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Spatial reasoning in a fuzzy region connection calculus

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ABSTRACT

Although the region connection calculus (RCC) offers an appealing framework for modelling topological relations, its application in real-world scenarios is hampered when spatial phenomena are affected by vagueness. To cope with this, we present a generalization of the RCC based on fuzzy set theory, and discuss how reasoning tasks such as satisfiability and entailment checking can be cast into linear programming problems. We furthermore reveal that reasoning in our fuzzy RCC is NP-complete, thus preserving the computational complexity of reasoning in the RCC, and we identify an important tractable subfragment. Moreover, we show how reasoning tasks in our fuzzy RCC can also be reduced to reasoning tasks in the original RCC. While this link with the RCC could be exploited in practical reasoning algorithms, we mainly focus on the theoretical consequences. In particular, using this link we establish a close relationship with the Egg–Yolk calculus, and we demonstrate that satisfiable knowledge bases can be realized by fuzzy regions in any dimension.

1. Introduction

Topological relations constitute an important facet of how humans perceive spatial configurations. Consequently, a large proportion of the spatial information conveyed in natural language discourse is related to topology: we may find, for instance, that a certain geographic region is adjacent to, contained in or overlapping with another. The region connection calculus (RCC; [39]) has been proposed as a means to model such topological relations, and to reason about available topological information (e.g., if a is adjacent to b, and b is a part of c, then c cannot be a part of a, regardless of how the regions a, b and c are defined). A core feature of this calculus, discriminating it from related approaches such as the 9-intersection model [12], is its generality. Starting from an arbitrary universe U of regions, topological relations are defined in terms of an arbitrary reflexive and symmetric relation C in U, called connection (see Table 1 in Section 2). The intuitive meaning of some of the RCC relations from Table 1 is illustrated in Fig. 1. In particular, note that *EC* (externally connected) models adjacency, while containment is modelled by *TPP* (tangential proper part), *NTPP* (non-tangential proper part) and *EQ* (equality). In different applications, regions can be modelled in different ways, and connection can be defined accordingly. Typically, regions are regular closed subsets of $\mathbb{R}^2$ or $\mathbb{R}^3$ and two regions a and b are called connected iff $a \cap b \neq \emptyset$, although, for instance, $\mathbb{Z}^2$ and $\mathbb{Z}^3$ are often of interest as well [29]. Furthermore, due to its generality, the RCC can also be applied in contexts where space is used in a metaphorical way (e.g., [38]). Note that frequently, the RCC is restricted to eight base relations, which have the property of being jointly exhaustive and pairwise disjoint: *DC* (disconnected), *EC*, *PO* (partially overlaps), *EQ*, *TPP*, *NTPP*, TPP^{-1} and $NTPP^{-1}$. The RCC restricted to (unions of) these eight relations is referred to as RCC-8.

When using the RCC in applications, it is usually assumed that regions are well-defined entities, e.g., characterized by precise boundaries. On the other hand, many geographical regions, for instance, are inherently ill-defined. For example,

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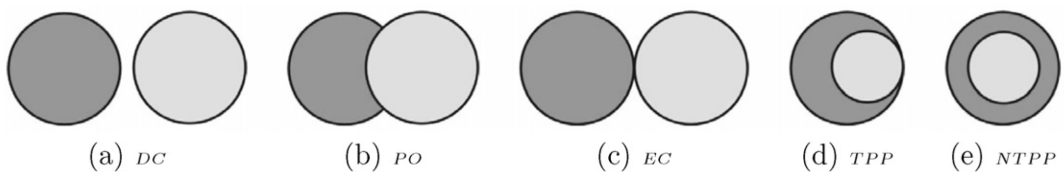

Fig. 1. Intuitive meaning of some RCC relations.

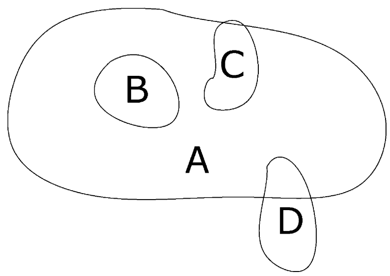

Fig. 2. Are the regions B, C, and D contained in region A?

although political regions, such as countries, states, and provinces, have officially defined—and therefore precise—boundaries, many of the places people refer to in everyday communication (i.e., vernacular places), do not (e.g., [1,2,13,15,33,57,58]). Even the names of political regions are often used in a way that is not in perfect accordance with their official definitions; a typical example are city neighborhoods, whose official boundaries, if they exist, are merely intended for administrative purposes (e.g., electoral divisions). Vague regions occur at very different scales (Ghent's city center, the Highlands, the Middle East), and a variety of techniques can be used to capture their spatial semantics. Approaches based on supervaluation semantics (e.g., [1,27,57]), for instance, associate a set of possible crisp *precisifications* with a vague language concept, and reason about assertions that are true in every precisification, in some precisification, etc., typically using first-order logic. They are mainly motivated by philosophical considerations about the nature of vagueness, and tend to be less suitable as workable, computational models. Particularly popular are techniques which represent a vague region as a pair of crisp sets (e.g., [2,4,5]). Their main idea is that a vague region can be approximated by defining a set of locations $\underline{a}$ which are definitely in the vague region, as well as a set of locations $\bar{a}$ which are in the vague region to some extent (where $\underline{a} \subseteq \bar{a}$); the complement of $\bar{a}$ is then the set of locations which are definitely not in the vague region. The resulting models are very efficient, and theoretical results (e.g., reasoning procedures) can usually be obtained relatively easily from existing results for crisp regions. Note that vague regions are in this case formally equivalent to ensembles flous [18] of locations. Finally, fuzzy set theory is frequently employed to model vague regions (e.g., [15,19,22,30,31]). In this case, the spatial extent of a vague region is modelled by a mapping A from locations (points) to the unit interval $[0,1]$, such that for any location l, $A(l)$ reflects the degree to which l belongs to the vague region. Although the resulting models may be somewhat less efficient than models based on pairs of crisp regions, their increased flexibility is often needed to accurately capture vague boundaries. Moreover, a pair $(\underline{a}, \bar{a})$ of crisp regions with $\underline{a} \subseteq \bar{a}$ can be seen as a special case of a fuzzy set, e.g., by assigning all points in $\underline{a}$ membership degree 1, all points in $\bar{a} \setminus \underline{a}$ membership degree 0.5, and all other points membership degree 0.

The existence of vague regions does not, as such, present any difficulties, as the RCC makes no assumptions on the representation of regions. However, when some of the regions involved are vague, topological relations can be vague as well. For instance, it is not entirely clear whether the Alps are included in, overlapping with, or disjoint from Southern Europe, as each of these relations seems defensible to some extent. This observation stands in contrast with the assumption that topological relations are defined in terms of first-order logic and a crisp relation C. Moreover, even if the regions involved are crisp, it may be desirable to define topological relations as fuzzy relations. In particular, the traditional, strict interpretation of topological relations (e.g., using point-set topology) does not always correspond very well to the way topological relations are used in natural language. For example, it is commonplace to say that a cabinet is located against a wall even if there is a gap of a few millimeters between the cabinet and the wall. In traditional frameworks, the cabinet and the wall would be considered disjoint, irrespective of the size of the gap. A more natural solution would be to define topological relations as fuzzy relations in which the cabinet and the wall are considered adjacent if they are actually touching, or located *very close* to each other. Note that adjacency then becomes a vague concept because it relies on nearness. A similar observation can be made for containment; consider, for instance, the regions depicted in Fig. 2. Clearly, B is contained in A and D is not. However, while C is in principle not a part of A, we could intuitively think of C as being a part of A to a large extent, because C is *almost* contained in A.

Our solution is to define C as a fuzzy relation, i.e., for each pair (u, v) of regions, $C(u, v)$ is a degree in $[0,1]$ reflecting to what extent u and v are connected. Keeping the generality of the RCC, other topological relations are still defined in terms of C, using fuzzy logic connectives, however, instead of classical first-order logic. Note that in this way, we make no commitment at all of why topological relations are vague (e.g., to define relations between vague regions, to model tolerant natural language relations, etc.). The central aim of this paper is to investigate how this fuzzification of the RCC affects spatial reasoning. First, Section 2 presents the relevant details of our fuzzification of the RCC. Among others, we illustrate how fuzzy RCC relations can be interpreted in terms of nearness between fuzzy sets. Next, in Section 3, we provide a number of use cases to further motivate the need for a fuzzy RCC, as well as the need for spatial reasoning in this context. Subsequently, we review some related work in Section 4, discussing shortcomings of existing approaches to handle fuzzy topological information. In Section 5, the main reasoning tasks in the fuzzy RCC are studied: satisfiability and entailment checking, calculating the best truth-value bound, and repairing inconsistencies. In Section 6, we show how satisfiability in the fuzzy RCC can alternatively be decided by reducing it to satisfiability checking in the original RCC. Using this reduction, we establish a close relationship with the Egg–Yolk calculus in Section 7 and we show in Section 8 that satisfiable knowledge bases in the fuzzy RCC can be realized by interpreting regions as fuzzy sets in $\mathbb{R}^n$, for an arbitrary dimension n.

2. Fuzzy topological relations

Before we discuss the problem of fuzzy spatial reasoning, this section familiarizes the reader with some necessary preliminaries. First, Section 2.1 recalls some basic concepts from fuzzy set theory and fuzzy relational calculus, which are used throughout the paper. Consequently, in Section 2.2, we summarize the most important results related to the fuzzy RCC presented in [50,51], and we introduce a number of definitions that are used below.

2.1. Preliminaries from fuzzy set theory

A fuzzy set [60] in a universe X is a mapping A from X to $[0,1]$. For x in X, $A(x)$ typically expresses the degree to which x exhibits some vague property. If $A(x)=1$ for some x in X, the fuzzy set A is called normalised. The support of A is the set of elements x from X for which $A(x)>0$. Similarly, for every α in $]0,1]$, the α-level set A_α of A is the (crisp) subset of X defined by

$$A_\alpha = \{x \mid x \in X \text{ and } A(x) \geqslant \alpha\}$$

The complement of A is the fuzzy set coA in X defined by $coA(x) = 1 - A(x)$ for all x in X. Throughout this paper, we will mainly be concerned with fuzzy sets in $\mathbb{R}^n$ representing vague regions. For p in $\mathbb{R}^n$, $A(p)$ then reflects to what extent point p belongs to the vague region. A fuzzy set R in $X \times X$ is called a fuzzy relation in X. For x and y in X, $R(x, y)$ is the degree to which a given vague relationship is satisfied between x and y. A fuzzy relation R in X is called reflexive if $R(x,x)=1$ for all x in X, and symmetric if $R(x,y)=R(y,x)$ for all x and y in X. The inverse of R is the fuzzy relation R^{-1} in X defined by $R^{-1}(x,y)=R(y,x)$ for all x and y in X.

As explained in the introduction, to define fuzzy topological relations, we start from a fuzzy relation C in a universe U of regions, modelling connection. In other words, for u and v in U, $C(u,v)$ reflects the degree in $[0,1]$ to which regions u and v could be considered connected. In analogy with the RCC, we furthermore require that C is a reflexive and symmetric fuzzy relation. By generalizing logical conjunction and implication to fuzzy logic connectives, and universal and existential quantification to the infima and suprema of truth degrees, definitions of RCC relations can be generalized to define fuzzy topological relations. Specifically, to generalize logical conjunction, a t-norm is typically used, i.e., an $[0,1]^2 - [0,1]$ mapping T which is symmetric, associative, increasing and which satisfies the boundary condition $T(a,1)=a$ for all a in $[0,1]$. It can be shown that such mappings intuitively behave as a conjunction operator. In this paper, we will mostly use the Łukasiewicz t-norm T_W, defined for a and b in $[0,1]$ as

$$T_W(a,b) = \max(0, a+b-1)$$

Other popular t-norms are the minimum T_M and product T_P defined by $T_M(a,b)=\min(a,b)$ and $T_P(a,b)=a\cdot b$ for all a and b in $[0,1]$. Usually, only left-continuous or continuous t-norms are considered [25,35]. Accordingly, we let T be a left-continuous t-norm throughout this paper. Next, fuzzy implication is typically defined in terms of the t-norm of interest. In particular, the residual implicator I_T corresponding to a t-norm T is defined for a and b in $[0,1]$ as

$$I_T(a,b) = \sup\{\lambda \mid \lambda \in [0,1] \text{ and } T(a,\lambda) \leqslant b\}$$

Residual implicators are known to satisfy many interesting properties, e.g. [56]:

$$I_T(a,b) = 1 \Leftrightarrow a \leqslant b$$

for all a and b in $[0,1]$.

The resulting definitions of the fuzzy RCC relations are presented in Table 1. It can be shown that when C is a crisp relation, i.e., $C(u,v)=1$ or $C(u,v)=0$ for all u and v in U, the fuzzy topological relations from Table 1 coincide with the original RCC relations [51].

Fuzzy logic connectives such as t-norms can also be used to define operations between fuzzy sets. For example, given a t-norm T, the degree of inclusion $incl(A,B)$ and the degree of overlap $overl(A,B)$ of two fuzzy sets A and B in a universe X is defined as [11]

Table 1
Definition of topological relations in the original RCC and the fuzzy RCC for regions a and b [51].

Name	Relation	RCC definition	Fuzzy RCC definition
Disconnected	DC	$\neg C(a,b)$	$1 - C(a,b)$
Part	P	$(\forall c \in U)(C(c,a) \Rightarrow C(c,b))$	$\inf_{c\in U} I_T(C(c,a), C(c,b))$
Proper Part	PP	$P(a,b) \wedge \neg P(b,a)$	$\min(P(a,b), 1 - P(b,a))$
Equals	EQ	$P(a,b) \wedge P(b,a)$	$\min(P(a,b), P(b,a))$
Overlaps	O	$(\exists c \in U)(P(c,a) \wedge P(c,b))$	$\sup_{c\in U} T(P(c,a), P(c,b))$
Discrete	DR	$\neg O(a,b)$	$1 - O(a,b)$
Partially Overlaps	PO	$O(a,b) \wedge \neg P(a,b) \wedge \neg P(b,a)$	$\min(O(a,b), 1 - P(a,b), 1 - P(b,a))$
Externally Connected	EC	$C(a,b) \wedge \neg O(a,b)$	$\min(C(a,b), 1 - O(a,b))$
Non-Tangential Part	NTP	$(\forall c \in U)(C(c,a) \Rightarrow O(c,b))$	$\inf_{c\in U} I_T(C(c,a), O(c,b))$
Tangential *PP*	TPP	$PP(a,b) \wedge \neg NTP(a,b)$	$\min(PP(a,b), 1 - NTP(a,b))$
Non-Tangential *PP*	$NTPP$	$PP(a,b) \wedge NTP(a,b)$	$\min(1 - P(b,a), NTP(a,b))$

$$incl(A,B) = \inf_{x\in X} I_T\big(A(x), B(x)\big)$$

$$overl(A,B) = \sup_{x\in X} T\big(A(x), B(x)\big)$$

In other words, $incl(A,B)$ is the degree to which every x which belongs to the fuzzy set A also belongs to the fuzzy set B, while $overl(A,B)$ is the degree to which some x belongs to both A and B. Finally, given a fuzzy set A in X and a symmetric fuzzy relation R in X, the upper approximation $R{\uparrow}A$ and lower approximation $R{\downarrow}A$ of A under R are defined by

$$(R{\uparrow}A)(x) = \sup_{y\in X} T\big(R(x,y), A(y)\big)$$

$$(R{\downarrow}A)(x) = \inf_{y\in X} I_T\big(R(x,y), A(y)\big)$$

for all x in X. Below, we will use these operators for the specific case where A is a vague region and R models nearness between locations. In this specific case, $(R{\uparrow}A)(x)$ is the degree to which x is located close to some point in vague region A, whereas $(R{\downarrow}A)(x)$ is the degree to which all points near x are located in A. For notational convenience, we will use expressions such as $R{\downarrow}{\uparrow}A$ as an abbreviation for $R{\downarrow}(R{\uparrow}A)$. The following two lemmas will be useful in the discussion below.

Lemma 1. *(See [3].) Let R be a symmetric fuzzy relation in X, A a fuzzy set in X and T a left-continuous t-norm. It holds that*

$$R{\uparrow}{\downarrow}{\uparrow}A = R{\uparrow}A \tag{1}$$

$$R{\downarrow}{\uparrow}{\downarrow}{\uparrow}A = R{\downarrow}{\uparrow}A \tag{2}$$

Lemma 2. *(See [3].) Let R be a reflexive and symmetric fuzzy relation in X, A a fuzzy set in X and T a left-continuous t-norm. It holds that*

$$R{\downarrow}A \subseteq R{\uparrow}{\downarrow}A \subseteq A \subseteq R{\downarrow}{\uparrow}A \subseteq R{\uparrow}A \tag{3}$$

2.2. Fuzzy region connection calculus

In [51], the properties of the fuzzy topological relations from Table 1 are examined in detail. Note that, from a theoretical perspective, C, O, P and NTP are the most important fuzzy topological relations; all the others can be expressed in terms of C, O, P and NTP without quantifying over the universe of regions U, using only the minimum and complement operations. We briefly recall some basic properties that will be of interest below.

Lemma 3 *(Reflexivity). (See [51].) For all u in U, it holds that*

$$C(u,u) = O(u,u) = P(u,u) = 1$$

Lemma 4 *(Symmetry). (See [51].) For all u and v in U, it holds that*

$$C(u,v) = C(v,u)$$

$$O(u,v) = O(v,u)$$

Lemma 5 *(Ordering). (See [51].) For all u and v in U, it holds that*

$$NTP(u,v) \leqslant P(u,v) \leqslant O(u,v) \leqslant C(u,v)$$

Table 2
Transitivity table for the generalized RCC relations (assuming that the t-norm T that is used in the definition of the fuzzy RCC relations satisfies $T_W(a,b) \leqslant T(a,b)$ for all a and b in $[0,1]$) [51].

