework can easily be extended to cope with such disjunctive relations by considering disjunctions of constraints of the form (8). Procedures to normalise, compose and check the consistency of such constraints follow straightforwardly from those discussed in this paper.

To analyse the time complexity of Procedure `Closure`, we assume that all lower bounds are initially taken from a finite set $M = \{0, \Delta, 2\Delta, \ldots, 1\}$. As long as all lower bounds are finitely representable, this assumption can always be met. It is easy to see that the lower bounds returned by Function `Compose` and the lower bounds resulting from Procedure `Normalise` are then contained in M as well. As a consequence, each constraint $C(x,y)$ can at most be changed $O(|M|)$ times. As, moreover, there are $O(n^2)$ such constraints, and each change adds $O(n)$ elements to the set *todo*, Procedure `Closure` takes $O(|M|n^3)$ time to complete.

In addition to the polynomial time approximation discussed above, we also have a complete exponential time algorithm at our disposal. For a detailed discussion about this algorithm, we refer to [7]. The main idea is that the problem of consistency checking for fuzzy time intervals can be reduced to consistency checking in a point algebra with disjunction [25]. Solving this last problem involves backtracking over disjunctive constraints between time points. We implemented two variants of this algorithm. The first variant, which we will refer to as `Complete`, is a standard backtracking implementation, which uses forward checking to detect inconsistencies as soon as possible, but uses no further optimizations. The second variant, which we will refer to as `Complete-optimized`, implements three optimizations that were proposed in [26] for temporal reasoning with disjunctive constraints: conflict–directed backjumping, removal of subsumed variables and semantic branching.

V. 2–CONSISTENCY

Let Θ, X, $C(x,y)$, $C_1(x,y)$ and $C_2(x,y)$ be defined as in Section IV. If for every pair of variables (x,y) from X^2, $C(x,y)$ is a consistent constraint, then Θ is called 2–consistent or arc–consistent. Clearly, if Θ is not 2–consistent, Θ cannot be consistent, hence we can sometimes detect inconsistencies by only checking 2–consistency. In particular, we would like to improve Function `Consistent` such that it returns only $true$ if the input is indeed a consistent constraint $C(x, y)$. First we establish a number of dependencies between fuzzy temporal relations, and, subsequently, we show that the consistency of $C(x, y)$ can always be decided by checking whether these dependencies, as well as the dependencies discussed in Section III-B, are violated. At the same time, these new dependencies will allow us to improve Procedure `Normalise`.

Lemma 4: Let A and B be fuzzy time intervals. It holds that

$$be^{\ll}(A,B) \geq T_W(be^{\preccurlyeq}(A,B), \min(bb^{\preccurlyeq}(A,B), ee^{\preccurlyeq}(A,B)), 1 - eb^{\preccurlyeq}(A,B)) \quad (11)$$

Using Lemma 1, we obtain the following corollary.

Corollary 1: Let A and B be fuzzy time intervals. It holds that

$$be^{\ll}(A,B) \leq S_W(1 - eb^{\preccurlyeq}(A,B), eb^{\ll}(A,B), \max(bb^{\ll}(A,B), ee^{\ll}(A,B)))$$

Lemma 5: Let A and B be fuzzy time intervals. It holds that

$$bb^{\ll}(A,B) \leq T_W(bb^{\preccurlyeq}(A,B), \max(be^{\ll}(A,B), 1 - eb^{\preccurlyeq}(A,B))) \quad (12)$$

$$ee^{\ll}(A,B) \leq T_W(ee^{\preccurlyeq}(A,B), \max(be^{\ll}(A,B), 1 - eb^{\preccurlyeq}(A,B))) \quad (13)$$

For crisp intervals $A = [a^-, a^+]$ and $B = [b^-, b^+]$, (12) corresponds to the trivial observation that if $a^- < b^-$ then $a^- \leq b^-$ and ($a^- < b^+$ or $b^- < a^+$).

Lemma 6: Let A and B be fuzzy time intervals. It holds that

$$eb^{\ll}(A,B) > 0 \Rightarrow be^{\ll}(A,B) = 1 \quad (14)$$

Lemma 6 becomes trivial when A and B are crisp intervals: $a^+ < b^- \Rightarrow a^- < b^+$. Using Lemma 1, we obtain the following corollary.

Corollary 2: Let A and B be fuzzy time intervals. It holds that

$$eb^{\preccurlyeq}(A,B) > 0 \Rightarrow be^{\preccurlyeq}(A,B) = 1 \quad (15)$$

The following proposition states that the dependencies introduced in this section, in addition to the dependencies from Lemma 2, are sufficient for checking the consistency of $C(x, y)$. Note that this implies that we have discovered all dependencies, i.e., every other dependency between the fuzzy temporal relations applied to the same variables x and y will be entailed by the aforementioned dependencies.

Proposition 1: Let $\alpha, \beta, \gamma, \delta, \alpha', \beta', \gamma', \delta' \in [0, 1]$. There exist fuzzy time intervals A and B such that $be^{\preccurlyeq}(A,B) = \alpha$, $bb^{\preccurlyeq}(A,B) = \beta$, $ee^{\preccurlyeq}(A,B) = \gamma$, $eb^{\preccurlyeq}(A,B) = \delta$, $be^{\ll}(A,B) = \alpha'$, $bb^{\ll}(A,B) = \beta'$, $ee^{\ll}(A,B) = \gamma'$ and $eb^{\ll}(A,B) = \delta'$ iff

$$\alpha \geq \beta \geq \delta \quad (16)$$
$$\alpha \geq \gamma \geq \delta \quad (17)$$
$$\alpha \geq \alpha' \quad (18)$$
$$\gamma \geq \gamma' \quad (19)$$
$$\alpha' \geq T_W(\alpha, \min(\beta, \gamma), 1 - \delta) \quad (20)$$
$$\beta' \leq T_W(\beta, \max(\alpha', 1 - \delta)) \quad (21)$$
$$\alpha' = 1 \vee \delta' = 0 \quad (22)$$
$$\alpha' \geq \beta' \geq \delta' \quad (23)$$
$$\alpha' \geq \gamma' \geq \delta' \quad (24)$$
$$\beta \geq \beta' \quad (25)$$
$$\delta \geq \delta' \quad (26)$$
$$\alpha' \leq S_W(1 - \delta, \delta', \max(\beta', \gamma')) \quad (27)$$
$$\gamma' \leq T_W(\gamma, \max(\alpha', 1 - \delta)) \quad (28)$$
$$\alpha = 1 \vee \delta = 0 \quad (29)$$

Given the lower bounds in $C(x, y)$, Proposition 1 can be used to specify a system of (disjunctions of) linear inequalities Σ which has a solution iff $C(x, y)$ is consistent. Function `Consistent-revised` shows how this can be done. The variables $a, b, \ldots, d'$ correspond to the unknown values of $be^{\preccurlyeq}(x,y)$, $bb^{\preccurlyeq}(x,y)$, …, $eb^{\ll}(x,y)$. The inequalities on lines 1–4 ensure that any solution of Σ satisfies the lower bounds in $C(x, y)$. Note that the lower bounds in $C_2(x, y)$ are converted into upper bounds using Lemma 1. The dependencies from Lemma 2 (i.e., (16)–(19) and (23)–(26)) are imposed by the inequalities on lines 5–6. Lines 7–8 corresponds to (20) and (27). To see this, consider for example (20):

$$\begin{aligned}
&\alpha' \geq T_W(\alpha, \min(\beta, \gamma), 1 - \delta) \\
&\Leftrightarrow \alpha' \geq \max(0, \alpha + T_W(\min(\beta, \gamma), 1 - \delta) - 1) \\
&\Leftrightarrow \alpha' \geq \max(0, \alpha + \max(0, \min(\beta, \gamma) - \delta) - 1) \\
&\Leftrightarrow \alpha' \geq 0 \wedge \alpha' \geq \alpha - 1 \wedge \alpha' \geq \alpha + \min(\beta, \gamma) - \delta - 1
\end{aligned}$$

As $\alpha' \geq 0$ and $\alpha' \geq \alpha - 1$ are trivially satisfied, the last expression is equivalent to $\alpha' \geq \alpha + \min(\beta, \gamma) - \delta - 1$. In the same way, lines 9–11 corresponds to (21)–(22) and (28)–(29). Checking whether a system of linear inequalities has a solution can be done using a linear programming solver. Since the number of variables and inequalities in Σ is constant, checking whether Σ has a solution can be done in constant time. Note that, as Σ contains disjunctions, more than one system of linear inequalities may need to be checked.