	C	DC	P	P^{-1}	coP	coP^{-1}	O	DR	NTP	NTP^{-1}	$coNTP$	$coNTP^{-1}$
C	1	coP	C	1	1	1	1	$coNTP$	O	1	1	1
DC	coP^{-1}	1	coP^{-1}	DC	1	1	coP^{-1}	1	coP^{-1}	DC	1	1
P	1	DC	P	1	1	coP^{-1}	1	DR	NTP	1	1	$coNTP^{-1}$
P^{-1}	C	coP	O	P^{-1}	coP	1	O	coP	O	NTP^{-1}	$coNTP$	1
coP	1	1	1	coP	1	1	1	1	1	coP	1	1
coP^{-1}	1	1	coP^{-1}	1	1	1	1	1	coP^{-1}	1	1	1
O	1	coP	O	1	1	1	1	coP	O	1	1	1
DR	$coNTP^{-1}$	1	coP^{-1}	DR	1	1	coP^{-1}	1	coP^{-1}	DC	1	1
NTP	1	DC	NTP	1	1	coP^{-1}	1	DC	NTP	1	1	coP^{-1}
NTP^{-1}	O	coP	O	NTP^{-1}	coP	1	O	coP	O	NTP^{-1}	coP	1
$coNTP$	1	1	1	$coNTP$	1	1	1	1	1	coP	1	1
$coNTP^{-1}$	1	1	$coNTP^{-1}$	1	1	1	1	1	coP^{-1}	1	1	1

Perhaps the most important properties of the fuzzy topological relations are transitivity properties. Table 2 summarizes the fuzzy transitivity rules involving C, O, P, NTP, as well as their complements and inverses. The entries in this table should be interpreted as follows. Let K be the fuzzy relation on the row corresponding to fuzzy relation R and the column corresponding to fuzzy relation S; it then holds that [51]

$$T_W\big(R(u,v), S(v,w)\big) \leqslant K(u,w)$$

for all u, v and w in U. For example, the entry on the first row, third column should be interpreted as

$$T_W\big(C(u,v), P(v,w)\big) \leqslant C(u,w)$$

which intuitively means that u is connected to w, at least to the degree that for some v, u is connected to v and v is a part of w. We refer to [51] for a detailed discussion on the equivalence between Table 2 and the classical RCC-8 composition table in the case that C is a crisp relation. Note that these transitivity rules are only valid if the t-norm T that is used satisfies the requirement $T_W(a,b) \leqslant T(a,b)$ for all a and b in $[0,1]$. Among others, this is the case for T_M, T_P and T_W.

Given the fuzzy topological relations from Table 1, the main reasoning task of interest is checking the satisfiability of a knowledge base such as $\Theta = \{NTPP(a,b) \geqslant 0.7, P(b,c) \leqslant 0.4, EC(a,c) \leqslant 0.5\}$. An important question here is what exactly we mean by satisfiability. Clearly, for Θ to be satisfiable, it should be possible to map the variables a, b, c to particular objects $a^{\mathcal{I}}$, $b^{\mathcal{I}}$, $c^{\mathcal{I}}$ from some interpretation domain D, and the relation C to some reflexive and symmetric fuzzy relation $C^{\mathcal{I}}$ in D such that all formulas in Θ hold, i.e., such that $NTPP^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \geqslant 0.7$, $P^{\mathcal{I}}(b^{\mathcal{I}}, c^{\mathcal{I}}) \leqslant 0.4$ and $EC^{\mathcal{I}}(a^{\mathcal{I}}, c^{\mathcal{I}}) \leqslant 0.5$, where $NTPP^{\mathcal{I}}$, $P^{\mathcal{I}}$ and $EC^{\mathcal{I}}$ are defined in terms of $C^{\mathcal{I}}$ (see below). However, should we impose additional restrictions on this interpretation domain D? For instance, we might require that variables be mapped to regions in some fuzzy topological space, or normalised sets in $\mathbb{R}^n$ for some n, possibly constrained by additional requirements like upper semi-continuity. If we choose to interpret variables as fuzzy sets in some universe, we may furthermore require that all the α-level sets be regular closed or regular open sets in some related topological space. As we will see below, however, all these definitions of satisfiability are equivalent. Although suitable restrictions on what objects can constitute a (fuzzy) region are definitely needed in applications, we will not impose such restrictions here. Formally, we are interested in the satisfiability of sets of fuzzy RCC formulas, defined as follows.

Definition 1 *(Atomic fuzzy RCC formula).* An atomic fuzzy RCC formula is a formula of the form $R(a,b) \leqslant \lambda$ or $R(a,b) \geqslant \lambda$, where R is either C or one of the fuzzy topological relations from Table 1 ($DC, P, \ldots, NTPP$), $\lambda \in [0,1]$, and a and b are elements from the universe of regions U.

Definition 2 *(Fuzzy RCC formula).* A fuzzy RCC formula is a formula of the form $f_1 \vee f_2 \vee \cdots \vee f_m$, where f_i is an atomic fuzzy RCC formula ($1 \leqslant i \leqslant m$).

In analogy, we will also refer to expressions such as $NTPP(a,b) \vee \neg PO(b,d)$ as (crisp) RCC formulas. Note that NTP is usually not considered in the RCC, we introduced it mainly as a shorthand to define and generalize the RCC-8 relations $NTPP$ and TPP. In contrast to other coarse spatial relations such as P, O and C, for example, NTP does not have an intuitive meaning other than that of $NTPP$. In applications, we will typically start from a set Θ of fuzzy RCC formulas which does not involve any (disjuncts of) fuzzy RCC formulas of the form $NTP(u,v) \geqslant \lambda$ or $NTP(u,v) \leqslant \lambda$ at all. Rather, formulas such as $NTPP(u,v) \geqslant \lambda$ or $TPP(u,v) \geqslant \lambda$ would occur. This observation leads to the definition of a standard set of fuzzy RCC formulas.

Definition 3 *(Standard).* A set of fuzzy RCC formulas is called standard if it does not contain (a disjunct of) a fuzzy RCC formula of the form $NTP(u,v) \geqslant \lambda$ or $NTP(u,v) \leqslant \lambda$ ($u, v \in U$ and $\lambda \in [0,1]$). Similarly, a set of crisp RCC formulas is called standard if it does not contain (a disjunct of) an RCC formula of the form $NTP(u,v)$ or $\neg NTP(u,v)$.

Definition 4 *(Interpretation).* An interpretation $\mathcal{I}$ is a mapping from the universe of regions U to some interpretation domain D, and from C to a reflexive and symmetric fuzzy relation $C^{\mathcal{I}}$ in D.

Since all fuzzy topological relations are defined in terms of C, we can extend the interpretation of C to interpretations of the other fuzzy topological relations; e.g., for all u and v in U, we define

$$P^{\mathcal{I}}\big(u^{\mathcal{I}}, v^{\mathcal{I}}\big) = \inf_{w \in U} I_T\big(C^{\mathcal{I}}\big(w^{\mathcal{I}}, u^{\mathcal{I}}\big), C^{\mathcal{I}}\big(w^{\mathcal{I}}, v^{\mathcal{I}}\big)\big)$$

Let D be the universe of all normalised fuzzy sets in $\mathbb{R}^n$ with a bounded support. A particularly interesting interpretation of connection between such fuzzy sets is based on nearness. Specifically, let $R_{(\alpha,\beta)}$ be the fuzzy relation in $\mathbb{R}^n$ defined for all p and q in $\mathbb{R}^n$ by ($\alpha, \beta \geqslant 0$)

$$R_{(\alpha,\beta)}(p,q) = \begin{cases} 1 & \text{if } d(p,q) \leqslant \alpha \\ 0 & \text{if } d(p,q) > \alpha + \beta \\ \frac{\alpha+\beta-d(p,q)}{\beta} & \text{otherwise } (\beta \neq 0) \end{cases} \tag{4}$$

where d is the Euclidean distance in $\mathbb{R}^n$. For two points p and q in $\mathbb{R}^n$, as well as suitable values of the parameters α and β, $R_{(\alpha,\beta)}(p,q)$ expresses the degree to which p is located near q. Note that if $d(p,q) \leqslant \alpha$, p and q are considered perfectly near, while for $d(p,q) > \alpha + \beta$, p and q are considered not near at all. In between there is a gradual, linear transition. For example, if $d(p,q) = \alpha + \frac{\beta}{2}$, p is considered near q to degree 0.5. Using this concept of nearness between points, we can define an interpretation in which connection is defined in terms of nearness between vague regions. In particular, let A and B be normalised fuzzy sets in $\mathbb{R}^n$ with a bounded support. The degree $C_{(\alpha,\beta)}(A, B)$ to which A is near B is defined as [50]

$$C_{(\alpha,\beta)}(A, B) = \sup_{p \in \mathbb{R}^n} T\Big(A(p), \sup_{q \in \mathbb{R}^n} T\big(R_{(\alpha,\beta)}(p,q), B(q)\big)\Big)$$

This is the degree to which some point in A is near some point in B. The reflexivity of $C_{(\alpha,\beta)}$ follows from the fact that only normalised fuzzy sets are considered. Indeed, let m in $\mathbb{R}^n$ be such that $A(m) = 1$, it holds that

$$C_{(\alpha,\beta)}(A, B) \geqslant T\big(A(m), T\big(R_{(\alpha,\beta)}(m,m), B(m)\big)\big) = T\big(1, T(1,1)\big) = 1$$

The symmetry of $C_{(\alpha,\beta)}$ follows from the symmetry of $R_{(\alpha,\beta)}$, the symmetry and associativity of T and the fact that $\sup_{x \in X} T(x, y) = T(\sup_{x \in X} x, y)$ for every $X \subseteq [0,1]$ and every y in $[0,1]$ (since T is a left-continuous t-norm; [56]). We find:

$$\begin{aligned} C_{(\alpha,\beta)}(A, B) &= \sup_{p \in \mathbb{R}^n} T\big(A(p), \sup_{q \in \mathbb{R}^n} T\big(R_{(\alpha,\beta)}(p,q), B(q)\big)\big) \\ &= \sup_{p \in \mathbb{R}^n} \sup_{q \in \mathbb{R}^n} T\big(A(p), T\big(R_{(\alpha,\beta)}(p,q), B(q)\big)\big) \\ &= \sup_{p \in \mathbb{R}^n} \sup_{q \in \mathbb{R}^n} T\big(B(q), T\big(R_{(\alpha,\beta)}(p,q), A(p)\big)\big) \\ &= \sup_{q \in \mathbb{R}^n} T\Big(B(q), \sup_{p \in \mathbb{R}^n} T\big(R_{(\alpha,\beta)}(p,q), A(p)\big)\Big) \\ &= \sup_{q \in \mathbb{R}^n} T\Big(B(q), \sup_{p \in \mathbb{R}^n} T\big(R_{(\alpha,\beta)}(q,p), A(p)\big)\Big) \\ &= C_{(\alpha,\beta)}(B, A) \end{aligned}$$

Furthermore let $P_{(\alpha,\beta)}$, $O_{(\alpha,\beta)}$ and $NTP_{(\alpha,\beta)}$ be defined as

$$P_{(\alpha,\beta)}(A, B) = incl(R_{(\alpha,\beta)}{\downarrow}{\uparrow}A, R_{(\alpha,\beta)}{\downarrow}{\uparrow}B)$$

$$O_{(\alpha,\beta)}(A, B) = overl(R_{(\alpha,\beta)}{\downarrow}{\uparrow}A, R_{(\alpha,\beta)}{\downarrow}{\uparrow}B)$$

$$NTP_{(\alpha,\beta)}(A, B) = incl(R_{(\alpha,\beta)}{\uparrow}A, R_{(\alpha,\beta)}{\downarrow}{\uparrow}B)$$

for all normalised fuzzy sets A and B with a bounded support. The fuzzy set $R_{(\alpha,\beta)}{\downarrow}{\uparrow}A$ is called the $R_{(\alpha,\beta)}$-closure of A. It can be shown that the concept of R-closure, for an arbitrary symmetric fuzzy relation R, bears close similarity to the notion of closure from mathematical topology [3]. Intuitively, in the context of vague regions, using $R_{(\alpha,\beta)}$-closures leverages the flexibility of using nearness in the definition of C, to a certain flexibility regarding the (vague) boundaries of regions when it comes to evaluating other fuzzy topological relations. Thus, for example, A is considered to be a part of B to the extent that the $R_{(\alpha,\beta)}$-closure of A is included in the $R_{(\alpha,\beta)}$-closure of B. As a second example, $NTP_{(\alpha,\beta)}(A, B)$ reflects the degree to which all points near A occur in the $R_{(\alpha,\beta)}$-closure of B. For a more detailed explanation about the intuition behind these definitions, we refer to [50]. For convenience, we will sometimes use expressions such as R_α, C_α and P_α to denote the fuzzy relations $R_{(\alpha,0)}$, $C_{(\alpha,0)}$ and $P_{(\alpha,0)}$ respectively. The following lemma reveals that $P_{(\alpha,\beta)}$, $O_{(\alpha,\beta)}$ and $NTP_{(\alpha,\beta)}$ correspond to interpretations of fuzzy RCC relations when connection is interpreted in terms of nearness.

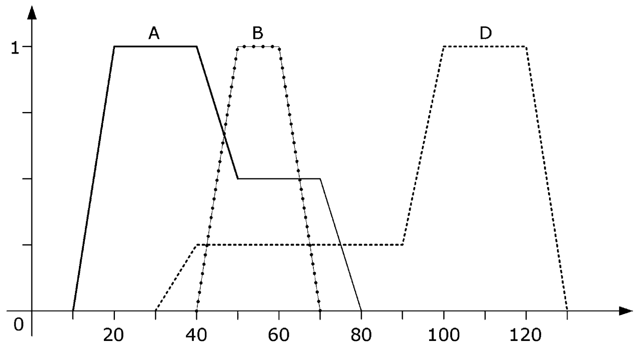

Fig. 3. An $(1; 10, 0)$-interpretation $\mathcal{I}$ of fuzzy RCC formulas involving the variables a, b and d, where $a^{\mathcal{I}} = A$, $b^{\mathcal{I}} = B$ and $d^{\mathcal{I}} = D$.

Lemma 6. *(See [50].) If $C^{\mathcal{I}} = C_{(\alpha,\beta)}$, it holds that*

$$P^{\mathcal{I}} = P_{(\alpha,\beta)}$$
$$O^{\mathcal{I}} = O_{(\alpha,\beta)}$$
$$NTP^{\mathcal{I}} = NTP_{(\alpha,\beta)}$$

We will refer to interpretations of this type as $(n; \alpha, \beta)$-interpretations, n being the dimension of the Euclidean space under consideration and (α, β) the parameters used to define nearness.

Example 1. An example of an $(1; 10, 0)$-interpretation $\mathcal{I}$ is illustrated in Fig. 3. Assuming $T = T_W$, we find that

$$\begin{aligned} C^{\mathcal{I}}\big(a^{\mathcal{I}}, b^{\mathcal{I}}\big) &= C_{(\alpha,\beta)}(A, B) = T_W\big(A(40), T_W\big(R_{(10,0)}(40, 50), B(50)\big)\big) \\ &= T_W\big(1, T_W(1, 1)\big) = 1 \end{aligned}$$

Similarly, we have that

$$C^{\mathcal{I}}\big(a^{\mathcal{I}}, d^{\mathcal{I}}\big) = C^{\mathcal{I}}\big(b^{\mathcal{I}}, d^{\mathcal{I}}\big) = 0.25$$

and hence, for example, also $DC^{\mathcal{I}}(a^{\mathcal{I}}, d^{\mathcal{I}}) = 0.75$. Furthermore, it can be shown that $R_{(10,0)}{\downarrow}{\uparrow}A = A$, $R_{(10,0)}{\downarrow}{\uparrow}B = B$ and $R_{(10,0)}{\downarrow}{\uparrow}D = D$, i.e., A, B and D are $R_{(10,0)}$-closed. This leads to

$$O^{\mathcal{I}}\big(a^{\mathcal{I}}, b^{\mathcal{I}}\big) = overl(A, B) = T_W\big(A(50), B(50)\big) = T_W(0.5, 1) = 0.5$$

and similarly

$$O^{\mathcal{I}}\big(a^{\mathcal{I}}, d^{\mathcal{I}}\big) = O^{\mathcal{I}}\big(b^{\mathcal{I}}, d^{\mathcal{I}}\big) = 0.25$$

We also have

$$P^{\mathcal{I}}\big(b^{\mathcal{I}}, a^{\mathcal{I}}\big) = incl(B, A) = I_W\big(B(60), A(60)\big) = I_W(1, 0.5) = 0.5$$

and

$$P^{\mathcal{I}}\big(b^{\mathcal{I}}, d^{\mathcal{I}}\big) = 0.25$$
$$P^{\mathcal{I}}\big(a^{\mathcal{I}}, b^{\mathcal{I}}\big) = P^{\mathcal{I}}\big(a^{\mathcal{I}}, d^{\mathcal{I}}\big) = P^{\mathcal{I}}\big(d^{\mathcal{I}}, a^{\mathcal{I}}\big) = P^{\mathcal{I}}\big(d^{\mathcal{I}}, b^{\mathcal{I}}\big) = 0$$

Finally, we can show that in this example

$$NTP^{\mathcal{I}}\big(b^{\mathcal{I}}, a^{\mathcal{I}}\big) = P^{\mathcal{I}}\big(b^{\mathcal{I}}, a^{\mathcal{I}}\big) = 0.5$$
$$NTP^{\mathcal{I}}\big(b^{\mathcal{I}}, d^{\mathcal{I}}\big) = P^{\mathcal{I}}\big(b^{\mathcal{I}}, d^{\mathcal{I}}\big) = 0.25$$

Note that while the definition of fuzzy topological relations in terms of nearness may be appropriate in some domains (e.g., treatment of vernacular regions in geographic information retrieval), it may be counterintuitive to use these definitions in other domains. For example, using the definitions above, the fuzzy topological relation O holds (to degree 1) between two regions, as soon as there is one point that belongs to both regions (to degree 1). In Section 7, we will therefore focus on another type of interpretations, which are based on the Egg–Yolk calculus. In these interpretations, regions are interpreted as fuzzy sets that only take a finite number of membership degrees. As a consequence, regions are completely determined by a finite number of α-level sets. Fuzzy topological relations can then be defined in terms of classical RCC relations between the α-level sets of these regions.

Definition 5 *(Satisfiability).* An interpretation $\mathcal{I}$ satisfies an atomic fuzzy RCC formula of the form $R(a,b) \leqslant \lambda$ (resp. $R(a,b) \geqslant \lambda$) iff $R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \leqslant \lambda$ (resp. $R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \geqslant \lambda$) holds. Furthermore, $\mathcal{I}$ satisfies a fuzzy RCC formula $f_1 \vee f_2 \vee \cdots \vee f_m$ iff it satisfies at least one of the atomic fuzzy RCC formulas $f_1, f_2, \ldots, f_m$. Finally, $\mathcal{I}$ satisfies a set of fuzzy RCC formulas Θ, written $\mathcal{I} \models \Theta$, iff it satisfies every fuzzy RCC formula in Θ. If such an interpretation $\mathcal{I}$ satisfying Θ exists, Θ is called satisfiable (or consistent) and $\mathcal{I}$ is called a model of Θ. An $(n;\alpha,\beta)$-interpretation which satisfies Θ is called an $(n;\alpha,\beta)$-model of Θ.

For example, let $\Theta = \{P(b,a) \geqslant 0.5,\ \ O(b,d) \geqslant 0.25,\ \ DC(a,d) \geqslant 0.75\}$. Intuitively, it might not be immediately clear whether or not Θ is satisfiable. In the original RCC, for example, $\{P(b,a), O(b,d), DC(a,d)\}$ is not satisfiable. However, the $(1;10,0)$-interpretation from Example 1 is a $(1;10,0)$-model of Θ, and Θ is therefore satisfiable.

Below, we will also talk about interpretations and models of sets of crisp RCC formulas. When there is cause for confusion, we will talk about F-satisfiability, F-interpretations and F-models to refer to the concepts introduced above, and about C-satisfiability, C-interpretations and C-models to refer to the corresponding concepts for crisp RCC formulas. The standard way to define C-interpretations is to map variables to regular closed subsets in $\mathbb{R}^n$ and to interpret C such that two regular closed subsets A and B are connected iff $A \cap B \neq \emptyset$. Below we will refer to such C-interpretations as standard interpretations, and denote the corresponding topological relations in $\mathbb{R}^n$ by C^n, P^n, O^n, etc. Note that P^n corresponds to the subset relation, and $O^n(A,B)$ holds iff $i(A) \cap i(B) \neq \emptyset$, where $i(A)$ and $i(B)$ are the interiors of A and B w.r.t. the Euclidean topology on $\mathbb{R}^n$.

While it can be convenient in applications to use fuzzy RCC formulas such as $EC(a,b) \geqslant 0.5$, every set Θ of fuzzy RCC formulas can equivalently be written as a set Θ' of fuzzy RCC formulas which only involve the fuzzy topological relations C, P, O and NTP. This observation follows from the fact that every fuzzy topological relation can be expressed in terms of C, P, O and NTP using only the minimum and complement operations. As an example, consider the fuzzy RCC formula $EC(a,b) \geqslant 0.4 \vee DC(a,b) \leqslant 0.3$. Using the definitions from Table 1, we find

$$\begin{aligned}
&EC(a,b) \geqslant 0.4 \vee DC(a,b) \leqslant 0.3\\
&\quad\Leftrightarrow \big(C(a,b) \geqslant 0.4 \wedge 1 - O(a,b) \geqslant 0.4\big) \vee 1 - C(a,b) \leqslant 0.3\\
&\quad\Leftrightarrow \big(C(a,b) \geqslant 0.4 \wedge O(a,b) \leqslant 0.6\big) \vee C(a,b) \geqslant 0.7\\
&\quad\Leftrightarrow \big(C(a,b) \geqslant 0.4 \vee C(a,b) \geqslant 0.7\big)\\
&\qquad \wedge \big(O(a,b) \leqslant 0.6 \vee C(a,b) \geqslant 0.7\big)\\
&\quad\Leftrightarrow C(a,b) \geqslant 0.4 \wedge \big(O(a,b) \leqslant 0.6 \vee C(a,b) \geqslant 0.7\big)
\end{aligned}$$

Thus, if $EC(a,b) \geqslant 0.4 \vee DC(a,b) \leqslant 0.3$ occurs in Θ, we could replace it by $\{C(a,b) \geqslant 0.4, O(a,b) \leqslant 0.6 \vee C(a,b) \geqslant 0.7\}$. For this reason, the following discussion will predominantly be restricted to sets of fuzzy RCC formulas involving only C, P, O and NTP.

Note that in general, such a set Θ does not completely specify, for every pair of variables, the degree to which each of the fuzzy topological relations C, P, O and NTP should hold. For example, consider the set $\Theta = \{C(a,b) \geqslant 0.5, O(b,a) \leqslant 0.7\}$, which is satisfied by an interpretation $\mathcal{I}$ if $C^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) = 0.5$ and $O^{\mathcal{I}}(b^{\mathcal{I}}, a^{\mathcal{I}}) = 0.6$, but also, among others, if $C^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) = O^{\mathcal{I}}(b^{\mathcal{I}}, a^{\mathcal{I}}) = 0.5$ or $C^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) = O^{\mathcal{I}}(b^{\mathcal{I}}, a^{\mathcal{I}}) = 0.6$. Formally, we have that every model of, for instance, $\Theta_1 = \{C(a,b) \geqslant 0.6,\ \ C(b,a) \leqslant 0.6,\ \ O(a,b) \geqslant 0.6,\ \ O(b,a) \leqslant 0.6\}$ is also a model of Θ, hence Θ_1 could be regarded as a refinement of the information in Θ.

Definition 6 *(Refinement).* Let Θ_1 and Θ_2 be sets of fuzzy RCC formulas. Θ_2 is called a refinement of Θ_1 iff every model of Θ_2 is also a model of Θ_1. If both Θ_1 is a refinement of Θ_2 and Θ_2 is a refinement of Θ_1, Θ_1 and Θ_2 are called equivalent.

A set of fuzzy RCC formulas Θ can always be refined such that the exact degree to which C, O, P and NTP should hold between every pair of variables is specified. If Θ is not satisfiable, this is trivial, as every set of fuzzy RCC formulas is then a refinement of Θ. On the other hand, if Θ is satisfiable, there exists some model $\mathcal{I}$ and we can specify a refinement of Θ by adding the formulas $C(u,v) \geqslant C^{\mathcal{I}}(u^{\mathcal{I}}, v^{\mathcal{I}})$, $C(u,v) \leqslant C^{\mathcal{I}}(u^{\mathcal{I}}, v^{\mathcal{I}})$, $O(u,v) \geqslant O^{\mathcal{I}}(u^{\mathcal{I}}, v^{\mathcal{I}})$, $O(u,v) \leqslant O^{\mathcal{I}}(u^{\mathcal{I}}, v^{\mathcal{I}})$, $P(u,v) \geqslant P^{\mathcal{I}}(u^{\mathcal{I}}, v^{\mathcal{I}})$, $P(u,v) \leqslant P^{\mathcal{I}}(u^{\mathcal{I}}, v^{\mathcal{I}})$, $NTP(u,v) \geqslant NTP^{\mathcal{I}}(u^{\mathcal{I}}, v^{\mathcal{I}})$ and $NTP(u,v) \leqslant NTP^{\mathcal{I}}(u^{\mathcal{I}}, v^{\mathcal{I}})$, for every pair of variables (u,v) occurring in Θ.

We will refer to such sets as normalised sets of fuzzy RCC relations.

Definition 7 *(Normalised).* Let Θ be a set of fuzzy RCC formulas, and let V be the subset of regions that are used in the formulas from Θ ($V \subseteq U$). Θ is called normalised iff

(1) for each fuzzy topological relation R in $\{C, P, O, NTP\}$ and all regions a and b in V, Θ contains a formula of the form $R(a,b) \leqslant 0$ or $R(a,b) \geqslant 1$, or both formulas of the form $R(a,b) \leqslant \lambda$ and $R(a,b) \geqslant \lambda$ for a given λ in $]0,1[$;
(2) Θ contains no other formulas.

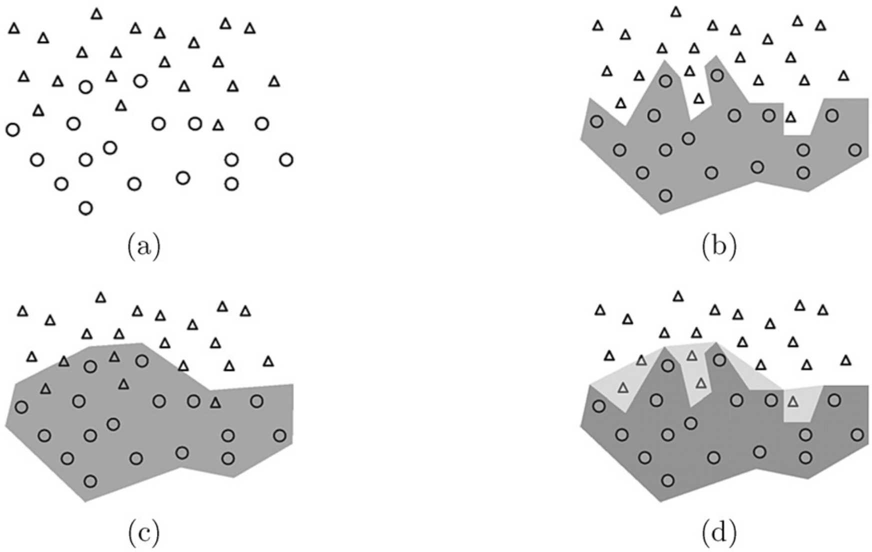

Fig. 4. Locations in region A (circles) and region B (triangles), and some possible resulting boundaries for A.

The first condition in the definition above guarantees that the membership degree of each of the fuzzy topological relations is uniquely determined for each pair of regions in V, while the second condition prevents the inclusion of superfluous additional information. Clearly, every satisfiable set of fuzzy RCC formulas Θ can be refined to a normalised set. Furthermore, Θ is satisfiable iff there exists at least one refinement which is satisfiable.

3. Motivating examples

Below we present a number of motivating examples to illustrate how reasoning about fuzzy topological relations might be useful in practice.

Use case 1: building quantitative geographical models In addition to qualitative spatial relations, often an abundance of quantitative information is available as well. Assume, for example, that we want to model the spatial extent of vague geographic regions such as city neighborhoods. From web documents, we may extract the addresses of places in these regions, which can subsequently be converted to their geographical coordinates through a process called geocoding. This is illustrated in Fig. 4(a), where a number of locations in a vague region A are shown (circles), as well as a number of locations in a vague region B (triangles). While, at first glance, the availability of this kind of quantitative information may seem to make any processing of topological spatial information redundant, the opposite is in fact true, i.e., topological information is often required to convert given quantitative information into reliable geographical models [48]. Given only the quantitative information from Fig. 4(a), what exactly would be a plausible boundary of region A? Knowing that A and B are adjacent, the boundary in Fig. 4(b) would be a good candidate. On the other hand, if A were overlapping with B, the boundary in Fig. 4(c) is more likely to be (approximately) correct. Hence, the actual boundaries that result from quantitative information about the location of a region depend on its topological relation with other regions. Note that qualitative relations are used in this case to help build quantitative models, rather than as a surrogate for them.

An additional level of difficulty arises when the topological relations between regions are only vaguely defined. Consider, for example, the following statements about the location of the Chiado and Baixa neighborhoods in Lisbon, Portugal:

(1) The Elevador de Santa Justa is an impressive steel lift built in 1900 to link the Baixa district to the Chiado.[2]
(2) Shops in the Baixa tend to be pricier than elsewhere, though. Chiado, the adjacent neighborhood, has …[3]
(3) Baixa, or downtown Lisbon, is the heart of the city.[4]
(4) Located in the heart of the historic Chiado quarter, in downtown Lisbon, Chiado Residence …[5]

In the RCC, we can encode the topological information from these statements as

[2] http://www.telegraph.co.uk/travel/main.jhtml?xml=/travel/2006/03/01/etmykind01.xml, accessed April 13, 2007.
[3] http://www.thisistravel.co.uk/travel/guides/city.htmlLisbon-what-to-buy_article.html?in_article_id=17537&in_page_id=1, accessed April 20, 2007.
[4] http://www.golisbon.com/sight-seeing/baixa.html, accessed April 13, 2007.
[5] http://www.chiadoresidence.com/location.htm, accessed April 13, 2007.

$$DC(Baixa, Chiado) \tag{5}$$

$$EC(Baixa, Chiado) \tag{6}$$

$$EQ(downtown\ Lisbon, Baixa) \tag{7}$$

$$O(Chiado, downtown\ Lisbon) \tag{8}$$

Note that this encoding into the RCC was done manually here. Automating this process is definitely not trivial, but it is outside the scope of the current discussion. While all four statements are true to some extent, the resulting description in the RCC is inconsistent. For example, since Baixa and Chiado are located very close to each other, but not actually touching, both the relations DC and EC are intuitively justified to some extent. As can be seen from (5) and (6), natural language statements expressing $EC(Baixa, Chiado)$ and $DC(Baixa, Chiado)$ are indeed both found in web documents. Another cause for the inconsistency of (5)–(8) is the fact that downtown Lisbon is a vague region. At least two of the four statements have to be discarded to make the resulting description in the RCC consistent. As we will see below, when using fuzzy topological relations, none of the four statements has to be fully rejected. We only need to weaken our interpretation by expressing that some of the topological relations only hold to some extent, e.g.

$$DC(Baixa, Chiado) \geqslant 0.5 \tag{9}$$

$$EC(Baixa, Chiado) \geqslant 0.5 \tag{10}$$

$$EQ(downtown\ Lisbon, Baixa) = 1 \tag{11}$$

$$O(Chiado, downtown\ Lisbon) \geqslant 0.5 \tag{12}$$

The reasoning tasks we solve below can be used to determine whether such a fuzzy interpretation of the natural language statements is satisfiable, whether there are stronger satisfiable fuzzy interpretations possible (i.e., higher lower bounds), etc. Returning to the example from Fig. 4, we may, for example, want to obtain a vague boundary for region A if it is known (or assumed) that $EC(A, B) = 0.5$ and $PO(A, B) = 0.5$. An example of such a vague boundary is given in Fig. 4(d), where the dark-gray points are points that belong to A to degree 1 and the light-gray points are points that belong to A to degree 0.5.

Use case 2: building qualitative geographical models In some situations, qualitative descriptions are all that is available. For example, when the region to be modelled is not a city neighborhood, but a park, forest, cultural heritage site or university campus, we will probably not find many addresses contained in it. As a consequence, we cannot rely on geocoding to acquire quantitative information. To illustrate this point, assume we want to acquire a spatial model of Stanley Park in Vancouver, B.C., Canada. Such a model may, for instance, be of interest in geographical information retrieval systems and question answering systems to help people locate tourist attractions such as the Vancouver Aquarium, Second Beach or Ceperley Meadows. Information about the spatial configuration of Stanley Park attractions and landmarks can be obtained by extracting spatial relations from texts:

(1) Second Beach Pool is appropriately located at Second Beach, which is the large waterfront park at the south-western entrance to Stanley Park … You can easily access the pool by walking down along the seawall towards Stanley Park … [6]
(2) English Bay is awesome, and all the beaches around Stanley Park are nice too (I really like Second Beach).[7]
(3) There are two beaches (called Third and Second) right in Stanley Park … [8]
(4) Second Beach is technically in Stanley Park.[9]
(5) An 8.8 kilometre (5.5 mile) seawall path circles the park, which is used … [10]
(6) The seawall in Vancouver, Canada is a stone wall that was constructed around the perimeter of Stanley Park … [11]

The spatial information conveyed in these statements can be translated into RCC formulas and, as before, the resulting description is inconsistent. In particular, the first statement suggests that Second Beach is either disconnected from or adjacent to Stanley Park. The second statement seems to indicate that Second Beach is adjacent to Stanley Park, whereas the third expresses that Second Beach is in fact contained in Stanley Park. Hence, there seems to be some vagueness about the exact boundaries of Stanley Park, i.e., about whether the boundaries encompass Second Beach or not. This is further exemplified in the fourth statement which conveys that Second Beach is located in Stanley Park, but, at the same time, not really considered to be a part of it. Similarly, it is unclear whether the Seawall is adjacent to or contained in Stanley Park,

[6] http://www.virtualtourist.com/travel/North_America/Canada/Province_of_British_Columbia/Vancouver-903183/Things_To_Do-Vancouver-Second_Beach-BR-1.html,accessed December 18, 2007.
[7] http://www.nextbody.com/forums/archive/index.php/t-4178.html, accessed December 18, 2007.
[8] http://www.whyvancouver.com/beaches.html, accessed December 18, 2007.
[9] http://members.virtualtourist.com/m/tt/8079/, accessed December 18, 2007.
[10] http://en.wikipedia.org/wiki/Stanley_Park, accessed December 18, 2007.
[11] http://en.wikipedia.org/wiki/Seawall_%28Vancouver%29, accessed December 18, 2007.

i.e., both *TPP* and *EC* hold between the Seawall and Stanley Park to some extent. Again, the fuzzy topological reasoning algorithms introduced below could be applied to find a satisfiable fuzzy interpretation of the natural language statements.

Use case 3: building non-geographical spatial models Spatial reasoning and information processing do not only occur in geography. For example, a significant fraction of the information in biomedical ontologies is topological [6,52,53]. Along similar lines, spatial relations occur in descriptions of various spatial scenes, e.g., specifications of multimedia documents [32], eye witness reports of a traffic accident, textual descriptions about the rooms' arrangement in a house, etc. Usually, spatial relations are initially expressed in a crisp way, but vagueness is introduced when descriptions of the same scene, either in structured form (e.g., ontologies) or textual form, from various sources are merged [59]. In this case, fuzzy spatial reasoning could be used to integrate conflicting spatial descriptions of a given scene. It is important to note here that this process does not only apply to the three-dimensional physical scenes found in reality or their two-dimensional abstractions, which we have used in our examples so far, but also to conceptual spaces of arbitrary dimension.