Function `Consistent-revised`

Input: $C(x, y) = \langle[\alpha_1, \beta_1, \gamma_1, \delta_1, \alpha'_1, \beta'_1, \gamma'_1, \delta'_1], [\alpha_2, \beta_2, \gamma_2, \delta_2, \alpha'_2, \beta'_2, \gamma'_2, \delta'_2]\rangle$
Output: $false$ if $C(x, y)$ cannot be satisfied by assigning a fuzzy time interval to x and y; $true$ otherwise

1 $\Sigma \leftarrow \{\alpha_1 \leq a \leq 1 - \delta'_2, \beta_1 \leq b \leq 1 - \beta'_2,$
2 $\gamma_1 \leq c \leq 1 - \gamma'_2, \delta_1 \leq d \leq 1 - \alpha'_2,$
3 $\alpha'_1 \leq a' \leq 1 - \delta_2, \beta'_1 \leq b' \leq 1 - \beta_2,$
4 $\gamma'_1 \leq c' \leq 1 - \gamma_2, \delta'_1 \leq d' \leq 1 - \alpha_2,$
5 $a \geq b \geq d, a \geq c \geq d, a' \geq b' \geq d', a' \geq c' \geq d',$
6 $a \geq a', b \geq b', c \geq c', d \geq d',$
7 $(a' \geq a + b - d - 1 \vee a' \geq a + c - d - 1),$
8 $(a' \leq 1 - d + d' + b' \vee a' \leq 1 - d + d' + c'),$
9 $(b' \leq b + a' - 1 \vee b' \leq b - d),$
10 $(c' \leq c + a' - 1 \vee c' \leq c - d),$
11 $(a \geq 1 \vee d \leq 0), (a' \geq 1 \vee d' \leq 0)\}$
12 **if** Σ *has a solution* **then**
13 **return** $true$
14 **else**
15 **return** $false$

The same dependencies can also be used to improve `Normalise`, yielding Procedure `Normalise-revised`.

Consider, for example, the dependency from Lemma 4. Using Lemma 1 we establish

$$\begin{aligned} & be^{\ll}(A,B) \geq T_W(be^{\preccurlyeq}(A,B), 1 - eb^{\preccurlyeq}(A,B), \\ & \qquad \min(bb^{\preccurlyeq}(A,B), ee^{\preccurlyeq}(A,B))) \\ \Leftrightarrow\ & be^{\ll}(A,B) \geq T_W(be^{\preccurlyeq}(A,B), be^{\ll}(B,A), \\ & \qquad \min(bb^{\preccurlyeq}(A,B), ee^{\preccurlyeq}(A,B))) \end{aligned}$$

which gives rise to line 8 in `Normalise-revised`. Hence, given the lower bounds for $be^{\preccurlyeq}(A,B)$, $bb^{\preccurlyeq}(A,B)$, $ee^{\preccurlyeq}(A,B)$ and $be^{\ll}(B,A)$, we can infer a lower bound for $be^{\ll}(A,B)$. Furthermore, by applying (1), (6) and Lemma 1 we find

$$\begin{aligned} & be^{\ll}(A,B) \geq T_W(be^{\preccurlyeq}(A,B), be^{\ll}(B,A), \\ & \qquad \min(bb^{\preccurlyeq}(A,B), ee^{\preccurlyeq}(A,B))) \\ \Leftrightarrow\ & I_W(be^{\ll}(B,A), be^{\ll}(A,B)) \\ & \quad \geq T_W(be^{\preccurlyeq}(A,B), \min(bb^{\preccurlyeq}(A,B), ee^{\preccurlyeq}(A,B))) \\ \Leftrightarrow\ & I_W(1 - be^{\ll}(A,B), 1 - be^{\ll}(B,A)) \\ & \quad \geq T_W(be^{\preccurlyeq}(A,B), \min(bb^{\preccurlyeq}(A,B), ee^{\preccurlyeq}(A,B))) \\ \Leftrightarrow\ & 1 - be^{\ll}(B,A) \geq T_W(be^{\preccurlyeq}(A,B), 1 - be^{\ll}(A,B), \\ & \qquad \min(bb^{\preccurlyeq}(A,B), ee^{\preccurlyeq}(A,B))) \\ \Leftrightarrow\ & eb^{\preccurlyeq}(A,B) \geq T_W(be^{\preccurlyeq}(A,B), eb^{\preccurlyeq}(B,A), \\ & \qquad \min(bb^{\preccurlyeq}(A,B), ee^{\preccurlyeq}(A,B))) \end{aligned}$$

which may allow to find a stronger lower bound for $eb^{\preccurlyeq}(A,B)$, as expressed in line 9. Similarly, we obtain

$$\begin{aligned} & eb^{\preccurlyeq}(A,B) \geq T_W(be^{\preccurlyeq}(A,B), eb^{\preccurlyeq}(B,A), \\ & \qquad \min(bb^{\preccurlyeq}(A,B), ee^{\preccurlyeq}(A,B))) \\ \Leftrightarrow\ & eb^{\ll}(B,A) \geq T_W(be^{\ll}(B,A), eb^{\preccurlyeq}(B,A), \\ & \qquad \min(bb^{\preccurlyeq}(A,B), ee^{\preccurlyeq}(A,B))) \end{aligned}$$

corresponding to line 10 in `Normalise-revised`. In the same way, lines 11–12 are valid updates due to Lemma 5. Finally, note that lines 4–5 correspond to Lemma 6 and lines 6–7 correspond to Corollary 2.

Procedure `Normalise-revised`

Data: $C(x,y) = \langle [\alpha_1, \beta_1, \gamma_1, \delta_1, \alpha'_1, \beta'_1, \gamma'_1, \alpha'_1], [\alpha_2, \beta_2, \gamma_2, \delta_2, \alpha'_2, \beta'_2, \gamma'_2, \alpha'_2] \rangle$
Result: If possible, the lower bounds in $C(x,y)$ are increased.

1 *Normalise*$(C(x,y))$
2 **while** *changes occur* **do**
3   **for** i *in* $\{1,2\}$ **do**
4   **if** $\delta_i > 0$ **then**
5       $\alpha_i = 1$
6   **if** $\delta'_i > 0$ **then**
7       $\alpha'_i = 1$
8     $\alpha'_i \leftarrow \max(\alpha'_i, T_W(\alpha_i, \min(\beta_i, \gamma_i), \alpha'_{1-i}))$
9     $\delta_i \leftarrow \max(\delta_i, T_W(\alpha_i, \min(\beta_i, \gamma_i), \delta_{1-i}))$
10     $\delta'_{1-i} \leftarrow \max(\delta'_{1-i}, T_W(\alpha'_{1-i}, \min(\beta_i, \gamma_i), \delta_{1-i}))$
11     $\beta_i \leftarrow \max(\beta_i, S_W(\beta'_i, \min(\delta_i, \delta_{1-i})))$
12     $\gamma_i \leftarrow \max(\gamma_i, S_W(\gamma'_i, \min(\delta_i, \delta_{1-i})))$

VI. Transitivity of fuzzy temporal relations

To further improve on Procedure `Closure`, in this section we investigate some transitivity properties which are stronger than those from Table II. To keep the problem manageable, we omit the fuzzy temporal relations $be^{\ll}$, $bb^{\ll}$, $ee^{\ll}$ and $eb^{\ll}$ from the following discussion. Note that these fuzzy temporal relations are somewhat less useful for applications like multi–document summarization or temporal question answering, as a natural language statement expressing that the end of A occurred before B, for example, does not always mean that the end of A is strictly before the beginning of B, i.e., it may still be possible that the end of A coincides with the beginning of B.

We want to derive the strongest lower bounds possible for $be^{\preccurlyeq}(x,z)$, $bb^{\preccurlyeq}(x,z)$, $ee^{\preccurlyeq}(x,z)$ and $eb^{\preccurlyeq}(x,z)$ given only the lower bounds for $be^{\preccurlyeq}(x,y)$, $bb^{\preccurlyeq}(x,y)$, $ee^{\preccurlyeq}(x,y)$, $eb^{\preccurlyeq}(x,y)$, $be^{\preccurlyeq}(y,z)$, $bb^{\preccurlyeq}(y,z)$, $ee^{\preccurlyeq}(y,z)$ and $eb^{\preccurlyeq}(y,z)$ for some variable y. First, in the following three lemmas, we investigate some transitivity properties which may sometimes yield stronger conclusions than the transitivity rules from Table II.

Lemma 7: Let A, B and C be fuzzy time intervals. It holds that

$$\begin{aligned} bb^{\preccurlyeq}(A,C) \geq \min(& be^{\preccurlyeq}(A,B) + T_W(ee^{\preccurlyeq}(A,B), eb^{\preccurlyeq}(B,C)), \\ & eb^{\preccurlyeq}(B,C), \\ & bb^{\preccurlyeq}(A,B) + T_W(eb^{\preccurlyeq}(B,C), be^{\preccurlyeq}(A,B))) \end{aligned} \tag{30}$$

$$\begin{aligned} ee^{\preccurlyeq}(A,C) \geq \min(& T_W(eb^{\preccurlyeq}(A,B), bb^{\preccurlyeq}(B,C)) + be^{\preccurlyeq}(B,C), \\ & eb^{\preccurlyeq}(A,B), \\ & T_W(eb^{\preccurlyeq}(A,B), be^{\preccurlyeq}(B,C)) + ee^{\preccurlyeq}(B,C)) \end{aligned} \tag{31}$$

Lemma 8: Let A, B and C be fuzzy time intervals. It holds that

$$T_W(be^{\preccurlyeq}(A,B), eb^{\preccurlyeq}(B,C)) > 0 \Rightarrow be^{\preccurlyeq}(A,C) \geq be^{\preccurlyeq}(A,B) \tag{32}$$

$$T_W(be^{\preccurlyeq}(B,C), eb^{\preccurlyeq}(A,B)) > 0 \Rightarrow be^{\preccurlyeq}(A,C) \geq be^{\preccurlyeq}(B,C) \tag{33}$$

If $T_W(be^{\preccurlyeq}(A,B), eb^{\preccurlyeq}(B,C)) > 0$ then $be^{\preccurlyeq}(B,C) = 1$ by Corollary 2, hence

$$be^{\preccurlyeq}(A,B) = \min(be^{\preccurlyeq}(A,B), be^{\preccurlyeq}(B,C)) \tag{34}$$

Similarly, (34) holds when $T_W(be^{\preccurlyeq}(B,C), eb^{\preccurlyeq}(A,B)) > 0$. This leads to the following corollary:

Corollary 3: Let A, B and C be fuzzy time intervals. It holds that

$$\begin{aligned} & T_W(be^{\preccurlyeq}(A,B), eb^{\preccurlyeq}(B,C)) > 0 \\ & \vee T_W(be^{\preccurlyeq}(B,C), eb^{\preccurlyeq}(A,B)) > 0 \\ & \Rightarrow be^{\preccurlyeq}(A,C) \geq \min(be^{\preccurlyeq}(A,B), be^{\preccurlyeq}(B,C)) \end{aligned}$$

Lemma 9: Let A, B and C be fuzzy time intervals. It holds that

$$eb^{\preccurlyeq}(A,C) \geq \min(eb^{\preccurlyeq}(A,B), eb^{\preccurlyeq}(B,C))$$

Note that if A, B and C are crisp intervals, the three previous lemmas become trivial.