Conceptual spaces have been developed in [16] as a powerful means to define the semantics of concepts. As opposed to symbolic approaches such as description logics, concepts are given precise definitions as convex regions in a (typically) Euclidean space, whose dimensions correspond to qualities of the objects under consideration. In [17], it is proposed to use the RCC for reasoning about concept definitions (i.e., regions in the conceptual space of interest). By generalizing this idea to regions with fuzzy boundaries, reasoning in our fuzzy RCC could be used to address challenging problems related to the modelling of vague concepts. Consider, for example, the well-known Wine Ontology.[12] In this ontology, many wine-related concepts are defined, e.g., *Dry Wine*, *Table Wine*, *Late Harvest Wine*, etc., as well as relations between these concepts. When thinking of concepts as regions in a conceptual space, such relations typically correspond to topological spatial relations. For example, the information that *Wine* is a subclass of *Potable Liquid* implies a relation *PP* between the corresponding regions, whereas the constraint that the properties *Dry*, *Off-Dry* and *Sweet* are mutually exclusive implies $DR(Dry, Off\text{-}Dry)$, $DR(Dry, Sweet)$ and $DR(Off\text{-}Dry, Sweet)$. Moreover, assuming that some prototypes are available for each concept, the actual region boundaries could be approximated using techniques from computational geometry such as Voronoi tessellations [17]. From these boundaries, relations such as *TPP*, *NTPP* and *EC* could be derived. However, whereas the wine ontology in itself is consistent, it is well-known that merging it with other wine-related knowledge bases easily leads to inconsistencies [36]. For example, many subclass relations are, in fact, not valid for certain particular cases: *Port* is defined as a subclass of *Red Wine*, while in reality there are some white port wines as well; the Wine Ontology claims that wines are made from at least one type of grape, which is clearly not the case for apple wine or rice wine; concepts such as *Dry Wine* are to some extent subjective, and different ontologies might use slightly different definitions (e.g., based on sugar content only, based on sugar content relative to acidity, based on taste, etc.). Using fuzzy topological reasoning, we may obtain consistent interpretations claiming that, for example, *Apple Wine* is a subclass of *Wine* to degree 0.3, meaning that although apple wine is a special kind of wine to some extent, it is not a prototypical wine, and that *Wine* is a subclass of *MadeFromGrapes* to degree 0.7, meaning that typical wines are made from grapes, but there might be borderline cases which are not.

4. Related work

Reasoning about topological information encoded in RCC-8 has been well studied. Most reasoning tasks of interest are NP-complete [42]. In consequence, and inspired by results about reasoning in the IA (interval algebra), considerable work has been devoted to finding tractable subfragments of RCC-8, i.e., subsets of the 2^8 relations that can be expressed in RCC-8 for which reasoning is tractable [20,34,40,42]. In [34], it was shown that reasoning in RCC-8 is tractable, provided only base relations are used (i.e., no disjunctive information such as $(TPP \cup EC)(a, b)$). Of special interest are subsets of RCC-8 relations that are maximally tractable, i.e., such that every proper superset of RCC-8 relations would result in NP-completeness. A first maximal tractable subfragment, containing 146 relations, was identified in [42]. Two additional maximal tractable subfragments were identified in [40], containing 158 and 160 relations. In [40] it was moreover shown that these three subfragments are the only maximal tractable subfragments of RCC-8 that contain all eight base relations. In all three subfragments of RCC-8, satisfiability can be decided using an $O(n^3)$ path-consistency algorithm. Experimental results in [43] indicate that using these maximal tractable subfragments for reasoning in RCC-8 has a significant impact on computation time: almost all problem instances up to 500 regions could be solved in a very efficient way (mostly less than 1 minute on a Sun Ultra 1 machine with 128 MB of internal memory).

Usually, a knowledge base of RCC-8 relations is called satisfiable (or consistent) if it can be realized in some topological space, i.e., if all variables can be interpreted by regions in some topological space such that all imposed relations hold [42]. In practice, however, it might be interesting to know whether a set of RCC-8 formulas can be realized by (regular closed) subsets of, for example, $\mathbb{R}^2$ or $\mathbb{Z}^2$ and, if so, which additional constraints on these subsets might be imposed (e.g., convexity, internal connectedness, etc.). In [41], it was shown that any satisfiable set of RCC-8 formulas can be realized in $\mathbb{R}^n$ for every n in $\mathbb{N} \setminus \{0\}$. In other words, satisfiability and realizability in $\mathbb{R}^n$ are equivalent. For $n \geqslant 3$, this result also holds when regions are constrained to be internally connected, and even if they are constrained to be polytopes. Unfortunately, for $n = 2$ and $n = 1$ this result does not hold in general. Furthermore, until recently, it was not even known if the problem of checking

[12] http://www.w3.org/TR/owl-guide/wine.rdf.

the realizability of a knowledge base of RCC-8 relations by internally connected two-dimensional regions is in NP, or even decidable. In particular, this problem can be related to the problem of recognizing a special class of graphs called string graphs [20,26]. In [45], it was shown that recognizing string graphs, and therefore deciding whether an RCC-8 knowledge base is realizable by internally connected regions in $\mathbb{R}^2$, is indeed in NP. Note that checking whether a knowledge base of RCC-8 relations can be realized by internally connected one-dimensional regions essentially corresponds to an undirected variant of the satisfiability problem in the IA, and is therefore in NP. In [28], it was shown that any satisfiable set of RCC-8 formulas can also be realized by subsets of $\mathbb{Z}^2$ (i.e., the digital plane).

Considerable work has been done on generalizing topological relations to cope with vague regions. Most models of topological relations between vague regions extend either the RCC or the 9-intersection model by treating a vague region a as a pair of two crisp regions $\underline{a}$ and $\bar{a}$ such that $\underline{a} \subseteq \bar{a}$. A well-known example is the Egg–Yolk calculus [5], which is based on the RCC. In [4], a similar approach, based on the notion of a thick boundary, is proposed as an extension of the 9-intersection model. Both models cause a significant increase in the number of possible relations: 46 and 44 relations respectively. For example, instead of specifying that two regions a and b overlap, we may specify that $\bar{a}$ and $\underline{b}$ overlap (but not $\underline{a}$ and $\underline{b}$), or that $\bar{a}$ and $\bar{b}$ overlap, or that $\underline{a}$ and $\underline{b}$ overlap, etc. Another possibility, which is adopted in [44], is to stay with the spatial relations of the RCC, but to use three-valued relations instead of classical two-valued relations.

Other approaches have been concerned with defining (fuzzy) spatial relations between vague regions represented as fuzzy sets. For example, in [61] and [46], generalizations of the 9-intersection model based on α-levels of fuzzy sets are suggested. In [55], a generalization of the 9-intersection model is introduced using concepts from fuzzy topology, yielding a set of 44 crisp spatial relations. Another generalization of the 9-intersection model, using similar fuzzy topological concepts, is proposed in [31], again obtaining 44 relations between fuzzy sets. On the other hand, [30] uses the RCC as a starting point to define crisp spatial relations between fuzzy sets. However, this approach can only be used when the membership values of the fuzzy sets are taken from a finite universe. The total number of relations is dependent on the cardinality of the finite set of membership values. In [21] degrees of appropriateness are assigned to RCC relations. These degrees could be interpreted as encoding, for instance, preferences or possibilistic uncertainty. In [37], definitions of fuzzy topological relations between vague regions in a discrete space (e.g., fuzzy sets of grid cells) are provided. Finally, [7] and [8] discuss fuzzy topological relations with the goal of modelling position uncertainty of region boundaries.

All of the aforementioned approaches have in common that certain assumptions are made on how vague regions are represented. Moreover, they are mainly applicable to geographical contexts, and can usually not be used in situations where, for example, RCC relations are used in a metaphorical way. The generality and much of the elegance of the RCC is lost in this way. A different possibility is to generalize the RCC relations directly, without making any assumptions on how regions should be represented. This idea has already been pursued, to some extent, in [14], where the starting point was to define connection as an arbitrary symmetric fuzzy relation C in the universe U of regions, satisfying a weak reflexivity property, namely $C(a,a) > 0.5$ for every region a in U. The fuzzy relation P (part of), for example, is defined by

$$P(a,b) = \inf_{z \in U} \max\bigl(1 - C(z,a), C(z,b)\bigr) \tag{13}$$

where a and b are regions in U. However, many properties of the original RCC relations are lost in this approach. For example, in correspondence with the reflexivity of P in the RCC, it would be desirable that $P(a,a) = 1$ for any region a in U. Unfortunately, this is, in general, not the case when (13) is used to define P, due to the particular way in which logical implication has been generalized. Similarly, many interesting transitivity properties are also lost, which makes these fuzzy relations unsuitable for spatial reasoning.

Note that the aforementioned approaches for incorporating vagueness in topological relations have largely neglected issues of reasoning. They are mainly concerned with calculating to what degree a topological relation is fulfilled, given the representations of all regions of interest (e.g., as fuzzy sets of locations). Finally, note that some of the results in this paper have already been summarized in [49].

5. Reasoning in the fuzzy RCC

To investigate the main reasoning properties of the fuzzy RCC, we proceed in two steps. First, in Section 5.1, we look at the satisfiability problem for normalised sets. In other words, we first solve the satisfiability problem for the special case where the exact fuzzy spatial relationship between each pair of regions is given. Next, we generalize the satisfiability checking procedure to arbitrary (standard) sets of fuzzy RCC formulas, relying heavily on the notion of refinement. In particular, we will start from the observation that a standard set of fuzzy RCC formulas is satisfiable iff it can be refined to a normalised set which is satisfiable. Subsequently, the computational complexity of the proposed procedures is investigated in more detail in Section 5.3. Finally, in Section 5.4, we illustrate how a number of additional reasoning tasks (e.g., entailment checking) can be solved in a similar way.

5.1. Satisfiability of normalised sets

First, we show how the satisfiability of a normalised set of fuzzy RCC formulas can be checked. A normalised set of fuzzy RCC formulas is completely characterized by four matrices, containing the membership degrees of the fuzzy topological relations C, P, O and NTP for each pair of regions in V. For example ($V = \{a, b, c\}$):

$$\begin{array}{cc} & \begin{array}{ccc} a & b & c \end{array} \\ C = \begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} 1 & 0.6 & 0.8 \\ 0.6 & 1 & 0.6 \\ 0.8 & 0.6 & 1 \end{pmatrix} \end{array} \qquad \begin{array}{cc} & \begin{array}{ccc} a & b & c \end{array} \\ P = \begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} 1 & 0.6 & 0 \\ 0 & 1 & 0 \\ 0.8 & 0.4 & 1 \end{pmatrix} \end{array} \tag{14}$$

$$\begin{array}{cc} & \begin{array}{ccc} a & b & c \end{array} \\ O = \begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} 1 & 0.6 & 0.8 \\ 0.6 & 1 & 0.4 \\ 0.8 & 0.4 & 1 \end{pmatrix} \end{array} \qquad \begin{array}{cc} & \begin{array}{ccc} a & b & c \end{array} \\ NTP = \begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} 0 & 0.6 & 0 \\ 0 & 0 & 0 \\ 0.4 & 0.4 & 0.6 \end{pmatrix} \end{array} \tag{15}$$

There are a number of necessary conditions for satisfiability that follow straightforwardly from elementary properties of the fuzzy topological relations. For instance, from Lemma 5, we have that

$$NTP(u,v) \leqslant P(u,v) \leqslant O(u,v) \leqslant C(u,v) \tag{16}$$

for every u and v in V. For normalised sets, (16) translates into restrictions on the corresponding matrix representation. In particular, this implies that the elements of the matrix for NTP should be smaller than the corresponding elements of the matrix for P, which should in turn be smaller than the elements of the matrix for O, which should be smaller than the elements of the matrix for C. It is easy to verify that this is indeed the case in the example (14)–(15). From Lemma 4, we know that for every u and v in V

$$C(u,v) = C(v,u) \tag{17}$$

$$O(u,v) = O(v,u) \tag{18}$$

In terms of the matrix representation, this means that the matrices for C and O should be symmetric. By Lemma 3, we have for every u in V

$$P(u,u) = O(u,u) = C(u,u) = 1 \tag{19}$$

implying that the elements on the diagonal of the matrices for C, O and P should all be 1. The requirements (16)–(19) can easily be checked by performing $O(|V|^2)$ simple arithmetic comparisons. Clearly, if any of these requirements is violated, the corresponding set of fuzzy RCC formulas is not satisfiable. Finally, a set of additional requirements follows from the transitivity rules in Table 2. For example, consider the transitivity rule

$$T_W\big(P(u,v), P(v,w)\big) \leqslant P(u,w) \tag{20}$$

generalizing the fact that in the original RCC, $P(u,v)$ and $P(v,w)$ together imply $P(u,w)$. From the matrices above, we find for $u = c$, $v = a$ and $w = b$

$$P(c,a) = 0.8 \qquad P(a,b) = 0.6 \qquad P(c,b) = 0.4$$

We obtain $T_W(P(c,a), P(a,b)) = T_W(0.8, 0.6) = 0.4$, hence (20) is satisfied. In total, the transitivity table summarizes 144 generalized transitivity rules. However, 84 of these rules are trivially satisfied because their right-hand side is 1 (e.g., $T_W(C(u,v), O(v,w)) \leqslant 1$). This means that at most 60 transitivity rules need to be checked. Moreover, most of these transitivity rules are the dual of another rule. For example, the transitivity table contains both of the following rules

$$T_W\big(P(u,v), P(v,w)\big) \leqslant P(u,w) \tag{21}$$

$$T_W\big(P^{-1}(u,v), P^{-1}(v,w)\big) \leqslant P^{-1}(u,w) \tag{22}$$

Clearly, if (21) holds for all u, v and w in V, then (22) will be satisfied as well. In consequence, the number of remaining transitivity rules can almost be halved: only the following 31 rules need to be checked.

$$T_W\big(C(u,v), DC(v,w)\big) \leqslant coP(u,w) \tag{23}$$

$$T_W\big(C(u,v), P(v,w)\big) \leqslant C(u,w) \tag{24}$$

$$T_W\big(C(u,v), DR(v,w)\big) \leqslant coNTP(u,w) \tag{25}$$

$$T_W\big(C(u,v), NTP(v,w)\big) \leqslant O(u,w) \tag{26}$$

$$T_W\big(DC(u,v), P(v,w)\big) \leqslant coP^{-1}(u,w) \tag{27}$$

$$T_W\big(DC(u,v), P^{-1}(v,w)\big) \leqslant DC(u,w) \tag{28}$$

$$T_W\big(DC(u,v), O(v,w)\big) \leqslant coP^{-1}(u,w) \tag{29}$$

$$T_W\big(DC(u,v), NTP(v,w)\big) \leqslant coP^{-1}(u,w) \tag{30}$$

$$T_W\big(DC(u,v), NTP^{-1}(v,w)\big) \leqslant DC(u,w) \tag{31}$$
$$T_W\big(P(u,v), P(v,w)\big) \leqslant P(u,w) \tag{32}$$
$$T_W\big(P(u,v), coP^{-1}(v,w)\big) \leqslant coP^{-1}(u,w) \tag{33}$$
$$T_W\big(P(u,v), DR(v,w)\big) \leqslant DR(u,w) \tag{34}$$
$$T_W\big(P(u,v), coNTP^{-1}(v,w)\big) \leqslant coNTP^{-1}(u,w) \tag{35}$$
$$T_W\big(P(u,v), NTP(v,w)\big) \leqslant NTP(u,w) \tag{36}$$
$$T_W\big(P^{-1}(u,v), NTP^{-1}(v,w)\big) \leqslant NTP^{-1}(u,w) \tag{37}$$
$$T_W\big(P^{-1}(u,v), NTP(v,w)\big) \leqslant O(u,w) \tag{38}$$
$$T_W\big(P^{-1}(u,v), coNTP(v,w)\big) \leqslant coNTP(u,w) \tag{39}$$
$$T_W\big(P^{-1}(u,v), P(v,w)\big) \leqslant O(u,w) \tag{40}$$
$$T_W\big(P^{-1}(u,v), DR(v,w)\big) \leqslant coP(u,w) \tag{41}$$
$$T_W\big(P^{-1}(u,v), coP(v,w)\big) \leqslant coP(u,w) \tag{42}$$
$$T_W\big(P^{-1}(u,v), O(v,w)\big) \leqslant O(u,w) \tag{43}$$
$$T_W\big(coP^{-1}(u,v), NTP(v,w)\big) \leqslant coP^{-1}(u,w) \tag{44}$$
$$T_W\big(coP(u,v), NTP^{-1}(v,w)\big) \leqslant coP(u,w) \tag{45}$$
$$T_W\big(O(u,v), DR(v,w)\big) \leqslant coP(u,w) \tag{46}$$
$$T_W\big(O(u,v), NTP(v,w)\big) \leqslant O(u,w) \tag{47}$$
$$T_W\big(DR(u,v), NTP^{-1}(v,w)\big) \leqslant DC(u,w) \tag{48}$$
$$T_W\big(DR(u,v), NTP(v,w)\big) \leqslant coP^{-1}(u,w) \tag{49}$$
$$T_W\big(NTP(u,v), NTP(v,w)\big) \leqslant NTP(u,w) \tag{50}$$
$$T_W\big(NTP(u,v), coNTP^{-1}(v,w)\big) \leqslant coP^{-1}(u,w) \tag{51}$$
$$T_W\big(NTP^{-1}(u,v), coNTP(v,w)\big) \leqslant coP(u,w) \tag{52}$$
$$T_W\big(NTP^{-1}(u,v), NTP(v,w)\big) \leqslant O(u,w) \tag{53}$$

Each of these transitivity rules has to be checked for all $|V|^3$ triples of variables from Θ. In summary, we have a number of necessary conditions for satisfiability, which can be checked in $O(|V|^3)$ time. As the next proposition reveals, these necessary conditions are also sufficient, provided Θ does not contain a fuzzy RCC formula of the form $NTP(v,v) \geqslant 1$.

Proposition 1. *Let Θ be a normalised, finite set of fuzzy RCC formulas, $T = T_W$ and let V be the set of variables occurring in Θ ($|V| = n$). Assume, moreover, that Θ does not contain fuzzy RCC formulas of the form $NTP(v,v) \geqslant 1$. If Θ satisfies* (16)–(19)*, as well as the transitivity rules* (23)–(53)*, for all u, v and w in V, it holds that Θ has an $(n;\alpha,0)$-model for every $\alpha > 0$.*

Proof. A sketch of the proof is provided in Appendix A. □

Corollary 1. *Under the assumptions of Proposition* 1*, it holds that Θ is satisfiable iff* (16)–(19) *and the transitivity rules* (23)–(53) *are satisfied for all u, v and w in V.*

Below, we will extend this result to $(n;\alpha,0)$-models (Section 8) for an arbitrary dimension n (rather than one particular dimension), as well as other types of models, including models from the Egg–Yolk calculus (Section 7). From the construction process in Appendix A, it follows furthermore that, if Θ is satisfiable, an $(n;\alpha,0)$-model can always be found in which the fuzzy sets satisfy a number of additional, natural requirements. For example, in this construction process, all α-level sets of the fuzzy sets are regular closed regions in the natural topology on $\mathbb{R}^n$. This is important, as it ensures that there are no normalised sets of fuzzy RCC formulas that can only be satisfied in counterintuitive models (e.g., where variables are mapped to points or lines). Thus, we also have that Proposition 1 remains valid if such additional restrictions are imposed on allowed interpretations.