The next proposition shows that the transitivity rules from Lemma 7–9, together with those from Table II, are the strongest transitivity rules possible given only a lower bound for $be^{\preccurlyeq}(x,y)$, $bb^{\preccurlyeq}(x,y)$, $ee^{\preccurlyeq}(x,y)$, $eb^{\preccurlyeq}(x,y)$, $be^{\preccurlyeq}(y,z)$, $bb^{\preccurlyeq}(y,z)$, $ee^{\preccurlyeq}(y,z)$ and $eb^{\preccurlyeq}(y,z)$ for some variable y. In particular, this proposition states that if one of the lower bounds on $be^{\preccurlyeq}(x,z)$, $bb^{\preccurlyeq}(x,z)$, $ee^{\preccurlyeq}(x,z)$ and $eb^{\preccurlyeq}(x,z)$ that are obtained from applying these transitivity rules would be further increased, there always exist fuzzy sets A, B and C corresponding to the variables x, y and z such that this lower bound is violated.

Proposition 2: Let $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2 \in [0,1]$ such that

$$\alpha_1 \geq \beta_1 \geq \delta_1 \qquad \alpha_1 \geq \gamma_1 \geq \delta_1 \tag{35}$$

$$\alpha_2 \geq \beta_2 \geq \delta_2 \qquad \alpha_2 \geq \gamma_2 \geq \delta_2 \tag{36}$$

$$\alpha_1 = 1 \vee \delta_1 = 0 \qquad \alpha_2 = 1 \vee \delta_2 = 0 \tag{37}$$

In other words, $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2$ satisfy the conditions from Proposition 1 involving only these values.

Let β_3, γ_3 and δ_3 be defined as

$$\begin{aligned}
\beta_3 &= \max(T_W(\beta_1,\beta_2), \\
&\qquad \min(\alpha_1 + T_W(\delta_2,\gamma_1), \delta_2, \beta_1 + T_W(\delta_2,\alpha_1))) \\
\gamma_3 &= \max(T_W(\gamma_1,\gamma_2), \\
&\qquad \min(\alpha_2 + T_W(\delta_1,\beta_2), \delta_1, \gamma_2 + T_W(\delta_1,\alpha_2))) \\
\delta_3 &= \max(T_W(\delta_1,\beta_2), T_W(\gamma_1,\delta_2), \min(\delta_1,\delta_2))
\end{aligned}$$

Furthermore, let α_3 be defined such that

$$\alpha_3 = 1$$

if $T_W(\gamma_1,\delta_2) > 0$ or $T_W(\delta_1,\beta_2) > 0$,

$$\alpha_3 = \min(\alpha_1,\alpha_2)$$

if $T_W(\gamma_1,\delta_2) = T_W(\delta_1,\beta_2) = 0$ and ($T_W(\alpha_1,\delta_2) > 0$ or $T_W(\delta_1,\alpha_2) > 0$), and

$$\alpha_3 = \max(T_W(\beta_1,\alpha_2), T_W(\alpha_1,\gamma_2))$$

otherwise.

There exist fuzzy time intervals A, B and C satisfying $be^{\preccurlyeq}(A,C) = \alpha_3$ and

$$be^{\preccurlyeq}(A,B) \geq \alpha_1 \tag{38}$$

$$bb^{\preccurlyeq}(A,B) \geq \beta_1 \tag{39}$$

$$ee^{\preccurlyeq}(A,B) \geq \gamma_1 \tag{40}$$

$$eb^{\preccurlyeq}(A,B) \geq \delta_1 \tag{41}$$

$$be^{\preccurlyeq}(B,C) \geq \alpha_2 \tag{42}$$

$$bb^{\preccurlyeq}(B,C) \geq \beta_2 \tag{43}$$

$$ee^{\preccurlyeq}(B,C) \geq \gamma_2 \tag{44}$$

$$eb^{\preccurlyeq}(B,C) \geq \delta_2 \tag{45}$$

Similarly, there exist fuzzy time intervals A, B and C satisfying $bb^{\preccurlyeq}(A,C) = \beta_3$ and (38)–(45). Furthermore, there exist fuzzy time intervals A, B and C satisfying $ee^{\preccurlyeq}(A,C) = \gamma_3$ and (38)–(45). Finally, there exist fuzzy time intervals A, B and C satisfying $eb^{\preccurlyeq}(A,C) = \delta_3$ and (38)–(45).

When A, B and C satisfy (38)–(45), we already know that they satisfy $be^{\preccurlyeq}(A,C) \geq \alpha_3$, $bb^{\preccurlyeq}(A,C) \geq \beta_3$, $ee^{\preccurlyeq}(A,C) \geq \gamma_3$ and $eb^{\preccurlyeq}(A,C) \geq \delta_3$ from Lemma 7–9 and the transitivity rules from Table II. The fact that $be^{\preccurlyeq}(A,C) = 1$ when $T_W(\gamma_1,\delta_2) > 0$ or $T_W(\delta_1,\beta_2) > 0$ follows from the fact that $eb^{\preccurlyeq}(A,C) > 0$ in this case, using Corollary 2.

Based on Proposition 2 we can improve Function `Compose` to Function `Compose-revised`.

Function `Compose-revised`

Input: $C_1(x,y) = [\alpha_1, \beta_1, \gamma_1, \delta_1, \alpha'_1, \beta'_1, \gamma'_1, \delta'_1]$,
$C_1(y,z) = [\alpha_2, \beta_2, \gamma_2, \delta_2, \alpha'_2, \beta'_2, \gamma'_2, \delta'_2]$
Output: A set S of lower bounds for $be^{\preccurlyeq}(x,z)$, $bb^{\preccurlyeq}(x,z)$, $\ldots$, $eb^{\ll}(x,z)$; $S = [\alpha, \beta, \gamma, \delta, \alpha', \beta', \gamma', \delta']$
if $T_W(\gamma_1,\delta_2) > 0 \vee T_W(\delta_1,\beta_2) > 0$ **then**
  $\alpha \leftarrow 1$
else if $T_W(\alpha_1,\delta_2) > 0 \vee T_W(\delta_1,\alpha_2) > 0$ **then**
  $\alpha \leftarrow \min(\alpha_1,\alpha_2)$
else
  $\alpha \leftarrow \max(T_W(\beta_1,\alpha_2), T_W(\alpha_1,\gamma_2))$
$\beta \leftarrow \max(T_W(\beta_1,\beta_2), \min(\alpha_1 + T_W(\delta_2,\gamma_1), \delta_2, \beta_1 + T_W(\delta_2,\alpha_1)))$
$\gamma \leftarrow \max(T_W(\gamma_1,\gamma_2), \min(\alpha_2 + T_W(\delta_1,\beta_2), \delta_1, \gamma_2 + T_W(\delta_1,\alpha_2)))$
$\delta \leftarrow \max(T_W(\delta_1,\beta_2), T_W(\gamma_1,\delta_2), \min(\delta_1,\delta_2))$
$\alpha' \leftarrow \max(T_W(\alpha'_1,\gamma_2), T_W(\beta'_1,\alpha_2), T_W(\alpha_1,\gamma'_2), T_W(\beta_1,\alpha'_2))$
$\beta' \leftarrow \max(T_W(\alpha'_1,\delta_2), T_W(\beta'_1,\beta_2), T_W(\alpha_1,\delta'_2), T_W(\beta_1,\beta'_2))$
$\gamma' \leftarrow \max(T_W(\gamma'_1,\gamma_2), T_W(\delta'_1,\alpha_2), T_W(\gamma_1,\gamma'_2), T_W(\delta_1,\alpha'_2))$
$\delta' \leftarrow \max(T_W(\gamma'_1,\delta_2), T_W(\delta'_1,\beta_2), T_W(\gamma_1,\delta'_2), T_W(\delta_1,\beta'_2))$

VII. Experimental Results

In this section, we compare the performance of the procedures `Closure`, `Complete` and `Complete-optimized`, which were described in Section IV, as well as the following variants on `Closure`:

1) Proc. `Closure-rev1` uses `Normalise-revised` and `Consistent-revised` instead of `Normalise` and `Consistent`.
2) Proc. `Closure-rev2` uses `Compose-revised` instead of `Compose`.
3) Proc. `Closure-rev3` uses `Normalise-revised`, `Consistent-revised` and `Compose-revised` instead of `Normalise`, `Consistent` and `Compose`.