The following proposition reveals that the additional constraint from Proposition 1 that formulas of the form $NTP(v,v) \geqslant 1$ do not occur, does not have to be considered in satisfiability checking procedures, provided only standard sets are allowed. In other words, to check the satisfiability of a standard set of fuzzy RCC formulas Θ, we only need to check whether the necessary conditions we identified above are satisfied in at least one refinement of Θ.

Proposition 2. *Let Θ_0 be a standard, finite set of fuzzy RCC formulas and let $T = T_W$. If Θ_0 can be refined to a normalised set of fuzzy RCC formulas Θ_1 which satisfies* (16)–(19)*, as well as the transitivity rules* (23)–(53)*, it holds that Θ_0 has an $(n;\alpha,0)$-model for every $\alpha > 0$.*

Proof. The proof is presented in Appendix B. $\square$

Corollary 2. *Let $T = T_W$. It holds that a standard, finite set of fuzzy RCC formulas Θ is satisfiable iff it can be refined to a normalised set Θ' which does not violate the transitivity rules* (23)–(53) *nor the requirements* (16)–(19)*.*

5.2. Satisfiability of standard sets

Let Θ be a standard set of fuzzy RCC formulas involving variables from a set $V = \{v_1, \ldots, v_n\}$. As explained above, we can rewrite the fuzzy RCC formulas in Θ such that only disjunctions of fuzzy RCC formulas involving C, P, O, and NTP occur, thus obtaining a set Θ'. Clearly, Θ' is equivalent to Θ, hence every refinement of Θ' is also a refinement of Θ. Note that although Θ' is not a standard set of fuzzy RCC formulas, we can still apply Proposition 2 and Corollary 2, because Θ' is equivalent to a standard set.

To decide whether Θ', and therefore Θ, is satisfiable, we need to find membership degrees for $C(v_i, v_j)$, $P(v_i, v_j)$, $O(v_i, v_j)$, and $NTP(v_i, v_j)$, for each pair of regions (v_i, v_j) in V^2, such that the transitivity rules and the requirements (16)–(19) are satisfied, as well as the inequalities imposed by the fuzzy RCC formulas in Θ'; or show that no such membership degrees exist. As we will show next, these requirements on the membership degrees can be formulated as a system of (disjunctions of) linear inequalities Σ, where both the number of variables and the number of inequalities is polynomial in the size of Θ. This means that the satisfiability of Θ' can be decided using a linear programming solver and backtracking (if Θ' contains disjunctions). In the following, let x_{ij}^C, x_{ij}^P, x_{ij}^O and x_{ij}^{NTP} be variables, corresponding to the values of $C(v_i, v_j)$, $P(v_i, v_j)$, $O(v_i, v_j)$, and $NTP(v_i, v_j)$ respectively. For every variable x_{ij}^R (where R is C, P, O or NTP), we add to Σ the constraint

$$0 \leqslant x_{ij}^R \leqslant 1$$

This ensures that every solution of Σ can be interpreted as a normalised set of fuzzy RCC formulas Θ^Σ, containing the formulas

$$R(v_i, v_j) \geqslant x_{ij}^R \qquad R(v_i, v_j) \leqslant x_{ij}^R$$

assuming for simplicity $x_{ij}^R \in]0,1[$. Next, the fuzzy RCC formulas in Θ' can all be written as disjunctions of linear inequalities involving these variables. For example, if Θ' contains the fuzzy RCC formula $NTP(v_1, v_2) \geqslant 0.4 \vee O(v_3, v_4) \leqslant 0.7$, we add the following constraint to Σ

$$x_{12}^{NTP} \geqslant 0.4 \vee x_{34}^O \leqslant 0.7$$

This ensures that if Σ has a solution, Θ^Σ is a refinement of Θ'. To ensure that Θ^Σ is satisfiable, we add linear inequalities corresponding to (16)–(19):

$$x_{ij}^{NTP} \leqslant x_{ij}^P \leqslant x_{ij}^O \leqslant x_{ij}^C \quad x_{ij}^C = x_{ji}^C \quad x_{ij}^O = x_{ji}^O \quad x_{ii}^P \geqslant 1$$

Finally, to guarantee that solutions of Σ do not violate the transitivity rules, we add an additional inequality for each of the 31 transitivity rules identified above and each triplet (v_i, v_j, v_k) in V^3. For example, because of the first of these transitivity rules, we know that

$$T_W\big(C(v_i, v_j), DC(v_j, v_k)\big) \leqslant coP(v_i, v_k)$$

which is equivalent to

$$\big(C(v_i, v_j) + 1 - C(v_j, v_k) - 1 \leqslant 1 - P(v_i, v_k)\big) \wedge \big(0 \leqslant 1 - P(v_i, v_k)\big)$$

or

$$\big(C(v_i, v_j) - C(v_j, v_k) \leqslant 1 - P(v_i, v_k)\big) \wedge \big(P(v_i, v_k) \leqslant 1\big)$$

Hence, we add the following inequality to Σ

$$x_{ij}^C - x_{jk}^C \leqslant 1 - x_{ik}^P$$

Note that $x_{ik}^P \leqslant 1$ is already in Σ and does, therefore, not have to be considered here anymore.

Thus we have constructed a system Σ of disjunctions of linear inequalities such that, if Σ has a solution, the corresponding normalised set of fuzzy RCC formulas Θ^Σ satisfies the assumptions from Proposition 2, implying that Θ and Θ' are satisfiable. If Σ does not contain any disjunctions, deciding whether Σ has a solution can be done in polynomial time using a linear programming solver [24] (assuming that the number of bits required to represent each of the lower and upper bounds in Θ is bounded by a constant). For example, let Θ' be defined as

$$\Theta' = \{C(b,c) \geqslant 0.6, P(c,a) \geqslant 0.8, O(a,c) \geqslant 0.4, \\ O(b,a) \leqslant 0.6, O(b,c) \leqslant 0.5, NTP(a,b) \geqslant 0.6, NTP(c,c) \geqslant 0.6\} \tag{54}$$

Solving the corresponding system of linear inequalities Σ using the lp_solve[13] linear programming solver yields the consistent, normalised set of fuzzy RCC formulas defined by (14)–(15). If Θ' contains disjunctions, we can use a backtracking algorithm to determine whether any choice of the disjuncts leads to a system of linear inequalities that has a solution.

5.3. Computational complexity

The construction above entails that the problem of checking the satisfiability of a set of fuzzy RCC formulas is in NP. We show that this problem is also NP-hard. Note that by restricting the fuzzy relations C_α, O_α, P_α and NTP_α to crisp subsets of $\mathbb{R}^n$, crisp spatial relations are obtained. The next lemma reveals that these crisp spatial relations correspond to interpretations of the RCC.

Lemma 7. *Let $\mathcal{I}$ be defined such that for every u in U, $u^\mathcal{I}$ is a crisp, non-empty, bounded subset of $\mathbb{R}^n$ for some n in $\mathbb{N} \setminus \{0\}$. It holds that*

$$P_\alpha(u^\mathcal{I}, v^\mathcal{I}) \equiv (\forall w \in U)(C_\alpha(w^\mathcal{I}, u^\mathcal{I}) \Rightarrow C_\alpha(w^\mathcal{I}, v^\mathcal{I}))$$
$$O_\alpha(u^\mathcal{I}, v^\mathcal{I}) \equiv (\exists w \in U)(P_\alpha(w^\mathcal{I}, u^\mathcal{I}) \wedge P_\alpha(w^\mathcal{I}, v^\mathcal{I}))$$
$$NTP_\alpha(u^\mathcal{I}, v^\mathcal{I}) \equiv (\forall w \in U)(C_\alpha(w^\mathcal{I}, u^\mathcal{I}) \Rightarrow O(w^\mathcal{I}, v^\mathcal{I}))$$

Proof. From Lemma 6 we know that

$$P_\alpha(u^\mathcal{I}, v^\mathcal{I}) = \inf_{w \in U} I_T(C_\alpha(w^\mathcal{I}, u^\mathcal{I}), C_\alpha(w^\mathcal{I}, v^\mathcal{I}))$$
$$O_\alpha(u^\mathcal{I}, v^\mathcal{I}) = \sup_{w \in U} T(P_\alpha(w^\mathcal{I}, u^\mathcal{I}), P_\alpha(w^\mathcal{I}, v^\mathcal{I}))$$
$$NTP_\alpha(u^\mathcal{I}, v^\mathcal{I}) = \inf_{w \in U} I_T(C_\alpha(w^\mathcal{I}, u^\mathcal{I}), O_\alpha(w^\mathcal{I}, v^\mathcal{I}))$$

Since $u^\mathcal{I}$, $v^\mathcal{I}$ and $w^\mathcal{I}$ are crisp sets, by assumption, for example, also $C_\alpha(w^\mathcal{I}, u^\mathcal{I})$ and $C_\alpha(w^\mathcal{I}, v^\mathcal{I})$ take crisp values. Therefore, we have that

$$\inf_{w \in U} I_T(C_\alpha(w^\mathcal{I}, u^\mathcal{I}), C_\alpha(w^\mathcal{I}, v^\mathcal{I})) = 1$$

holds iff

$$(\forall w \in U)(C_\alpha(w^\mathcal{I}, u^\mathcal{I}) \Rightarrow C_\alpha(w^\mathcal{I}, v^\mathcal{I}))$$

and $\inf_{w \in U} I_T(C_\alpha(w^\mathcal{I}, u^\mathcal{I}), C_\alpha(w^\mathcal{I}, v^\mathcal{I})) = 0$ otherwise. The expressions for $O_\alpha(u^\mathcal{I}, v^\mathcal{I})$ and $NTP_\alpha(u^\mathcal{I}, v^\mathcal{I})$ follow in entirely the same manner. □

It follows from Proposition 2 that every standard, C-satisfiable set Θ of RCC formulas can be interpreted by an $(n;\alpha,0)$-model. Indeed, crisp RCC formulas such as $NTPP(v_1, v_2) \vee \neg EC(v_1, v_3)$ can be interpreted as fuzzy RCC formulas with upper and lower bounds in $\{0, 1\}$, i.e., $NTPP(v_1, v_2) \geqslant 1 \vee EC(v_1, v_3) \leqslant 0$. Moreover, since all upper and lower bounds come from $\{0, 1\}$, it follows from the construction process in the proof of Proposition 1 that the fuzzy RCC formulas can be interpreted by crisp sets. This leads to the following corollary.

Corollary 3. *Every finite, standard, C-satisfiable set of RCC formulas can be interpreted by an $(n;\alpha,0)$-model in which every variable is interpreted as a crisp, non-empty, bounded subset of $\mathbb{R}^n$.*

Note that, from [41] we know that every C-satisfiable set of RCC formulas can be interpreted by a standard RCC model in any dimension. Thus, we also have the following corollary.

Corollary 4. *Let Θ be a set of RCC formulas. If Θ has an $(n;\alpha,0)$-model, Θ also has a standard model in $\mathbb{R}^m$ for every m in $\mathbb{N} \setminus \{0\}$.*

[13] http://sourceforge.net/projects/lpsolve.

In particular, we have that checking the C-satisfiability of a set of RCC formulas can be reduced to checking the F-satisfiability of a set of fuzzy RCC formulas. Since the former problem is known to be NP-hard [42], we thus find that also the latter problem is NP-hard. Hence, we have the following proposition.

Proposition 3. *Checking the F-satisfiability of a set of fuzzy RCC formulas is NP-complete* ($T = T_W$).

Note that the F-satisfiability problem for fuzzy RCC formulas is in the same complexity class as the C-satisfiability problem in RCC-8. Furthermore, note that notwithstanding this NP-hardness result, reasoning about fuzzy topological relations is often tractable in practice. In particular, if a knowledge base Θ can be written in terms of fuzzy RCC formulas involving only C, P, O and *NTP*, such that no disjunctions occur, the procedure outlined in Section 5.2 only takes polynomial time.

5.4. Other reasoning tasks

Besides satisfiability checking, also other interesting reasoning tasks, such as entailment checking, finding the best truth-value bound, and inconsistency repairing, can be cast into systems of linear inequalities.

5.4.1. Entailment

Definition 8. Let Θ be a set of fuzzy RCC formulas and γ a fuzzy RCC formula; Θ is said to entail γ, written $\Theta \models \gamma$, if γ is satisfied in every model of Θ.

Without loss of generality, we can assume that Θ only contains (disjunctions of) formulas involving C, O, P and *NTP*. Furthermore, note that

$$\Theta \cup \{f_1 \vee f_2 \vee \cdots \vee f_m\} \models \gamma \Leftrightarrow \big(\Theta \cup \{f_1\} \models \gamma\big) \wedge \cdots \wedge \big(\Theta \cup \{f_m\} \models \gamma\big) \tag{55}$$

As a consequence, it is sufficient to show how $\Theta \models \gamma$ can be checked for the case where Θ does not contain any disjunctions. In other words, we can assume that all formulas in Θ are of the form $R(u, v) \geqslant \lambda$ or $R(u, v) \leqslant \lambda$ where R can be C, O, P or *NTP*. Let Σ furthermore be the corresponding system of linear inequalities, having a solution iff Θ is satisfiable. We can now extend Σ to a system of linear inequalities Σ_γ such that Σ_γ does not have a solution iff $\Theta \models \gamma$. First assume that γ does not contain disjunctions, e.g., $\gamma \equiv P(u, v) \geqslant 0.7$. Clearly, it holds that $\Theta \models \gamma$ iff $\Theta \cup \{\neg\gamma\}$ is not satisfiable, i.e., iff $\Theta_\gamma = \Theta \cup \{P(v_i, v_j) < 0.7\}$ is not satisfiable. If v_i or v_j do not occur in Θ, then Θ_γ is satisfiable iff Θ is satisfiable, i.e., we can take $\Sigma_\gamma = \Sigma$. Typically, v_i and v_j will already occur in Θ however. In that case, we can take

$$\Sigma_\gamma = \Sigma \cup \big\{x^P_{ij} < 0.7\big\}$$

As before, we can show that Σ_γ has a solution iff Θ_γ is satisfiable. Note that we do not have to add additional inequalities corresponding to (16)–(19) and the transitivity rules, since these are already contained in Σ (as v_i and v_j occur in Θ as well).

Note that Σ_γ contains a strict inequality, and can therefore not directly be solved using a linear programming solver. However, we can equivalently write Σ_γ as $\Sigma'_\gamma = \Sigma \cup \{x^P_{ij} \leqslant 0.7,\ x^P_{ij} \neq 0.7\}$. In [23], techniques for solving systems of linear inequalities with additional disequalities are introduced. Clearly, if $\Sigma \cup \{x^P_{ij} \leqslant 0.7\}$ does not have a solution, then neither has Σ'_γ. Therefore, assume that $\Sigma \cup \{x^P_{ij} \leqslant 0.7\}$ does have a solution. To find whether also Σ'_γ has a solution, we can make use of the fact that linear programming solvers can find the solution of a system of linear inequalities which minimizes or maximizes a given objective function. In particular, we can thus find the minimal and maximal values of the objective function x^P_{ij}, given the system of inequalities $\Sigma \cup \{x^P_{ij} \leqslant 0.7\}$. It holds that Σ'_γ does not have a solution iff both this minimal and maximal value is 0.7.

Example 2. As an example, consider again the set Θ' from (54), and suppose we want to check whether $\Theta' \models \gamma$ for $\gamma \equiv C(a, c) \geqslant 0.9$. We have

$$\Sigma'_\gamma = \Sigma \cup \big\{x^C_{ac} \leqslant 0.9,\ x^C_{ac} \neq 0.9\big\}$$

where x^C_{ac} is the variable in Σ corresponding to the value of $C(a, c)$. Maximizing the objective function x^C_{ac} subject to the constraints in $\Sigma \cup \{x^C_{ac} \leqslant 0.9\}$, we obtain $x^C_{ac} = 0.9$, whereas we obtain $x^C_{ac} = 0.8$ when minimizing x^C_{ac}. Since $0.8 \neq 0.9$, we have that Σ'_γ has a solution, and therefore, that $\Theta' \not\models C(a, c) \geqslant 0.9$. For $\gamma \equiv C(a, c) \geqslant 0.8$, we obtain

$$\Sigma'_\gamma = \Sigma \cup \big\{x^C_{ac} \leqslant 0.8,\ x^C_{ac} \neq 0.8\big\}$$

Both when minimizing x^C_{ac} and maximizing x^C_{ac}, subject to $\Sigma \cup \{x^C_{ac} \leqslant 0.8\}$, we obtain $x^C_{ac} = 0.8$. Therefore Σ'_γ does not have a solution, and $\Theta' \models C(a, c) \geqslant 0.8$.

If γ is the disjunction of more than one atomic fuzzy RCC formula, we can proceed in a similar way. For example, let $\gamma \equiv NTP(v_1, v_2) \leqslant 0.7 \vee P(v_3, v_4) \geqslant 0.5 \vee O(v_1, v_4) \geqslant 0.9$. For each disjunct, we now have to add a strict inequality to Σ, corresponding to its negation. We obtain

$$\Sigma_\gamma = \Sigma \cup \{x_{12}^{NTP} > 0.7,\ x_{34}^{P} < 0.5,\ x_{14}^{O} < 0.9\}$$

or, using disequalities

$$\Sigma'_\gamma = \Sigma \cup \{x_{12}^{NTP} \geqslant 0.7,\ x_{34}^{P} \leqslant 0.5,\ x_{14}^{O} \leqslant 0.9,\ x_{12}^{NTP} \neq 0.7,\ x_{34}^{P} \neq 0.5,\ x_{14}^{O} \neq 0.9\}$$

To solve, Σ'_γ, we can apply the following lemma.