Given particular values of the parameters n and Δ, representing the number of variables and the precision which is used to encode the various lower bounds as before, and a constant p in $]0,1]$, we randomly generate sets of constraints as follows. At most, $8n(n-1)$ lower bounds can be specified to constrain the possible fuzzy time spans corresponding to each of the n variables. When randomly generating constraints, we need to ensure that none of these sets are trivially inconsistent, e.g., if $be^{\preccurlyeq}(x,y) \geq 0.8$ is imposed, the lower bound for $eb^{\ll}(y,x)$ should be at most 0.2. Therefore, we do not choose the lower bounds of $be^{\preccurlyeq}(x,y)$ and $eb^{\ll}(y,x)$ independent of each other. In particular, for each i and j in $\{0,1,\ldots,n\}$ such that $i \neq j$, we randomly select two values r_1 and r_2 from M (using a uniform distribution). With a probability p we add the formula $be^{\preccurlyeq}(x,y) \geq \min(r_1,r_2)$, and with probability $1-p$ we specify no lower bound for $be^{\preccurlyeq}(x,y)$ at all. Similarly, with probability p we add the formula $eb^{\ll}(y,x) \geq 1 - \max(r_1,r_2)$ and with probability $1-p$ we specify no lower bound for $eb^{\ll}(y,x)$. For the other fuzzy temporal relations, we proceed in the same manner. Thus we obtain a set Θ in which approximately $8n(n-1)p$ lower bounds are specified.

In a first experiment, we keep $n = 5$ and $p = 0.1$ fixed, and analyse the behaviour of both algorithms for varying

TABLE III
NUMBER OF INCONSISTENCIES DETECTED FOR $n = 5$, $p = 0.1$ AND VARYING VALUES OF Δ

$\frac{1}{\Delta}$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
`Complete`	219	275	289	319	328	357	399	401	419	417	438	398	422	427	444	455	448
`Closure`	171	187	181	219	193	196	230	221	227	229	251	221	257	241	252	232	229
`Closure-rev1`	209	260	275	295	299	319	369	359	382	378	397	372	372	395	402	414	407
`Closure-rev2`	198	211	208	222	224	232	274	257	285	279	302	258	309	277	303	285	276
`Closure-rev3`	217	275	289	317	326	352	396	398	419	415	436	397	420	422	443	453	442

values of Δ. For each Δ in $\{\frac{1}{18}, \frac{1}{17}, \frac{1}{16}, \ldots, \frac{1}{2}\}$, we generated 1000 sets of constraints using the method described above. Table III shows how many of these sets are found to be inconsistent for each of the reasoning procedures. Recall that `Complete` and `Complete-optimized` will both identify every inconsistent set. Therefore, the results for `Complete-optimized` are omitted from this table. Table III reveals that many inconsistencies are not detected by Procedure `Closure`, especially for small values of Δ; e.g., for $\Delta = \frac{1}{18}$, only 51% of the inconsistent sets are identified. Both `Closure-rev1` and `Closure-rev2` improve on `Closure` significantly. Procedure `Closure-rev3`, which combines the improvements used in `Closure-rev1` and `Closure-rev2`, provides even better results: inconsistencies are detected in all but a few cases.

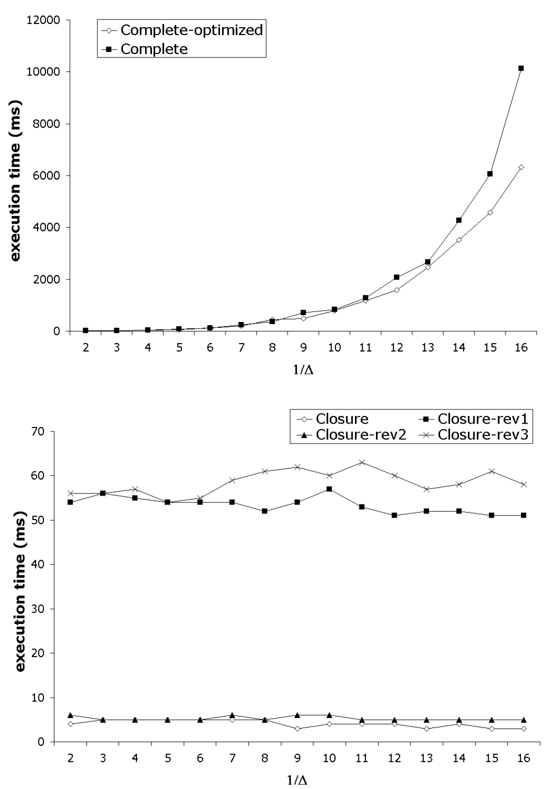


Fig. 1. Average execution time needed to detect inconsistencies for $n = 5$, $p = 0.1$ and varying values of Δ.

Figure 1 depicts the execution time needed on average for each of the 1000 sets of constraints, while Figure 2 depicts the maximal execution time that was needed to check the consistency of a set of constraints[1]. From these figures, it becomes clear that the execution time of `Closure`, `Closure-rev1`, `Closure-rev2`, and `Closure-rev3` is, in practice, largely independent of the value of Δ, whereas the exectution time of `Complete` and `Complete-optimized` depends heavily on this value. These results furthermore suggest that `Complete-optimized` may be useful in practice, as long as the size of M is relatively small (e.g., $\Delta = \frac{1}{3}$ or $\Delta = \frac{1}{4}$). For smaller values of Δ (i.e., when M is larger), the execution time of `Closure` and its variants is significantly less than that of `Complete-optimized`. Note that the maximal execution time of `Complete-optimized` is significantly lower than that of `Complete`.

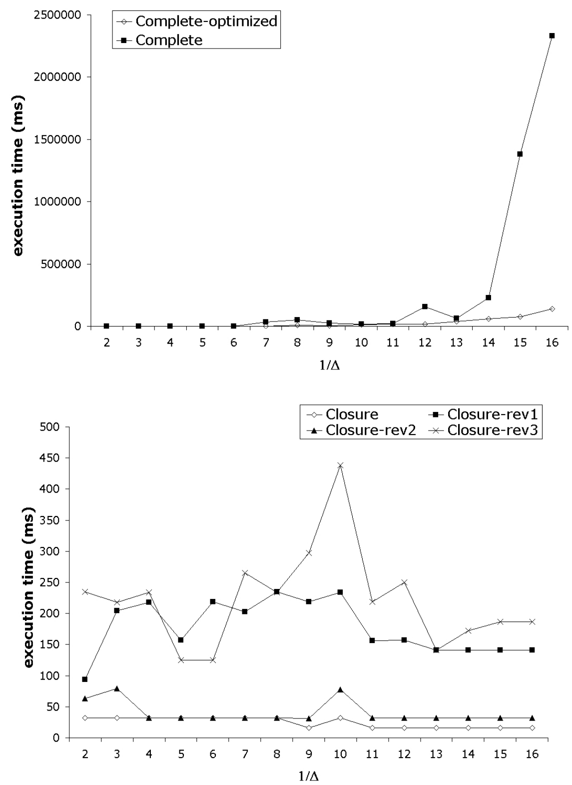


Fig. 2. Maximal execution time needed to detect inconsistencies for $n = 5$, $p = 0.1$ and varying values of Δ.

Table IV displays the number of inconsistencies that are found for various values of n when $p = 0.1$ and $\Delta = 0.1$ are fixed. Similarly, Table V shows the number of inconsistencies for $\Delta = 0.1$ and $n = 5$ fixed, and a varying value of p. Again,

[1]All algorithms were implemented in Java and executed on a 2.80 GHz Pentium 4 system running Windows XP, SP2. The JVM was allowed to use 400 MB of internal memory.

TABLE IV

NUMBER OF INCONSISTENCIES DETECTED FOR $p = 0.1$, $\Delta = 0.1$ AND A VARYING NUMBER OF VARIABLES n

n	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
`Complete`	120	243	419	588	759	865	932	979	992	998	1000	1000	1000	1000	1000	1000	1000
`Closure`	78	130	237	334	476	523	651	748	828	882	926	945	965	988	993	996	998
`Closure-rev1`	118	225	382	542	698	804	888	945	980	990	996	1000	1000	1000	1000	1000	1000
`Closure-rev2`	82	145	285	412	573	675	811	912	940	981	998	996	1000	1000	1000	1000	1000
`Closure-rev3`	120	241	419	586	756	860	927	976	991	997	1000	1000	1000	1000	1000	1000	1000

TABLE V

NUMBER OF INCONSISTENCIES DETECTED FOR $n = 5$, $\Delta = 0.1$ AND A VARYING VALUE OF p

p	0.04	0.08	0.12	0.16	0.20	0.24	0.28	0.32	0.34	0.36	0.40	0.44	0.48	0.52	0.56	0.60
`Complete`	44	290	534	753	882	951	984	989	999	1000	1000	1000	1000	1000	1000	1000
`Closure`	21	170	303	495	633	755	844	909	954	974	989	991	993	997	999	1000
`Closure-rev1`	40	260	488	691	847	925	972	983	997	998	999	1000	1000	999	1000	1000
`Closure-rev2`	25	198	375	580	721	839	922	957	978	994	996	998	998	999	1000	1000
`Closure-rev3`	44	287	531	747	881	949	981	988	999	1000	1000	1000	1000	1000	1000	1000

1000 sets of constraints were generated for each combination of the parameters. Both the results in Table IV and Table V confirm our observations from Table III. In Figure 3, the average execution time for `Complete-optimized` is shown. It becomes clear from this figure that the computation time needed for detecting inconsistencies follows an easy–hard–easy pattern, where under–constrained and over–constrained problems, i.e., problems corresponding to very high or very low values of p and/or n, are easy to solve, and in between there is a class of problems which are computationally very hard. This is further illustrated in Figure 4(a), which shows the effect of changing both the values of p and n.