Lemma 8. *Let Σ be a system of linear inequalities, and let $\{\gamma_1, \ldots, \gamma_k\}$ be a set of linear disequalities. It holds that $\Sigma \cup \{\gamma_1, \ldots, \gamma_k\}$ has a solution iff $\Sigma \cup \{\gamma_1\}$, $\Sigma \cup \{\gamma_2\}, \ldots, \Sigma \cup \{\gamma_k\}$ all have a solution.*

Proof. This lemma follows straightforwardly from Lemma 17 in [23]. □

5.4.2. Best truth-value bound

The notion of the best truth value bound was originally introduced in [54] in the context of fuzzy description logics. Here it consists of finding the strongest possible lower and upper bound for the values of $C(a,b)$, $P(a,b)$, $O(a,b)$, or $NTP(a,b)$, given that a set Θ of fuzzy RCC formulas is satisfied. Formally, we want to find the values of $lub_R(a,b;\Theta)$ and $glb_R(a,b;\Theta)$ for R either C, P, O, or NTP:

Definition 9. Let R be one of the fuzzy topological relations C, P, O, or NTP. Moreover, let Θ be a set of fuzzy RCC formulas. For regions a and b, $lub_R(a,b;\Theta)$ and $glb_R(a,b;\Theta)$ are defined as

$$lub_R(a,b;\Theta) = \inf\{\lambda \mid \lambda \in [0,1] \wedge \Theta \models R(a,b) \leqslant \lambda\}$$
$$glb_R(a,b;\Theta) = \sup\{\lambda \mid \lambda \in [0,1] \wedge \Theta \models R(a,b) \geqslant \lambda\}$$

Finding the best truth-value bounds can be done very analogously to entailment checking. Let Θ be a set of fuzzy RCC formulas. Moreover, let V, Θ and Σ be defined as before, i.e., Σ is a system of linear inequalities which has a solution iff Θ is consistent and V is the set of variables occurring in Θ. If $a \notin V$ or $b \notin V$, we immediately find that $lub_C(a,b;\Theta) = 1$, as the possible values of $C(a,b)$ are not constrained by the formulas in Θ. Therefore, assume that $a, b \in V$. A linear programming solver can be used to find a solution of Σ that maximizes $C(a,b)$. This maximal value of $C(a,b)$ is equal to $lub_C(a,b;\Theta)$. In the same way, the solution of Σ which minimizes $C(a,b)$ yields the value of $glb_C(a,b;\Theta)$. For R equal to P, O, or NTP, the values of $lub_R(a,b;\Theta)$ and $glb_R(a,b;\Theta)$ can be found in entirely the same way.

For example, again considering the set Θ' defined by (54), we obtain that $lub_C(a,c;\Theta') = 0.9$ and $glb_C(a,c;\Theta') = 0.8$. This means that for any model $\mathcal{I}$ of Θ', it holds that

$$C^{\mathcal{I}}(a^{\mathcal{I}}, c^{\mathcal{I}}) \in [0.8, 0.9]$$

5.4.3. Inconsistency repairing

In real-world applications, available topological information is often inconsistent. For example, although none of the four assertions (5)–(8) is clearly wrong, the resulting set is inconsistent because some assertions are only partially true. To remedy such inconsistencies, we weaken the interpretation of the available topological information. For example, (5)–(8) could be interpreted as

$$DC(Baixa, Chiado) \geqslant \lambda_1 \tag{56}$$
$$EC(Baixa, Chiado) \geqslant \lambda_2 \tag{57}$$
$$EQ(downtown\ Lisbon, Baixa) \geqslant \lambda_3 \tag{58}$$
$$O(Chiado, downtown\ Lisbon) \geqslant \lambda_4 \tag{59}$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in [0,1]$. Usually, we want the interpretation to be as strong as possible, i.e., we want to choose $\lambda_1, \lambda_2, \lambda_3$, and λ_4 such that increasing one of these values leads to inconsistency (i.e., Pareto optimality). This can again be formulated as a linear programming problem. First, we rewrite (56)–(59) as a set Θ' of fuzzy RCC formulas containing only (disjunctions of) atomic fuzzy RCC formulas involving C, P, O, or NTP:

$$C(Baixa, Chiado) \leqslant 1 - \lambda_1 \tag{60}$$
$$C(Baixa, Chiado) \geqslant \lambda_2 \tag{61}$$
$$O(Baixa, Chiado) \leqslant 1 - \lambda_2 \tag{62}$$

$$P(downtown\ Lisbon, Baixa) \geqslant \lambda_3 \quad (63)$$

$$P(Baixa, downtown\ Lisbon) \geqslant \lambda_3 \quad (64)$$

$$O(Chiado, downtown\ Lisbon) \geqslant \lambda_4 \quad (65)$$

We now consider the corresponding system of inequalities Σ. As before, each solution of Σ corresponds to a model of (56)–(59). Unlike before, however, the membership degrees λ_1, λ_2, λ_3 and λ_4 are additional variables, rather than constants. By specifying $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$ as the objective function (to be maximized), we obtain an interpretation that cannot be strengthened without introducing inconsistencies. Using lp_solve, we find $\lambda_1 = \lambda_2 = \lambda_3 = 0.5$ and $\lambda_4 = 1$. If a priori information is available about the reliability of each of the statements, this idea can be further extended by using weights in the objective function. For example, if the first and last statement are assumed to be somewhat less reliable, $0.5\lambda_1 + 0.8\lambda_2 + 0.8\lambda_3 + 0.5\lambda_4$ could be used as the objective function, yielding $\lambda_1 = \lambda_2 = \lambda_4 = 0.5$ and $\lambda_3 = 1$.

Another possibility is to first establish the minimal degree to which all statements can be satisfied. In this case, we first consider the fuzzy RCC formulas

$$DC(Baixa, Chiado) \geqslant \lambda \quad (66)$$

$$EC(Baixa, Chiado) \geqslant \lambda \quad (67)$$

$$EQ(downtown\ Lisbon, Baixa) \geqslant \lambda \quad (68)$$

$$O(Chiado, downtown\ Lisbon) \geqslant \lambda \quad (69)$$

In solving the corresponding system of linear inequalities Σ, we can now look for a solution that maximizes λ. This maximal value of λ is given by $\lambda = 0.5$. Next, within the set of solutions that satisfy Σ with $\lambda = 0.5$, we can try to find the best one, according to some application-dependent criterium. Note that this way of working is very similar to how fuzzy constraint satisfaction problems are solved [9]. In general, there are a large number of alternative strategies that may be pursued to define such optimal solutions; see, for example, Section 2.3 of [10]. For instance, assuming that the last two statements are more reliable (or more important) than the former two, we may be interested to find solutions that maximize λ' in the system of linear inequalities defined by

$$DC(Baixa, Chiado) \geqslant 0.5$$

$$EC(Baixa, Chiado) \geqslant 0.5$$

$$EQ(downtown\ Lisbon, Baixa) \geqslant 0.5$$

$$O(Chiado, downtown\ Lisbon) \geqslant 0.5$$

$$EQ(downtown\ Lisbon, Baixa) \geqslant \lambda'$$

$$O(Chiado, downtown\ Lisbon) \geqslant \lambda'$$

where the first four fuzzy RCC formulas correspond to (66)–(69), with λ instantiated by its optimal value 0.5. The maximal value of λ' for which these fuzzy RCC formulas are satisfiable is given by $\lambda' = 0.75$, which can again be found using a linear programming solver. Thus we obtain a solution in which the first two statements are true to degree 0.5 and the last two statements are true to degree 0.75. It is easy to verify (using a linear programming solver) that this solution can not be strengthened anymore without introducing inconsistency.

6. Reduction to the RCC

In the remainder of this paper, we provide a number of additional, important properties of our approach to fuzzy topological reasoning. We start by demonstrating how reasoning in the fuzzy RCC can be reduced to reasoning in the original RCC. On one hand, this allows to use optimized RCC reasoners for reasoning in the fuzzy RCC, adopting for example the heuristic strategies introduced in [43]. On the other hand, this reduction is particularly useful to leverage theoretical properties of the RCC to corresponding properties of the fuzzy RCC. Illustrating this point, we reveal a relationship between our framework and the Egg–Yolk calculus in Section 7. Finally, in Section 8 we use this relationship with the RCC to prove that any standard, satisfiable set Θ of fuzzy RCC formulas can be interpreted using fuzzy sets and nearness in any dimension, thus generalizing the observation from Proposition 2 that Θ can be realized in one particular dimension n. In particular, this implies that we can always find models in two-dimensional and three-dimensional Euclidean space.

The cornerstone in the reduction to the RCC is the observation that a satisfiable set of fuzzy RCC formulas can always be satisfied by models taking only finitely different membership degrees. Let $\Delta \in]0, 1[$ such that $\Delta = \frac{1}{\rho}$ for a certain ρ in $\mathbb{N} \setminus \{0\}$. Furthermore let M_Δ and $M_{\frac{\Delta}{2}}$ be defined as

$$M_\Delta = \{0, \Delta, 2\Delta, \ldots, 1 - \Delta, 1\}$$

$$M_{\frac{\Delta}{2}} = \left\{0, \frac{\Delta}{2}, \Delta, \ldots, 1 - \frac{\Delta}{2}, 1\right\}$$

Henceforth, we will always assume that all the upper and lower bounds in the set of fuzzy RCC formulas Θ are taken from M_Δ. While being a theoretical restriction, this has no practical consequences as computers only deal with finite precision anyway. Below, we state that if Θ is satisfiable, it has an $(n;\alpha,0)$-model in which only membership degrees from $M_{\frac{\Delta}{2}}$ are used.

Proposition 4. *Let $T = T_W$ and let Θ be a standard, satisfiable set of fuzzy RCC formulas whose upper and lower bounds are all in M_Δ. Furthermore, let V be the set of variables occurring in Θ and $|V| = n$. There exists a model of Θ mapping every variable v occurring in Θ to a normalised, bounded fuzzy set in $\mathbb{R}^n$ which only takes membership degrees from $M_{\frac{\Delta}{2}}$.*

Proof. The proof is presented in Appendix C. □

Note that from the construction process in the proof of Proposition 1, we know that when Θ is a normalised set in which all bounds are taken from M_Δ, Θ has a model in which all membership degrees are taken from M_Δ, which is a stronger result than what is expressed in the proposition above. From Proposition 4, we have that an arbitrary set of fuzzy RCC formulas Θ can be refined to a normalised set in which all bounds come from $M_{\frac{\Delta}{2}}$, which implies that a model can be found in which all membership degrees are taken from $M_{\frac{\Delta}{2}}$. Whether or not a model can always be found involving only membership degrees from M_Δ is currently still an open problem.

As Proposition 5 below shows, when the fuzzy topological relations C_α, O_α, P_α and NTP_α are applied to fuzzy sets which only take membership degrees from a finite set, their value can be found by checking whether a finite number of crisp spatial relations are satisfied.

Proposition 5. *Let A and B be normalised, bounded fuzzy sets in $\mathbb{R}^n$ which only take membership degrees from $M_{\frac{\Delta}{2}}$, let λ be in $M_{\frac{\Delta}{2}} \setminus \{0\}$ and let λ' be in $M_{\frac{\Delta}{2}} \setminus \{1\}$. It holds that $(\alpha > 0)$*[14]

$$C_\alpha(A,B) \geqslant \lambda \Leftrightarrow C_\alpha(A_1,B_\lambda) \vee C_\alpha(A_{1-\frac{\Delta}{2}},B_{\lambda+\frac{\Delta}{2}}) \vee \cdots \vee C_\alpha(A_\lambda,B_1) \tag{70}$$

$$C_\alpha(A,B) \leqslant \lambda' \Leftrightarrow DC_\alpha(A_1,B_{\lambda'+\frac{\Delta}{2}}) \wedge DC_\alpha(A_{1-\frac{\Delta}{2}},B_{\lambda'+\Delta}) \wedge \cdots \wedge DC_\alpha(A_{\lambda'+\frac{\Delta}{2}},B_1) \tag{71}$$

$$O_\alpha(A,B) \geqslant \lambda \Leftrightarrow O_\alpha(A_1,B_\lambda) \vee O_\alpha(A_{1-\frac{\Delta}{2}},B_{\lambda+\frac{\Delta}{2}}) \vee \cdots \vee O_\alpha(A_\lambda,B_1) \tag{72}$$

$$O_\alpha(A,B) \leqslant \lambda' \Leftrightarrow DR_\alpha(A_1,B_{\lambda'+\frac{\Delta}{2}}) \wedge DR_\alpha(A_{1-\frac{\Delta}{2}},B_{\lambda'+\Delta}) \wedge \cdots \wedge DR_\alpha(A_{\lambda'+\frac{\Delta}{2}},B_1) \tag{73}$$

$$P_\alpha(A,B) \geqslant \lambda \Leftrightarrow P_\alpha(A_1,B_\lambda) \wedge P_\alpha(A_{1-\frac{\Delta}{2}},B_{\lambda-\frac{\Delta}{2}}) \wedge \cdots \wedge P_\alpha(A_{1-\lambda+\frac{\Delta}{2}},B_{\frac{\Delta}{2}}) \tag{74}$$

$$P_\alpha(A,B) \leqslant \lambda' \Leftrightarrow \neg P_\alpha(A_1,B_{\lambda'+\frac{\Delta}{2}}) \vee \neg P_\alpha(A_{1-\frac{\Delta}{2}},B_{\lambda'}) \vee \cdots \vee \neg P_\alpha(A_{1-\lambda'},B_{\frac{\Delta}{2}}) \tag{75}$$

$$NTP_\alpha(A,B) \geqslant \lambda \Leftrightarrow NTPP_\alpha(A_1,B_\lambda) \wedge NTPP_\alpha(A_{1-\frac{\Delta}{2}},B_{\lambda-\frac{\Delta}{2}}) \wedge \cdots \wedge NTPP_\alpha(A_{1-\lambda+\frac{\Delta}{2}},B_{\frac{\Delta}{2}}) \tag{76}$$

$$NTP_\alpha(A,B) \leqslant \lambda' \Leftrightarrow \neg NTPP_\alpha(A_1,B_{\lambda'+\frac{\Delta}{2}}) \vee \neg NTPP_\alpha(A_{1-\frac{\Delta}{2}},B_{\lambda'}) \vee \cdots \vee \neg NTPP_\alpha(A_{1-\lambda'},B_{\frac{\Delta}{2}}) \tag{77}$$

Proof. The proof is given in Appendix D. □

Recall from Lemma 7 that C_α, O_α, P_α and NTP_α can be used to define C-interpretations of RCC formulas. This observation suggests how we can reason about fuzzy RCC formulas by translating them first to crisp RCC formulas, thereby applying Proposition 5. In particular, we can relate the regions from the crisp RCC formulas to α-level sets of the regions from the fuzzy RCC formulas. Let Θ be a standard set of fuzzy RCC formulas, and let V be the set of variables (regions) occurring in Θ. We construct a set of RCC formulas Γ over the set of variables V', containing for each v in V, the variables $v_{\frac{\Delta}{2}}$, v_Δ, $\ldots$, v_1. In particular, we add the following RCC formulas to Γ, for each v in V

$$\{P(v_{\frac{\Delta}{2}},v_\Delta), P(v_\Delta,v_{\Delta+\frac{\Delta}{2}}),\ldots,P(v_{1-\frac{\Delta}{2}},v_1)\} \tag{78}$$

This ensures that we can always associate F–interpretations of Θ with C-interpretations of Γ and vice versa. In particular, given a C-interpretation $\mathcal{I}'$ of Γ which maps every v in V' to a non-empty, bounded region in $\mathbb{R}^n$, we can define an F-interpretation $\mathcal{I}$ of Θ as

$$v^{\mathcal{I}}(p) = \begin{cases} \max\{\lambda \mid \lambda \in M_{\frac{\Delta}{2}} \setminus \{0\} \wedge p \in v_\lambda^{\mathcal{I}'}\} & \text{if } p \in v_{\frac{\Delta}{2}}^{\mathcal{I}'} \\ 0 & \text{otherwise} \end{cases} \tag{79}$$

[14] Note that an expression such as $C_\alpha(A_1,B_\lambda)$ denotes a value in $\{0,1\}$. Slightly abusing notation, we therefore use $C_\alpha(A_1,B_\lambda)$ as a shorthand for $C_\alpha(A_1,B_\lambda)=1$, thus identifying 0 and 1 with the classical notions of false and true.

for all v in V and p in $\mathbb{R}^n$. Conversely, we can define a C-interpretation $\mathcal{I}'$ of Γ, given an F-interpretation $\mathcal{I}$ of Θ which maps every variable to a normalised, bounded fuzzy set in $\mathbb{R}^n$ taking only membership degrees from $M_{\frac{\Delta}{2}}$:

$$p \in v_\lambda^{\mathcal{I}'} \equiv p \in \{q \mid q \in \mathbb{R}^n \wedge v^{\mathcal{I}}(q) \geqslant \lambda\} \tag{80}$$

for $\lambda \in M_{\frac{\Delta}{2}} \setminus \{0\}$. In addition to (78), we also add (sets of) RCC formulas to Γ corresponding with each of the fuzzy RCC formulas in Θ. For example, if Θ contains the fuzzy RCC formula $EC(a,b) \geqslant 0.5 \vee P(c,d) \leqslant 0$, being equivalent to $(C(a,b) \geqslant 0.5 \wedge O(a,b) \leqslant 0.5) \vee P(c,d) \leqslant 0$, we obtain the following expression by Proposition 5 (assuming $\Delta = 0.5$)

$$\begin{aligned}&\big(\big(C(a_1,b_{0.5}) \vee C(a_{0.75},b_{0.75}) \vee C(a_{0.5},b_1)\big) \wedge \neg O(a_1,b_{0.75}) \wedge \neg O(a_{0.75},b_1)\big)\\&\quad \vee \neg P(c_1,d_{0.25}) \vee \neg P(c_{0.75},d_{0.5}) \vee \neg P(c_{0.5},d_{0.75}) \vee \neg P(c_{0.25},d_1)\end{aligned}$$

which corresponds to the following set of RCC formulas

$$\begin{aligned}&\{C(a_1,b_{0.5}) \vee C(a_{0.75},b_{0.75}) \vee C(a_{0.5},b_1) \vee \neg P(c_1,d_{0.25}) \vee \neg P(c_{0.75},d_{0.5}) \vee \neg P(c_{0.5},d_{0.75}) \vee \neg P(c_{0.25},d_1),\\&\ \neg O(a_1,b_{0.75}) \vee \neg P(c_1,d_{0.25}) \vee \neg P(c_{0.75},d_{0.5}) \vee \neg P(c_{0.5},d_{0.75}) \vee \neg P(c_{0.25},d_1),\\&\ \neg O(a_{0.75},b_1) \vee \neg P(c_1,d_{0.25}) \vee \neg P(c_{0.75},d_{0.5}) \vee \neg P(c_{0.5},d_{0.75}) \vee \neg P(c_{0.25},d_1)\}\end{aligned}$$

If $\mathcal{I}'$ is a C-model of Γ and $\mathcal{I}$ is the F–interpretation of Θ defined by (79), we know from Proposition 5 that $\mathcal{I}$ will be an F-model of Θ, and vice versa, if $\mathcal{I}$ is an F-model of Θ, we know that the C-interpretation $\mathcal{I}'$ of Γ, defined by (80), will be a C-model of Γ. Note that if Θ is F-satisfiable, there always exists an F-model of Θ which maps variables to normalised, bounded fuzzy sets in $\mathbb{R}^n$ (Proposition 2), and if Γ is C-satisfiable, there always exists a C-model of Γ which maps variables to non-empty, bounded sets in $\mathbb{R}^n$ [41]. We thus have the following corollary of Proposition 5.