Procedure `Closure-rev3` is much more efficient than `Complete-optimized`, as can be seen in Figure 4(b). When considering maximal execution time, this difference is even more pronounced: for the sets of constraints that were used for the results in Figure 3, the maximal execution time for `Closure-optimized` was over 448 seconds ($p = 0.04$, $n = 14$), while for `Closure-rev3` this was less than 10 seconds ($p = 0.08$, $n = 12$). Hence, `Closure-rev3` seems to be a good compromise between completeness and scalability. However, although most inconsistencies can be detected using `Closure-rev3`, there will always be inconsistent sets of constraints for which this procedure fails. One example involving only three variables is $\Theta = \{ee^{\preccurlyeq}(y,z) \geq 0.2, eb^{\preccurlyeq}(x,z) \geq 0.1, eb^{\ll}(x,y) \geq 0.3, ee^{\ll}(z,x) \geq 0.8\}$.

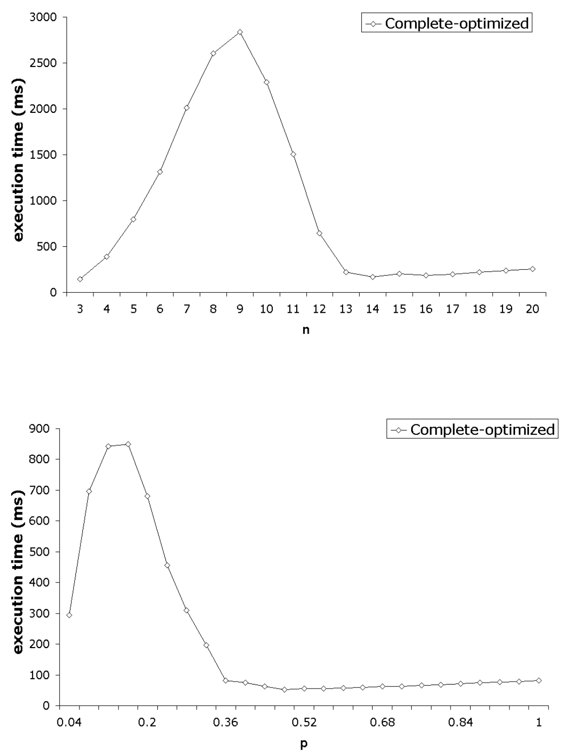
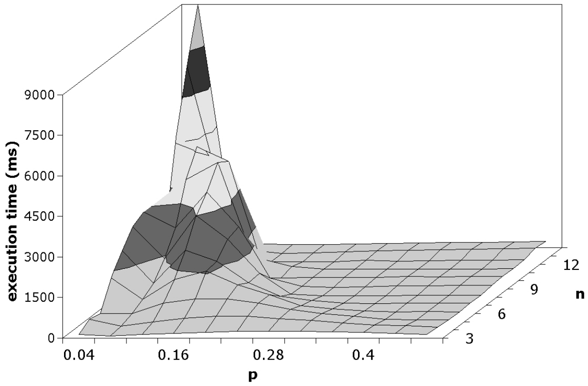


Fig. 3. Average execution time needed to detect inconsistencies for (a) $p = 0.1$, $\Delta = 0.1$ and a varying value of n, and for (b) $\Delta = 0.1$, $n = 5$ and a varying value of p.

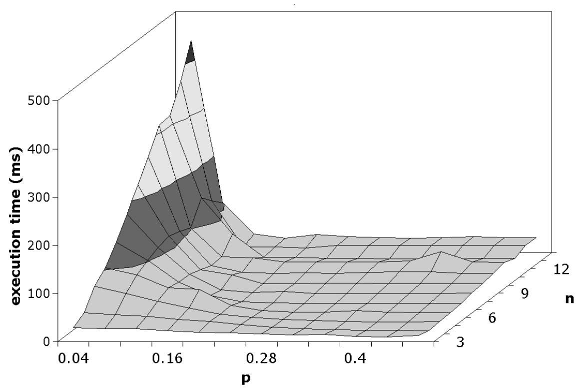
VIII. CONCLUDING REMARKS

In this paper, we have introduced a novel algorithm for fuzzy temporal reasoning which is efficient and, although incomplete, detects inconsistencies in all but a few cases. Thus it forms the first practical approach to fuzzy temporal reasoning as previous algorithms are either too time consuming — complete algorithms may require exponential time — or too weak to derive interesting conclusions. The groundwork for this algorithm is laid by two important properties of fuzzy temporal relations. First, we have proven a number of dependencies between fuzzy temporal relations and shown how these can be used to decide the 2–consistency of a set of constraints, defined as upper and lower bounds of fuzzy temporal relations. Second, we have provided transitivity rules and shown that these are the strongest transitivity rules possible using only lower bounds on the fuzzy temporal relations $be^{\preccurlyeq}$, $bb^{\preccurlyeq}$, $ee^{\preccurlyeq}$ and $eb^{\preccurlyeq}$, which would allow to detect even more inconsistencies.

Besides their practical value, the properties shown in this paper are also important from a theoretical point of view. In particular, they reveal that fuzzy temporal relations satisfy properties that are far more subtle than those resulting from a straightforward generalization of their crisp counterparts. The restriction that fuzzy time intervals need to be normalised



(a) `Complete-optimized`



(b) `Closure-rev3`

Fig. 4. Average execution time needed to detect inconsistencies for $\Delta = 0.1$ and a varying value of p and n.

seems to play a pivotal role in this. One possible direction for future work is to derive transitivity rules involving also the fuzzy temporal relations $be^{\ll}$, $bb^{\ll}$, $ee^{\ll}$ and $eb^{\ll}$.

In [10], it was shown that the fuzzy temporal reasoning task discussed in this paper is useful in the context of question answering systems to deal with temporal constraints involving historical events. Future work will also focus on applying this model in other information retrieval tasks, such as multi-document summarization (e.g., generating overview timelines) or image retrieval (e.g., retrieving images of a given event).

APPENDIX I
PROOFS

In the proofs below, fuzzy time intervals whose membership function is a step function will sometimes be used. In particular, we will use expressions like

$$\{[p_1, p_2[/\lambda_1, \ldots, p_k/1,]p_k, p_{k+1}]/\lambda_k, \ldots,]p_{n-1}, p_n]/\lambda_n\}$$

to denote the fuzzy time interval A defined as

$$A(p) = \begin{cases} \lambda_1 & \text{if } p \in [p_1, p_2[\\ \ldots & \\ 1 & \text{if } p = p_k \\ \lambda_k & \text{if } p \in]p_k, p_{k+1}] \\ \ldots & \\ \lambda_n & \text{if } p \in]p_{n-1}, p_n] \\ 0 & \text{otherwise} \end{cases}$$

for all p in $\mathbb{R}$, where $\lambda_1 \leq \lambda_2 \cdots \leq \lambda_{k-1}$, $\lambda_k \geq \cdots \geq \lambda_{n+1} \geq \lambda_n$. For such fuzzy time intervals, the fuzzy temporal relations can be evaluated in a very convenient way.

Lemma 10: Let A and B be fuzzy time intervals which are constant over $]p_1, p_2[$, $]p_2, p_3[$, $\ldots]p_{n-1}, p_n[$. Assume, moreover, that the support of A and B is contained in $[p_1, p_n]$, and that p_a and p_b are modal values of A and B respectively ($1 \leq a, b \leq n$). It holds that

$$be^{\preccurlyeq}(A, B) = \max_{i \in \{1,2,\ldots,a\}} T_W(A(p_i), B(\max(p_i, p_b))) \quad (46)$$

$$bb^{\preccurlyeq}(A, B) = \min_{i \in \{1,\ldots,\min(a,b)\}} I_W(B(p_i), A(p_i)) \quad (47)$$

$$ee^{\preccurlyeq}(A, B) = \min_{i \in \{\max(a,b),\ldots,n\}} I_W(A(p_i), B(p_i)) \quad (48)$$

$$eb^{\preccurlyeq}(A, B) = \min_{i \in \{a,\ldots,n\}} (1 - T_W(A(p_i), B(\min(p_{i-1}, p_b)))) \quad (49)$$

$$be^{\ll}(A, B) = \max_{i \in \{b,b+1,\ldots,n\}} T_W(B(p_i), A(\min(p_{i-1}, p_a))) \quad (50)$$

$$bb^{\ll}(A, B) = \max_{i \in \{1,\ldots,\min(a,b)\}} T_W(A(p_i), 1 - B(p_i)) \quad (51)$$

$$ee^{\ll}(A, B) = \max_{i \in \{\max(a,b),\ldots,n\}} T_W(B(p_i), 1 - A(p_i)) \quad (52)$$

$$eb^{\ll}(A, B) = \min_{i \in \{1,\ldots,b\}} (1 - T_W(B(p_i), A(\max(p_i, p_a)))) \quad (53)$$

where $p_0 < p_1 \leq p_2 \leq \cdots \leq p_n$ such that $A(p_0) = B(p_0) = 0$.

Proof: As an example, we show (46). The proof of (47)–(49) is analogous. Note that (50)–(53) follow straightforwardly from (46)–(49) by Lemma 1 and (3)–(4).

Let the $\mathbb{R} - \mathbb{R}$ mappings l and r be defined as

$$l(p) = \begin{cases} p & \text{if } p < p_1 \\ \max\{p_i | i \in \{1, 2, \ldots, n\} \wedge p_i \leq p\} & \text{otherwise} \end{cases}$$

$$r(p) = \begin{cases} p & \text{if } p > p_n \\ \min\{p_i | i \in \{1, 2, \ldots, n\} \wedge p_i \geq p\} & \text{otherwise} \end{cases}$$

for all p in $\mathbb{R}$. Since A (resp. B) is increasing for values smaller than p_a (resp. p_b) and decreasing for values greater than p_a (resp. p_b), we have

$$\begin{aligned} be^{\preccurlyeq}(A, B) &= \sup_{p \in \mathbb{R}} T_W(A(p), \sup_{q \in \mathbb{R}} T_W(B(q), L^{\preccurlyeq}(p, q))) \\ &= \sup_{p \leq m_a} T_W(A(p), \sup_{q \geq m_b} T_W(B(q), L^{\preccurlyeq}(p, q))) \\ &= \sup_{p \leq m_a} T_W(A(p), \sup_{q \geq m_b, p \leq q} B(q)) \\ &= \sup_{p \leq m_a} \sup_{q \geq m_b, p \leq q} T_W(A(p), B(q)) \end{aligned}$$

By definition of a fuzzy time interval, if $p \leq m_a$, $A(l(p)) = A(p)$, and if $p \geq m_b$, $B(r(p)) = B(p)$. Based in this observation, we obtain

$$\begin{aligned} &\sup_{p \leq m_a} \sup_{q \geq m_b, p \leq q} T_W(A(p), B(q)) \\ &= \sup_{p \leq m_a} \sup_{q \geq m_b, p \leq q} T_W(A(l(p)), B(r(q))) \\ &= \max_{i \in \{1,2,\ldots,a\}} \max_{\substack{j \in \{b,b+1,\ldots,n\} \\ i \leq j}} T_W(A(p_i), B(p_j)) \end{aligned}$$

As B is decreasing for values greater than p_b, the maximum for j is attained by the smallest value in $\{b, b+1, \ldots, n\}$ satisfying $i \le j$, i.e., $\max(b, i)$. ■

A. Proof of Proposition 1

From Lemma 2, Lemma 4, Corollary 1, Lemma 5, Lemma 6 and Corollary 2, we already know that when A and B exist, all constraints on $\alpha, \beta, \gamma, \delta, \alpha', \beta', \gamma', \delta'$ will be satisfied. Conversely, we will provide a constructive proof, showing that whenever these constraints are satisfied, corresponding fuzzy time intervals A and B can be found. The proof proceeds by case analysis.

First assume that $\delta > 0$, $\delta' > 0$ and $\gamma' \ge \beta'$. Let $p_1 < p_2 < p_3 < p_4 < p_5 < p_6$ be real numbers, and let A and B be defined as

$$\begin{aligned} A &= \{[p_2, p_3[/I_W(\beta, \beta'), p_3/1,]p_3, p_4]/I_W(\delta, \delta'), \\ &\qquad]p_4, p_5]/1-\gamma',]p_5, p_6]/1-\gamma\} \\ B &= \{[p_1, p_3[/1-\beta, [p_3, p_5[/1-\delta', p_5/1\} \end{aligned}$$

First, we verify that A and B are indeed fuzzy time intervals. In particular, it has to hold that $I_W(\beta, \beta') \le 1$, $1-\gamma \le 1-\gamma' \le I_W(\delta, \delta') \le 1$ and $1-\beta \le 1-\delta' \le 1$. Most of these inequalities are trivial or follow straightforwardly from (16)–(19) and (23)–(26). To see why $1-\gamma' \le I_W(\delta, \delta')$, first note that $\alpha' = 1$ due to our assumption that $\delta' > 0$ and (22). From (27) we therefore find that $1 \le S_W(1-\delta, \delta', \max(\beta', \gamma'))$ and hence $1 \le 1-\delta+\delta'+\max(\beta', \gamma')$. Using our assumption that $\gamma' \ge \beta'$ yields $1-\gamma' \le 1-\delta+\delta'$ from which we find $1-\gamma' \le I_W(\delta, \delta')$.

Next, we show that the fuzzy temporal relations evaluate to the required values $\alpha, \beta, \ldots, \delta'$ for A and B. Using Lemma 10, we find

$$\begin{aligned} be^{\ll}(A,B) &= \max(T_W(A(p_3), B(p_5)), T_W(A(p_3), B(p_6))) \\ &= \max(T_W(1,1), T_W(1,0)) \\ &= 1 \end{aligned}$$

Recall that $\alpha' = 1$ because of (22) and our assumption that $\delta' > 0$. Using Lemma 2, we immediately find

$$be^{\preccurlyeq}(A,B) \ge be^{\ll}(A,B) = 1$$

Note that $\alpha = 1$ because of (29) and our assumption that $\delta > 0$.

$$\begin{aligned} & bb^{\preccurlyeq}(A,B) \\ &= \min(I_W(B(p_1), A(p_1)), I_W(B(p_2), A(p_2)), \\ &\qquad I_W(B(p_3), A(p_3))) \\ &= \min(I_W(1-\beta, 0), I_W(1-\beta, I_W(\beta, \beta')), I_W(1-\delta', 1)) \\ &= \min(I_W(1-\beta, 0), I_W(1-\beta, I_W(\beta, \beta'))) \\ &= I_W(1-\beta, 0) \\ &= \beta \end{aligned}$$

Using (5), (25) and (23), we find (still using Lemma 10)

$$\begin{aligned} & bb^{\ll}(A,B) \\ &= \max(T_W(A(p_1), 1-B(p_1)), T_W(A(p_2), 1-B(p_2)), \\ &\qquad T_W(A(p_3), 1-B(p_3))) \\ &= \max(T_W(0, \beta), T_W(I_W(\beta, \beta'), \beta), T_W(1, \delta')) \\ &= \max(T_W(I_W(\beta, \beta'), \beta), \delta') \\ &= \max(\min(\beta, \beta'), \delta') \\ &= \beta' \end{aligned}$$

$$\begin{aligned} ee^{\preccurlyeq}(A,B) &= \min(I_W(A(p_5), B(p_5)), I_W(A(p_6), B(p_6))) \\ &= \min(I_W(1-\gamma', 1), I_W(1-\gamma, 0)) \\ &= \min(1, \gamma) \\ &= \gamma \end{aligned}$$

$$\begin{aligned} & ee^{\ll}(A,B) \\ &= \max(T_W(B(p_5), 1-A(p_5)), T_W(B(p_6), 1-A(p_6))) \\ &= \max(T_W(1, \gamma'), T_W(0, \gamma)) \\ &= \max(\gamma', 0) \\ &= \gamma' \end{aligned}$$

Using (26), (16) and (17) and the fact that $1-\gamma' \le I_W(\delta, \delta')$, we find

$$\begin{aligned} & eb^{\preccurlyeq}(A,B) \\ &= \min(1-T_W(A(p_3), B(p_2)), 1-T_W(A(p_4), B(p_3)), \\ &\qquad 1-T_W(A(p_5), B(p_4)), 1-T_W(A(p_6), B(p_5))) \\ &= \min(1-T_W(1, 1-\beta), 1-T_W(I_W(\delta, \delta'), 1-\delta'), \\ &\qquad 1-T_W(1-\gamma', 1-\delta'), 1-T_W(1-\gamma, 1)) \\ &= \min(\beta, 1-T_W(I_W(\delta, \delta'), 1-\delta'), \gamma) \end{aligned}$$

and by (6) and (5)

$$\begin{aligned} &= \min(\beta, 1-T_W(I_W(1-\delta', 1-\delta), 1-\delta'), \gamma) \\ &= \min(\beta, 1-\min(1-\delta', 1-\delta), \gamma) \\ &= \min(\beta, \delta, \gamma) \\ &= \delta \end{aligned}$$

Finally, by (23) and (24), we establish

$$\begin{aligned} & eb^{\ll}(A,B) \\ &= \min(1-T_W(A(p_3), B(p_1)), 1-T_W(A(p_3), B(p_2)), \\ &\qquad 1-T_W(A(p_3), B(p_3)), 1-T_W(A(p_4), B(p_4)), \\ &\qquad 1-T_W(A(p_5), B(p_5))) \\ &= \min(1-T_W(1, 1-\beta), 1-T_W(1, 1-\delta'), \\ &\qquad 1-T_W(I_W(\delta, \delta'), 1-\delta'), 1-T_W(1-\gamma', 1)) \\ &= \min(\beta, \delta', \gamma') \\ &= \delta' \end{aligned}$$

There are eight more cases to consider, whose proofs are similar to that of the first case. Therefore, we only provide the definitions of A and B in these remaining cases and omit further details.