Corollary 5. *Let Θ be a standard set of fuzzy RCC formulas whose upper and lower bounds are all in M_Δ, and let Γ be the corresponding set of (crisp) RCC formulas, obtained through the construction process outlined above. It holds that Θ is F-satisfiable iff Γ is C-satisfiable.*

7. Relationship with the Egg–Yolk calculus

Until now, we have mainly considered interpretations of fuzzy topological relations in terms of fuzzy sets and nearness, i.e., $(n;\alpha,\beta)$-interpretations. On the other hand, as our generalization of the RCC is not explicitly tied to this type of interpretations, it seems that it should also encompass other models of vague topological information. In this section, we demonstrate that this is indeed the case by revealing a relationship between our approach and the Egg–Yolk calculus, the latter being neither based on nearness nor on fuzzy sets.

In the most general form of the Egg–Yolk calculus, a vague region A is represented as k nested (crisp) sets $(A^1, A^2, \ldots, A^k)$, where A^1 contains the points that definitely belong to the vague region, and $co(A^k)$ contains the points that definitely do not ($A^1 \subseteq A^2 \subseteq \cdots \subseteq A^k$). In other words, the sets $A^1, A^2, A^3, \ldots, A^k$ are increasingly more tolerant boundaries for the vague region. Without loss of generality, we can assume that A^i is a non-empty, bounded, regular closed subset of $\mathbb{R}^n$, for some n in $\mathbb{N} \setminus \{0\}$. Note that typically $k = 2$, in which case A^1 is called the yolk and A^2 is called the white of vague region A. We will refer to k nested sets that represent a vague region, as an Egg–Yolk region. Egg–Yolk relations, i.e., topological relations between two Egg–Yolk regions A and B, are defined by expressing which are the possible RCC relations that may hold between the corresponding nested sets $(A^1, A^2, \ldots, A^k)$ and $(B^1, B^2, \ldots, B^k)$. For example, to express that A is a part of B, we may require

$$P(A^1,B^1) \wedge P(A^2,B^2) \wedge \cdots \wedge P(A^k,B^k) \tag{81}$$

or, adopting a stricter interpretation, that $P(A^k, B^1)$. On the other hand, we may also define fuzzy topological relations between Egg–Yolk regions, imposing, for instance, that A and B are connected to degree 1 if A^1 and B^1 are connected, and to some lower degree if A^1 and B^1 are not connected but A^i and B^j are connected for some i and j in $\{1,2,\ldots,k\}$.

The k nested sets $(A^1, A^2, \ldots, A^k)$ can naturally be regarded as the α-level sets of a fuzzy set A, i.e., $A^1 = A_1$, $A^2 = A_{\frac{k-1}{k}}$, …, $A^k = A_{\frac{1}{k}}$. Thus, given an $(n;\alpha,0)$-interpretation $\mathcal{I}_0$ of a set of fuzzy RCC formulas Θ taking only membership degrees from $M_{\frac{1}{k}}$, we can straightforwardly construct a new interpretation $\mathcal{I}$ in which each variable v is interpreted as an Egg–Yolk region $v^{\mathcal{I}} = (v^1, \ldots, v^k)$, where

$$v^i = (v^{\mathcal{I}_0})_{\frac{k-i+1}{k}} \tag{82}$$

Moreover, if we define $C^{\mathcal{I}}$ such that (assuming $\max\emptyset = 0$)

$$\begin{aligned}C^{\mathcal{I}}(u^{\mathcal{I}}, v^{\mathcal{I}}) &= C^{\mathcal{I}}((u^1,\ldots,u^k),(v^1,\ldots,v^k))\\&= \max\left\{\frac{k+1-i}{k} \,\middle|\, i \in \{1,\ldots,k\} \wedge \big(C_\alpha(u^1,v^i) \vee C_\alpha(u^2,v^{i-1}) \vee \cdots \vee C_\alpha(u^i,v^1)\big)\right\}\end{aligned} \tag{83}$$

it holds by Proposition 5 that for all regions u and v

$$C^{\mathcal{I}_0}(u^{\mathcal{I}_0}, v^{\mathcal{I}_0}) = C^{\mathcal{I}}(u^{\mathcal{I}}, v^{\mathcal{I}}) \tag{84}$$

Indeed, since $u^{\mathcal{I}_0}$ and $v^{\mathcal{I}_0}$ are, by assumption, fuzzy sets taking only membership degrees from $M_{\frac{1}{k}}$, we know that $C^{\mathcal{I}_0}(u^{\mathcal{I}_0}, v^{\mathcal{I}_0}) = C_\alpha(u^{\mathcal{I}_0}, v^{\mathcal{I}_0}) \in M_{\frac{1}{k}}$. Assuming $C^{\mathcal{I}_0}(u^{\mathcal{I}_0}, v^{\mathcal{I}_0}) > 0$, this means that the value of $C^{\mathcal{I}_0}(u^{\mathcal{I}_0}, v^{\mathcal{I}_0})$ is the largest element λ from $M_{\frac{1}{k}}$ such that $C^{\mathcal{I}_0}(u^{\mathcal{I}_0}, v^{\mathcal{I}_0}) \geqslant \lambda$ holds, or equivalently by Proposition 5, such that

$$C_\alpha\big((u^{\mathcal{I}_0})_1, (v^{\mathcal{I}_0})_\lambda\big) \vee C_\alpha\big((u^{\mathcal{I}_0})_{1-\frac{1}{k}}, (v^{\mathcal{I}_0})_{\lambda+\frac{1}{k}}\big) \vee \cdots \vee C_\alpha\big((u^{\mathcal{I}_0})_\lambda, (v^{\mathcal{I}_0})_1\big)$$

For $\lambda \in M_{\frac{1}{k}}$, it holds that $\lambda = \frac{k-i+1}{k}$ for some i in $\{1, \ldots, k\}$. Thus we find that $C^{\mathcal{I}_0}(u^{\mathcal{I}_0}, v^{\mathcal{I}_0}) = \frac{k-i+1}{k}$ where i is the smallest element in $\{1, \ldots, k\}$ satisfying

$$C_\alpha\big((u^{\mathcal{I}_0})_1, (v^{\mathcal{I}_0})_{\frac{k-i+1}{k}}\big) \vee C_\alpha\big((u^{\mathcal{I}_0})_{1-\frac{1}{k}}, (v^{\mathcal{I}_0})_{\frac{k-i+2}{k}}\big) \vee \cdots \vee C_\alpha\big((u^{\mathcal{I}_0})_{\frac{k-i+1}{k}}, (v^{\mathcal{I}_0})_1\big)$$

By the correspondence (82), we therefore have that $C^{\mathcal{I}_0}(u^{\mathcal{I}_0}, v^{\mathcal{I}_0}) = \frac{k-i+1}{k}$, where i is the smallest value in $\{1, \ldots, k\}$ such that

$$C_\alpha(u^1, v^i) \vee C_\alpha(u^2, v^{i-1}) \vee \cdots \vee C_\alpha(u^i, v^1)$$

in other words (84). If $C^{\mathcal{I}_0}(u^{\mathcal{I}_0}, v^{\mathcal{I}_0}) = 0$, then $DC_\alpha(u^1, v^i), DC_\alpha(u^2, v^{i-1}), \ldots, DC_\alpha(u^i, v^1)$ will hold for all i in $\{1, \ldots, k\}$, and we therefore find (84) again.

Note that (84) entails that $\mathcal{I}_0$ is an F-model of Θ iff $\mathcal{I}$ is an F-model of Θ, and, moreover, that

$$P^{\mathcal{I}_0}(u^{\mathcal{I}_0}, v^{\mathcal{I}_0}) = P^{\mathcal{I}}(u^{\mathcal{I}}, v^{\mathcal{I}})$$
$$O^{\mathcal{I}_0}(u^{\mathcal{I}_0}, v^{\mathcal{I}_0}) = O^{\mathcal{I}}(u^{\mathcal{I}}, v^{\mathcal{I}})$$
$$NTP^{\mathcal{I}_0}(u^{\mathcal{I}_0}, v^{\mathcal{I}_0}) = NTP^{\mathcal{I}}(u^{\mathcal{I}}, v^{\mathcal{I}})$$

Following a similar line of reasoning as for $C^{\mathcal{I}}$, we thus obtain from Proposition 5 that (assuming $\max \emptyset = 0$)

$$\begin{aligned}
P^{\mathcal{I}}(u^{\mathcal{I}}, v^{\mathcal{I}}) &= P^{\mathcal{I}}\big((u^1, \ldots, u^k), (v^1, \ldots, v^k)\big) \\
&= \max\left\{\frac{k+1-i}{k} \;\Big|\; i \in \{1, \ldots, k\} \wedge P_\alpha(u^1, v^i) \wedge P_\alpha(u^2, v^{i+1}) \wedge \cdots \wedge P_\alpha(u^{k+1-i}, v^k)\right\} \\
O^{\mathcal{I}}(u^{\mathcal{I}}, v^{\mathcal{I}}) &= O^{\mathcal{I}}\big((u^1, \ldots, u^k), (v^1, \ldots, v^k)\big) \\
&= \max\left\{\frac{k+1-i}{k} \;\Big|\; i \in \{1, \ldots, k\} \wedge \big(O_\alpha(u^1, v^i) \vee O_\alpha(u^2, v^{i-1}) \vee \cdots \vee O_\alpha(u^i, v^1)\big)\right\} \\
NTP^{\mathcal{I}}(u^{\mathcal{I}}, v^{\mathcal{I}}) &= NTP^{\mathcal{I}}\big((u^1, \ldots, u^k), (v^1, \ldots, v^k)\big) \\
&= \max\left\{\frac{k+1-i}{k} \;\Big|\; i \in \{1, \ldots, k\} \wedge NTPP_\alpha(u^1, v^i) \wedge \cdots \wedge NTPP_\alpha(u^{k+1-i}, v^k)\right\}
\end{aligned}$$

Although $\mathcal{I}$ maps variables to Egg–Yolk regions, the fuzzy topological relations between them are still defined using nearness, in contrast to typical Egg–Yolk interpretations, where spatial relations are defined in terms of standard RCC relations such as C^n, O^n, P^n and NTP^n. Recall, for example, that for A and B two subsets of $\mathbb{R}^n$, $C^n(A, B) \equiv A \cap B \neq \emptyset$.

Definition 10. Let Θ be a set of fuzzy RCC formulas and let V be the set of regions used. An F-interpretation $\mathcal{I}$ of Θ is called an Egg–Yolk interpretation of Θ w.r.t. (k, n) if it maps every v in V to an Egg–Yolk region, and C is interpreted for Egg–Yolk regions $(u^1, \ldots, u^k)$ and $(v^1, \ldots, v^k)$ by $(k, n \in \mathbb{N} \setminus \{0\}, \max \emptyset = 0)$:

$$\begin{aligned}
&C^{\mathcal{I}}\big((u^1, \ldots, u^k), (v^1, \ldots, v^k)\big) \\
&\quad = \max\left\{\frac{k+1-i}{k} \;\Big|\; i \in \{1, \ldots, k\} \wedge \big(C^n(u^1, v^i) \vee C^n(u^2, v^{i-1}) \vee \cdots \vee C^n(u^i, v^1)\big)\right\}
\end{aligned} \tag{85}$$

An Egg–Yolk interpretation which is also a model of Θ is called an Egg–Yolk model of Θ.

Let $\mathcal{I}$ be defined as in (83). If $\mathcal{I}$ is an F-model of a standard set of fuzzy RCC formulas Θ, then Θ must also have an Egg–Yolk model $\mathcal{I}'$. Indeed, identifying Egg–Yolk regions with their corresponding fuzzy sets, we can use the construction process from Corollary 5 to obtain a set of RCC formulas Γ which is C-satisfiable iff Θ is F-satisfiable. The variables occurring in Γ correspond to the different nested sets of the Egg–Yolk regions (or equivalently, to the α-level sets of the corresponding fuzzy sets), and $\mathcal{I}$ naturally corresponds to an $(n; \alpha, 0)$-model of Γ. Specifically, the variables occurring in Γ are mapped to the crisp regions $u^1, u^2, \ldots, u^k$, where $(u^1, \ldots, u^k)$ is the Egg–Yolk region corresponding to the interpretation of region u under $\mathcal{I}$, i.e., $u^{\mathcal{I}} = (u^1, \ldots, u^k)$. From Corollary 4, we know that Γ must also have a standard model. In this standard model, the variables from Γ are mapped to crisp regions, which can again be interpreted as the nested sets of Egg–Yolk regions. Specifically, this leads to an Egg–Yolk interpretation $\mathcal{I}'$ where $u^{\mathcal{I}'} = (u'^1, \ldots, u'^k)$, i.e., the variable from Γ which was initially mapped to crisp region u^i is in the new interpretation mapped to another crisp region u'^i. Because $\mathcal{I}'$ moreover corresponds to a standard model of Γ, we have that $C^n(u'^i, v'^j) \equiv C_\alpha(u^i, v^j)$ for all regions u and v occurring in Θ and all i and j in $\{1, 2, \ldots, k\}$. In other words, interpreting connection as in (85), $\mathcal{I}'$ corresponds to an Egg–Yolk model of Θ. Together with Proposition 4, this leads to the following proposition.

Proposition 6. *Let Θ be an F-satisfiable standard set of fuzzy RCC formulas in which all upper and lower bounds are taken from $\{0, \frac{2}{k}, \frac{4}{k}, \ldots, 1\}$ for some $k \in \mathbb{N} \setminus \{0\}$ ($T = T_W$). It holds that Θ has an Egg–Yolk model w.r.t. (k, n) for every n in $\mathbb{N} \setminus \{0\}$.*

Note that because of the construction process above, it holds that

$$C_\alpha\big(u^i, v^j\big) \equiv C^n\big(u'^i, v'^j\big)$$
$$P_\alpha\big(u^i, v^j\big) \equiv P^n\big(u'^i, v'^j\big)$$
$$O_\alpha\big(u^i, v^j\big) \equiv O^n\big(u'^i, v'^j\big)$$
$$NTP_\alpha\big(u^i, v^j\big) \equiv NTP^n\big(u'^i, v'^j\big)$$

for all variables u and v occurring in Θ, and all i and j in $\{1, 2, \ldots, k\}$. Since, moreover, $C^{\mathcal{I}}(u^{\mathcal{I}}, v^{\mathcal{I}}) = C^{\mathcal{I}'}(u^{\mathcal{I}'}, v^{\mathcal{I}'})$ for all regions u and v, it holds that ($\max\emptyset = 0$)

$$\begin{aligned}
&P^{\mathcal{I}'}\big(\big(u'^1, \ldots, u'^k\big), \big(v'^1, \ldots, v'^k\big)\big)\\
&\quad = \max\left\{\frac{k+1-i}{k} \;\middle|\; i \in \{1, \ldots, k\} \wedge P^n\big(u'^1, v'^i\big) \wedge P^n\big(u'^2, v'^{i+1}\big) \wedge \cdots \wedge P^n\big(u'^{k+1-i}, v'^k\big)\right\}\\
&O^{\mathcal{I}'}\big(\big(u'^1, \ldots, u^k\big), \big(v'^1, \ldots, v'^k\big)\big)\\
&\quad = \max\left\{\frac{k+1-i}{k} \;\middle|\; i \in \{1, \ldots, k\} \wedge \big(O^n\big(u'^1, v'^i\big) \vee O^n\big(u'^2, v'^{i-1}\big) \vee \cdots \vee O^n\big(u'^i, v'^1\big)\big)\right\}\\
&NTP^{\mathcal{I}'}\big(\big(u'^1, \ldots, u'^k\big), \big(v'^1, \ldots, v'^k\big)\big)\\
&\quad = \max\left\{\frac{k+1-i}{k} \;\middle|\; i \in \{1, \ldots, k\} \wedge NTPP^n\big(u'^1, v'^i\big) \wedge \cdots \wedge NTPP^n\big(u'^{k+1-i}, v'^k\big)\right\}
\end{aligned}$$

Note that $P^{\mathcal{I}'}(u^{\mathcal{I}'}, v^{\mathcal{I}'}) = 1$ corresponds to the notion of containment between Egg–Yolk regions expressed in (81). For $i < k$, $P^{\mathcal{I}'}(u^{\mathcal{I}'}, v^{\mathcal{I}'}) = \frac{i}{k}$ corresponds to a more flexible notion of containment. In other words, a fuzzy topological relation R corresponds to a list of Egg–Yolk relations $(R^1, R^2, \ldots, R^k)$, where $R^i(u, v)$ corresponds to $R(u, v) \geqslant \frac{k+1-i}{k}$. For example, if $k = 2$, $O(u, v) = 1$ means that the yolk of u overlaps with the yolk of v, while $O(u, v) \geqslant 0.5$ means that either the yolk of u overlaps with the white of v, or the white of u overlaps with the yolk of v. Similarly, $P(u, v) = 1$ means that the yolk of u is contained in the yolk of v, and the white of u is contained in the white of v, whereas $P(u, v) \geqslant 0.5$ means that the yolk of u is contained in the white of v. We conclude this section with an example illustrating the relationship between $(n; \alpha, 0)$-interpretations and Egg–Yolk interpretations.