1) If $\delta > 0$, $\delta' > 0$ and $\beta' > \gamma'$, A and B are defined as

$$\begin{aligned}A &= \{p_2/1,]p_2, p_4]/1-\delta',]p_4, p_6]/1-\gamma\}\\ B &= \{[p_1, p_2[/1-\beta, [p_2, p_3[/1-\beta', [p_3, p_4[/I_W(\delta, \delta'),\\ &\quad p_4/1,]p_4, p_5]/I_W(\gamma, \gamma')\}\end{aligned}$$

2) If $\delta = 0$ and $\beta < \gamma$, A and B are defined as

$$\begin{aligned}A &= \{[p_1, p_2[/\beta', [p_2, p_3[/\beta, [p_3, p_4[/I_W(\alpha, \alpha'),\\ &\quad p_4/1,]p_4, p_5]/I_W(\gamma, \gamma')\}\\ B &= \{p_2/1,]p_2, p_4]/\alpha,]p_4, p_6]/\gamma'\}\end{aligned}$$

3) If $\delta = 0$ and $\beta \geq \gamma$, A and B are defined as

$$\begin{aligned}A &= \{[p_1, p_3[/\beta', [p_3, p_5[/\alpha, p_5/1\}\\ B &= \{[p_2, p_3[/I_W(\beta, \beta'), p_3/1,]p_3, p_4]/I_W(\alpha, \alpha'),\\ &\quad]p_4, p_5]/\gamma,]p_5, p_6]/\gamma'\}\end{aligned}$$

4) If $\delta > 0$, $\delta' = 0$, $\alpha' \geq 1-\delta$ and $\alpha' > S_W(1-\delta, \beta')$, A and B are defined as

$$\begin{aligned}A &= \{[p_2, p_3[/I_W(\delta, \beta'), p_3/1,]p_3, p_5]/T_W(\alpha', 1-\gamma')\}\\ B &= \{[p_1, p_2[/1-\beta, [p_2, p_3[/1-\delta, p_3/1,]p_3, p_4]/\alpha',\\ &\quad]p_4, p_5]/T_W(\gamma, 1-\gamma', \alpha')\}\end{aligned}$$

5) If $\delta > 0$, $\delta' = 0$, $\alpha' \geq 1-\delta$ and $\alpha' > S_W(1-\delta, \gamma')$, A and B are defined as

$$\begin{aligned}A &= \{[p_1, p_2[/T_W(\beta, 1-\beta', \alpha'), [p_2, p_3[/\alpha', p_3/1,\\ &\quad]p_3, p_4]/1-\delta,]p_4, p_5]/1-\gamma\}\\ B &= \{[p_1, p_3[/T_W(\alpha', 1-\beta'), p_3/1,]p_3, p_4]/I_W(\delta, \gamma')\}\end{aligned}$$

6) If $\delta > 0$, $\delta' = 0$, $\alpha' \geq 1-\delta$ and $\alpha' \leq S_W(1-\delta, \min(\beta', \gamma'))$, A and B are defined as

$$\begin{aligned}A &= \{[p_2, p_3[/I_W(\beta, \beta'), [p_3, p_4[/\alpha', p_4/1,\\ &\quad]p_4, p_6]/1-\gamma\}\\ B &= \{[p_1, p_3[/1-\beta, [p_3, p_4[/1-\delta, p_4/1,\\ &\quad]p_4, p_5]/I_W(\gamma, \gamma')\}\end{aligned}$$

7) If $\delta > 0$, $\delta' = 0$, $\alpha' < 1-\delta$ and $\beta \leq \gamma$, A and B are defined as

$$\begin{aligned}A &= \{[p_1, p_2[/\beta', [p_2, p_3[/T_W(\beta, 1-\delta), [p_3, p_4[/\alpha',\\ &\quad p_4/1,]p_4, p_5]/I_W(\gamma, \gamma')\}\\ B &= \{[p_2, p_4[/1-\delta, p_4/1,]p_4, p_6]/\gamma'\}\end{aligned}$$

8) If $\delta > 0$, $\delta' = 0$, $\alpha' < 1-\delta$ and $\beta > \gamma$, A and B are defined as

$$\begin{aligned}A &= \{[p_1, p_3[/\beta', p_3/1,]p_3, p_5]/1-\delta\}\\ B &= \{[p_2, p_3[/I_W(\beta, \beta'), p_3/1,]p_3, p_4]/\alpha',\\ &\quad]p_4, p_5]/T_W(\gamma, 1-\delta),]p_5, p_6]/\gamma'\}\end{aligned}$$

B. Proof of Proposition 2

The proof proceeds by a case analysis on the values of δ_1 ($\delta_1 = 0$ or $\delta_1 > 0$) and δ_2 ($\delta_2 = 0$ or $\delta_2 > 0$). First assume that $\delta_1 = \delta_2 = 0$. Let A, B and C be defined as follows

$$\begin{aligned}A &= \{[p_1, p_4[/T_W(\beta_1, \alpha_2), [p_4, p_7[/\beta_1, p_7/1\}\\ B &= \{[p_2, p_5[/\alpha_2, p_5/1,]p_5, p_8]/\alpha_1\}\\ C &= \{p_3/1,]p_3, p_6]/\gamma_2,]p_6, p_9]/T_W(\alpha_1, \gamma_2)\}\end{aligned}$$

where $p_1 < p_2 < \cdots < p_9$ are real numbers. Clearly, A, B and C are fuzzy time intervals. First, we verify that (38)–(45) are satisfied. Using Lemma 10, we find

$$\begin{aligned}&be^{\preccurlyeq}(A, B)\\ &= \max(T_W(T_W(\beta_1, \alpha_2), 1), T_W(\beta_1, 1), T_W(1, \alpha_1))\\ &= \max(T_W(\beta_1, \alpha_2), \beta_1, \alpha_1)\\ &= \alpha_1\end{aligned}$$

(39)–(45) are shown in completely the same way. Next, we show that A and C satisfy the restrictions for $be^{\preccurlyeq}(A, C)$ and $eb^{\preccurlyeq}(A, C)$. By Lemma 10 and Lemma 2, we have

$$\begin{aligned}&be^{\preccurlyeq}(A, C)\\ &= \max(T_W(T_W(\beta_1, \alpha_2), 1), T_W(\beta_1, \gamma_2), T_W(1, T_W(\alpha_1, \gamma_2)))\\ &= \max(T_W(\beta_1, \alpha_2), T_W(\beta_1, \gamma_2), T_W(\alpha_1, \gamma_2))\\ &= \max(T_W(\beta_1, \alpha_2), T_W(\alpha_1, \gamma_2))\end{aligned}$$

which equals α^* since $T_W(\alpha_1, \delta_2) = T_W(\alpha_1, 0) = 0$ and $T_W(\delta_1, \alpha_2) = T_W(0, \alpha_2) = 0$. Using Lemma 10, we obtain

$$eb^{\preccurlyeq}(A, C) \leq 1 - T_W(A(p_7), C(p_3)) = 0$$

To show that the restrictions for $bb^{\preccurlyeq}(A, C)$ and $ee^{\preccurlyeq}(A, C)$ can be satisfied as well, we define A, B and C as follows

$$\begin{aligned}A &= \{[p_1, p_4[/T_W(\beta_1, \beta_2), [p_4, p_8[/\beta_1, [p_8, p_{10}[/\alpha_1,\\ &\quad [p_{10}, p_{12}]/1\}\\ B &= \{[p_2, p_6[/\beta_2, [p_6, p_9]/1,]p_9, p_{13}]/\gamma_1\}\\ C &= \{[p_3, p_5]/1,]p_5, p_7]/\alpha_2,]p_7, p_{11}]/\gamma_2,\\ &\quad]p_{11}, p_{14}]/T_W(\gamma_1, \gamma_2)\}\end{aligned}$$

where $p_1 < p_2 < \cdots < p_{14}$ are real numbers. As before, we can verify that (38)–(45) are satisfied. Moreover, using Lemma 10, we obtain

$$\begin{aligned}&bb^{\preccurlyeq}(A, C)\\ &= \min(I_W(0, T_W(\beta_1, \beta_2)), I_W(1, T_W(\beta_1, \beta_2)))\\ &= T_W(\beta_1, \beta_2)\\ &= \max(T_W(\beta_1, \beta_2), \min(\alpha_1 + T_W(\delta_2, \gamma_1), 0,\\ &\qquad \beta_1 + T_W(\delta_2, \alpha_1)))\\ &= \max(T_W(\beta_1, \beta_2), \min(\alpha_1 + T_W(\delta_2, \gamma_1), \delta_2,\\ &\qquad \beta_1 + T_W(\delta_2, \alpha_1)))\end{aligned}$$

and

$$\begin{aligned}
&ee^{\preccurlyeq}(A,C)\\
&=\min(I_W(1,T_W(\gamma_1,\gamma_2)),I_W(0,T_W(\gamma_1,\gamma_2)))\\
&=T_W(\gamma_1,\gamma_2)\\
&=\max(T_W(\gamma_1,\gamma_2),\min(\alpha_2+T_W(\delta_1,\beta_2),\delta_1,\\
&\qquad\qquad\gamma_2+T_W(\delta_1,\alpha_2)))
\end{aligned}$$

Calculations for the remaining cases are rather similar. Therefore, we only provide the definitions of the fuzzy time intervals in each case, and omit the details.

1) Assume $\delta_1 = 0$, $\delta_2 > 0$. If $1-\delta_2 \geq \alpha_1$, the restriction for $be^{\preccurlyeq}(A,C)$ is satisfied by

$$\begin{aligned}
A &= \{[p_1,p_6[/\beta_1, p_6/1\}\\
B &= \{p_2/1,]p_2,p_4]/1-\delta_2,]p_4,p_7]/\alpha_1\}\\
C &= \{p_3/1,]p_3,p_5]/\gamma_2,]p_5,p_8]/T_W(\alpha_1,\gamma_2)\}
\end{aligned}$$

If $1-\delta_2 < \alpha_1$ and $1-\delta_2 < \gamma_1$, it holds that $\alpha^* = 1$, since $1-\delta_2 < \gamma_1$ implies $T_W(\gamma_1,\delta_2) > 0$, and the restriction for $be^{\preccurlyeq}(A,C)$ thus becomes trivial.
If $1-\delta_2 < \alpha_1$ and $1-\delta_2 \geq \gamma_1$, the restriction for $be^{\preccurlyeq}(A,C)$ is satisfied by

$$\begin{aligned}
A &= \{[p_1,p_4[/\alpha_1, p_4/1\}\\
B &= \{p_2/1,]p_2,p_5]/\gamma_1\}\\
C &= \{p_3/1,]p_3,p_5]/T_W(\gamma_1,\gamma_2)\}
\end{aligned}$$

If $\beta_1 + T_W(\alpha_1,\delta_2) \geq \alpha_1 + T_W(\gamma_1,\delta_2)$, the restrictions for $bb^{\preccurlyeq}(A,C)$ and $ee^{\preccurlyeq}(A,C)$ are satisfied by

$$\begin{aligned}
A &= \{[p_1,p_3[/T_W(\beta_1,\beta_2,1-\delta_2), [p_3,p_5[/\alpha_1, p_5/1,\\
&\qquad]p_5,p_7]/1-T_W(\gamma_1,\gamma_2)\}\\
B &= \{[p_1,p_3[/T_W(\beta_2,1-\delta_2), p_3/1,]p_3,p_5]/\gamma_1,\\
&\qquad]p_5,p_7]/\min(\gamma_1,1-\gamma_2)\}\\
C &= \{[p_2,p_4[/1-\delta_2, [p_4,p_6[/1-T_W(\delta_2,\gamma_1), p_6/1\}
\end{aligned}$$

If $\beta_1 + T_W(\alpha_1,\delta_2) < \alpha_1 + T_W(\gamma_1,\delta_2)$, the restrictions for $bb^{\preccurlyeq}(A,C)$ and $ee^{\preccurlyeq}(A,C)$ are satisfied by

$$\begin{aligned}
A &= \{[p_1,p_3[/T_W(\beta_1,\beta_2,1-\delta_2), [p_3,p_5[/\beta_1, p_5/1,\\
&\qquad]p_5,p_7]/1-T_W(\gamma_1,\gamma_2)\}\\
B &= \{[p_1,p_3[/T_W(\beta_2,1-\delta_2), p_3/1,]p_3,p_5]/\alpha_1,\\
&\qquad]p_5,p_7]/\min(\gamma_1,1-\gamma_2)\}\\
C &= \{[p_2,p_4[/1-\delta_2, [p_4,p_6[/1-T_W(\delta_2,\alpha_1), p_6/1\}
\end{aligned}$$

Finally, the restriction for $eb^{\preccurlyeq}(A,C)$ is satisfied by

$$\begin{aligned}
A &= \{[p_1,p_3[/\beta_1, [p_3,p_5[/\alpha_1, p_5/1\}\\
B &= \{[p_1,p_3[/\beta_2, p_3/1,]p_3,p_6]/\gamma_1\}\\
C &= \{[p_2,p_4[/1-\delta_2, [p_4,p_7[/1-T_W(\gamma_1,\delta_2), p_7/1\}
\end{aligned}$$

2) Assume $\delta_1 > 0$, $\delta_2 = 0$. If $1-\delta_1 \geq \alpha_2$, the restriction for $be^{\preccurlyeq}(A,C)$ is satisfied by

$$\begin{aligned}
A &= \{[p_1,p_3[/T_W(\beta_1,\alpha_2), [p_3,p_5[/\beta_1, p_5/1\}\\
B &= \{[p_1,p_4[/\alpha_2, [p_4,p_6[/1-\delta_1, p_6/1\}\\
C &= \{p_2/1,]p_2,p_7]/\gamma_2\}
\end{aligned}$$

If $1-\delta_1 < \alpha_1$ and $1-\delta_1 < \beta_2$, it holds that $\alpha^* = 1$, and the restriction for $be^{\preccurlyeq}(A,C)$ thus becomes trivial. If $1-\delta_1 < \alpha_1$ and $1-\delta_1 \geq \beta_2$, the restriction for $be^{\preccurlyeq}(A,C)$ is satisfied by

$$\begin{aligned}
A &= \{[p_1,p_3[/T_W(\beta_1,\beta_2), p_3/1\}\\
B &= \{[p_1,p_4[/\beta_2, p_4/1\}\\
C &= \{p_2/1,]p_2,p_5]/\alpha_2\}
\end{aligned}$$

If $\alpha_2 + T_W(\delta_1,\beta_2) \geq \gamma_2 + T_W(\delta_1,\alpha_2)$, the restrictions for $bb^{\preccurlyeq}(A,C)$ and $ee^{\preccurlyeq}(A,C)$ are satisfied by

$$\begin{aligned}
A &= \{p_2/1,]p_2,p_4]/1-T_W(\delta_1,\alpha_2),]p_4,p_7]/1-\delta_1\}\\
B &= \{[p_1,p_3[/\min(1-\beta_1,\beta_2), [p_3,p_5[/\alpha_2, p_5/1,\\
&\qquad]p_5,p_8]/T_W(\gamma_1,1-\delta_1)\}\\
C &= \{[p_1,p_3[/1-T_W(\beta_1,\beta_2), p_3/1,]p_3,p_6]/\gamma_2,\\
&\qquad]p_6,p_8]/T_W(\gamma_1,\gamma_2,1-\delta_1)\}
\end{aligned}$$

If $\alpha_2 + T_W(\delta_1,\beta_2) < \gamma_2 + T_W(\delta_1,\alpha_2)$, the restrictions for $bb^{\preccurlyeq}(A,C)$ and $ee^{\preccurlyeq}(A,C)$ are satisfied by

$$\begin{aligned}
A &= \{p_2/1,]p_2,p_4]/1-T_W(\delta_1,\beta_2),]p_4,p_7]/1-\delta_1\}\\
B &= \{[p_1,p_3[/\min(1-\beta_1,\beta_2), [p_3,p_4[/\beta_2, p_4/1,\\
&\qquad]p_4,p_8]/T_W(\gamma_1,1-\delta_1)\}\\
C &= \{[p_1,p_3[/1-T_W(\beta_1,\beta_2), p_3/1,]p_3,p_4]/\alpha_2,\\
&\qquad]p_4,p_6]/\gamma_2,]p_6,p_8]/T_W(\gamma_1,\gamma_2,1-\delta_1)\}
\end{aligned}$$

Finally, the restriction for $eb^{\preccurlyeq}(A,C)$ is satisfied by

$$\begin{aligned}
A &= \{p_1/1,]p_1,p_4]/1-T_W(\delta_1,\beta_2),]p_4,p_6]/1-\delta_1\}\\
B &= \{[p_2,p_5[/\beta_2, p_5/1,]p_5,p_7]/\gamma_1\}\\
C &= \{p_3/1,]p_3,p_5]/\alpha_2,]p_5,p_7]/\gamma_2\}
\end{aligned}$$

3) Assume $\delta_1 > 0$, $\delta_2 > 0$. The restrictions on $bb^{\preccurlyeq}(A,C)$ and $ee^{\preccurlyeq}(A,C)$ are satisfied by

$$\begin{aligned}
A &= \{p_2/1,]p_2,p_5]/1-\max(\delta_1,T_W(\gamma_1,\gamma_2))\}\\
B &= \{[p_1,p_3[/1-\beta_1, p_3/1,]p_3,p_5]/1-\gamma_2\}\\
C &= \{[p_1,p_4[/1-\max(\delta_2,T_W(\beta_1,\beta_2)), p_4/1\}
\end{aligned}$$

Finally, the restrictions on $be^{\preccurlyeq}(A,C)$ and $eb^{\preccurlyeq}(A,C)$ are satisfied by

$$\begin{aligned}
A &= \{[p_1,p_3]/1,]p_3,p_6]/1-\max(\delta_1,T_W(\gamma_1,\delta_2))\}\\
B &= \{[p_1,p_4[/\min(1-\delta_1,\beta_2), p_4/1,\\
&\qquad]p_4,p_7]/\min(1-\delta_2,\gamma_1)\}\\
C &= \{[p_2,p_5[/1-\max(\delta_2,T_W(\delta_1,\beta_2)), [p_5,p_7]/1\}
\end{aligned}$$

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