Example 3. Let Θ be the normalised set of fuzzy RCC formulas, defined by the following four matrices:

$$C = \begin{array}{c c} & \begin{array}{cc} a & b \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{pmatrix} 1 & 0.75 \\ 0.75 & 1 \end{pmatrix} \end{array} \qquad P = \begin{array}{c c} & \begin{array}{cc} a & b \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{pmatrix} 1 & 0 \\ 0.5 & 1 \end{pmatrix} \end{array}$$

$$O = \begin{array}{c c} & \begin{array}{cc} a & b \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix} \end{array} \qquad NTP = \begin{array}{c c} & \begin{array}{cc} a & b \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{pmatrix} 1 & 0 \\ 0.25 & 1 \end{pmatrix} \end{array}$$

This set Θ is satisfied by the $(1; 20, 0)$-interpretation $\mathcal{I}$ which maps the variables a and b to the fuzzy sets A and B in $\mathbb{R}$ from Fig. 5(a), i.e., $a^{\mathcal{I}} = A$ and $b^{\mathcal{I}} = B$. Since Θ is normalised and contains only upper and lower bounds from $M_{0.25} = \{0, 0.25, 0.5, 0.75, 1\}$, there also exists an $(n; \alpha, 0)$-model which only uses membership degrees from $M_{0.25}$. Such a model is shown in Fig. 5(b) for $n = 1$ and $\alpha = 20$. Furthermore, this model is closely related to a model $\mathcal{I}'$ in which regions are interpreted as Egg–Yolk regions and $C^{\mathcal{I}'}$ is given by (83):

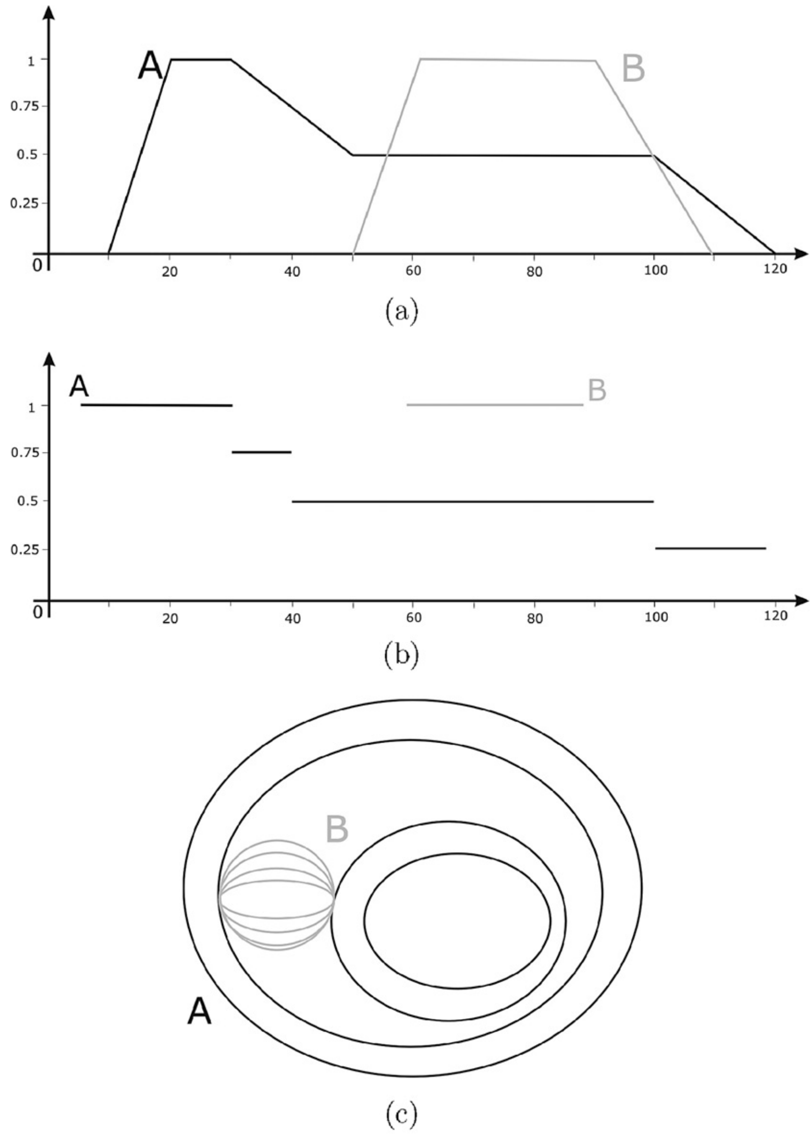

Fig. 5. An $(n; \alpha, 0)$-model can naturally be linked to an Egg–Yolk model.

$$a^{\mathcal{I}'} = \big([5, 30], [5, 40], [5, 100], [5, 120]\big)$$

$$b^{\mathcal{I}'} = \big([60, 90], [60, 90], [60, 90], [60, 90]\big)$$

Note that under interpretation $\mathcal{I}'$, for example, $[5, 40]$ and $[60, 90]$ are connected, because $d(40, 60) = |60 - 40| \leqslant 20$. In other words, because connection is interpreted in terms of nearness, the intervals need not actually have a point in common to be connected.

Finally, it is also possible to construct an Egg–Yolk model $\mathcal{I}''$ in which C is interpreted as in (85); for example

$$a^{\mathcal{I}''} = \big([100, 120], [90, 130], [60, 140], [50, 150]\big)$$

$$b^{\mathcal{I}''} = \big([60, 90], [60, 90], [60, 90], [60, 90]\big)$$

As Egg–Yolk models are not interpreted in terms of nearness, two intervals need to have a point in common to be connected, e.g., $[60, 90]$ and $[90, 130]$. Note that Proposition 6 guarantees the existence of an Egg–Yolk model in any dimension. In Fig. 5(c), for instance, a model is depicted in $\mathbb{R}^2$.

8. Realizability in any dimension

From the previous discussion, we already know that standard, satisfiable sets of fuzzy RCC formulas Θ have an Egg–Yolk model in any dimension (provided the upper and lower bounds are finitely representable), and an $(n;\alpha,0)$-model in at least one particular dimension n. In this section, we prove that an $(m;\alpha,0)$-model of Θ can be found in any dimension m as well. Specifically, we show that an $(1;\alpha,0)$-model can always be found, and, subsequently, that this $(1;\alpha,0)$-model can be converted into an $(m;\alpha,0)$-model for any m in $\mathbb{N}\setminus\{0\}$.

Proposition 7. *Let Θ be an F-satisfiable standard set of fuzzy RCC formulas whose upper and lower bounds are finitely representable. It holds that Θ has an $(1;\alpha,0)$-model for some $\alpha>0$.*

Proof. The proof is presented in Appendix E. □

Note that the $(1;\alpha,0)$-model $\mathcal{I}'$ constructed in the proof above maps every variable v to a fuzzy set taking only membership degrees from $M_{\frac{1}{k}}$. This fuzzy set is characterized by the k corresponding α-level sets which are all the union of a finite number of closed, non-degenerate intervals. In particular, let the $\frac{i}{k}$-level set of the fuzzy set $v^{\mathcal{I}'}$ be given by

$$\left[v_{i1}^-,v_{i1}^+\right]\cup\left[v_{i2}^-,v_{i2}^+\right]\cup\left[v_{in_i}^-,v_{in_i}^+\right]$$

We now define a new interpretation $\mathcal{I}''$ in which v is mapped to the fuzzy set taking only membership degrees from $M_{\frac{1}{k}}$, whose $\frac{i}{k}$-level set is given by

$$\left[\frac{v_{i1}^-\alpha'}{\alpha},\frac{v_{i1}^+\alpha'}{\alpha}\right]\cup\left[\frac{v_{i2}^-\alpha'}{\alpha},\frac{v_{i2}^+\alpha'}{\alpha}\right]\cup\left[\frac{v_{in_i}^-\alpha'}{\alpha},\frac{v_{in_i}^+\alpha'}{\alpha}\right]$$

Note that for p and q in $\mathbb{R}$, it holds that $d(\frac{p\alpha'}{\alpha},\frac{q\alpha'}{\alpha})=|\frac{p\alpha'}{\alpha}-\frac{q\alpha'}{\alpha}|=\frac{\alpha'}{\alpha}|p-q|=\frac{\alpha'}{\alpha}d(p,q)$. Therefore, for all regions u and v, $C_\alpha(u^{\mathcal{I}'},v^{\mathcal{I}'})$ implies $C_{\alpha'}(u^{\mathcal{I}''},v^{\mathcal{I}''})$. Hence, as $\mathcal{I}'$ is an $(1;\alpha,0)$-model of Θ, it holds that $\mathcal{I}''$ is an $(1;\alpha',0)$-model of Θ. We therefore have the following corollary.

Corollary 6. *Let Θ be an F-satisfiable standard set of fuzzy RCC formulas whose upper and lower bounds are finitely representable. It holds that Θ has an $(1;\alpha,0)$-model for every $\alpha>0$.*

Intuitively, it seems obvious that when a set of fuzzy RCC formulas can be realized in one dimension, it can be realized in all higher dimensions as well. The next proposition reveals that this is indeed the case. The key observation is that the role of closed, non-degenerate intervals in $\mathbb{R}$ can be generalized by m-dimensional hypercubes. In particular, given an interval $[a,b]$ $(a<b)$, we define the associated hypercube as the hypercube with center $(\frac{a+b}{2},0,0,\ldots,0)$ whose edges have length $b-a$ and are all parallel to one of the axes. As an example, the hypercubes H and G corresponding to the intervals $[80,120]$ and $[80,200]$ are depicted in Fig. 6 for $m=3$.

Proposition 8. *Let Θ be an F-satisfiable standard set of fuzzy RCC formulas whose upper and lower bounds are finitely representable. It holds that Θ has an $(m;\alpha,0)$-model for every $\alpha>0$ and every m in $\mathbb{N}\setminus\{0\}$.*

Proof. The proof is presented in Appendix F. □

Let Θ be a finite, standard set of fuzzy RCC formulas whose upper and lower bounds are finitely representable. We have established that the following statements are all equivalent (Γ being the set of crisp RCC formulas from Corollary 5).

(1) Θ is F-satisfiable;
(2) Γ is C-satisfiable;
(3) Θ can be refined to a normalised set of fuzzy RCC formulas satisfying (16)–(19), as well as the transitivity rules for fuzzy topological relations;
(4) Θ has an Egg–Yolk model in at least one dimension;
(5) Θ has an Egg–Yolk model in any dimension;
(6) Θ has an $(m;\alpha,0)$-model for at least one $\alpha>0$ and one m in $\mathbb{N}\setminus\{0\}$;
(7) Θ has an $(m;\alpha,0)$-model for any $\alpha>0$ and any m in $\mathbb{N}\setminus\{0\}$.

9. Conclusions

In this paper, we have discussed reasoning tasks in a fuzzy region connection calculus. One of our most important results is that satisfiability checking in this fuzzy RCC essentially corresponds to verifying that no transitivity, reflexivity

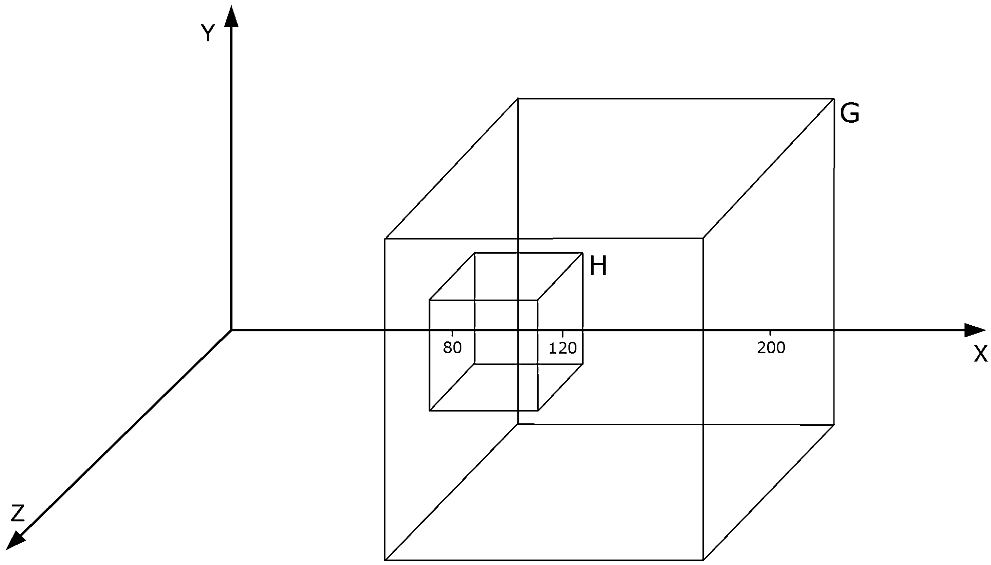

Fig. 6. Hypercubes H and G corresponding to the intervals $[80, 120]$ and $[80, 200]$.

and symmetry rules are violated (Proposition 2). We furthermore proved that any satisfiable knowledge base of fuzzy RCC relations can be realized by interpreting regions as fuzzy sets in $\mathbb{R}^n$, for an arbitrary n in $\mathbb{N} \setminus \{0\}$, and by interpreting connection in terms of nearness. For practical reasoning, we showed how satisfiability checking can be reduced to solving systems of linear inequalities and backtracking. Each system of linear inequalities can be solved in polynomial time using a linear programming solver, leading to an overall NP-complete time complexity. Related reasoning tasks such as entailment checking and inconsistency repairing can moreover be solved in a similar way. Thus we find that adding vagueness does not alter the computational complexity of topological reasoning. Next, we proved how reasoning in the fuzzy RCC can be reduced to reasoning in the original RCC. The cornerstone in this reduction was the observation that an interpretation is guaranteed to exist which only involves fuzzy sets taking a finite number of different membership degrees (Proposition 4). By this reduction, important theoretical results from the RCC can be leveraged to our fuzzy RCC. As an important example, it allowed us to show the aforementioned result of realizability in any dimension n. The reduction to the RCC furthermore allowed us to establish a close relationship with the Egg–Yolk calculus for topological reasoning under vagueness. Among others, this led to an alternative interpretation of connection as a fuzzy relation, supporting our claim of generality.

Appendix A. Proof of Proposition 1 (sketch)

Here we only provide a sketch; the complete details of the proof can be found in [47]. In the following, we assume $\alpha > 0$ and $\alpha_0 > 0$, where α_0 is a sufficiently large constant ($\alpha \ll \alpha_0$). Moreover, let Θ be a normalised set of RCC formulas over a set of variables V satisfying requirements (16)–(19) and the transitivity rules for fuzzy topological relations. For every pair of variables (a, b) in V^2, the values of $C^{\mathcal{I}}(a^{\mathcal{I}}, a^{\mathcal{I}})$, $P^{\mathcal{I}}(a^{\mathcal{I}}, a^{\mathcal{I}})$, $O^{\mathcal{I}}(a^{\mathcal{I}}, a^{\mathcal{I}})$ and $NTP^{\mathcal{I}}(a^{\mathcal{I}}, a^{\mathcal{I}})$ are then identical in any model $\mathcal{I}$ of Θ. Below, we will refer to these values as λ^C_{ab}, λ^P_{ab}, λ^O_{ab} and λ^{NTP}_{ab}. In other words, we assume that for every (a, b) in V^2, Θ contains the fuzzy RCC formulas

$$\begin{array}{llll} C(a,b) \geqslant \lambda^C_{ab} & C(a,b) \leqslant \lambda^C_{ab} & P(a,b) \geqslant \lambda^P_{ab} & P(a,b) \leqslant \lambda^P_{ab} \\ O(a,b) \geqslant \lambda^O_{ab} & O(a,b) \leqslant \lambda^O_{ab} & NTP(a,b) \geqslant \lambda^{NTP}_{ab} & NTP(a,b) \leqslant \lambda^{NTP}_{ab} \end{array}$$

where, by assumption, $\lambda^{NTP}_{aa} < 1$ for every a in V. We will prove that there exists a mapping f from V to normalised, bounded fuzzy sets in $\mathbb{R}^n$ such that

$$C_\alpha\big(f(a), f(b)\big) = \lambda^C_{ab} \tag{A.1}$$

$$O_\alpha\big(f(a), f(b)\big) = \lambda^O_{ab} \tag{A.2}$$

$$P_\alpha\big(f(a), f(b)\big) = \lambda^P_{ab} \tag{A.3}$$

$$NTP_\alpha\big(f(a), f(b)\big) = \lambda^{NTP}_{ab} \tag{A.4}$$

for all (a, b) in V^2, in other words, that f defines an $(n; \alpha, 0)$-interpretation which satisfies Θ. In particular, we choose $n = |V|$, i.e., the dimension of the Euclidean space is the same as the number of different variables in Θ. Furthermore, we can always define a total ordering on the elements of V; we write $V = \{v_1, v_2, \ldots, v_n\}$. For the ease of presentation, we will write λ_{ij}^C instead of $\lambda_{v_i v_j}^C$, λ_{ij}^O instead of $\lambda_{v_i v_j}^O$, etc. Rather than constructing the function f at once, we first construct a function f^C from V to the class of fuzzy sets in $\mathbb{R}^n$, satisfying (A.1) but not necessarily (A.2)–(A.4). Subsequently, we will define a function f^O satisfying (A.1) and (A.2), but not necessarily (A.3) and (A.4), etc.

For each point a in $\mathbb{R}^n$, we let P_a and L_a be the fuzzy sets in $\mathbb{R}^n$ defined as

$$P_a(p) = \begin{cases} 1 & \text{if } R(a,p) \geqslant \alpha_0 \\ 0 & \text{otherwise} \end{cases}$$

$$L_a(p) = \begin{cases} 1 & \text{if } R(a,p) \geqslant \alpha \\ 0 & \text{otherwise} \end{cases}$$

for all p in $\mathbb{R}^n$. Note that P_a and L_a are n-dimensional spheres in $\mathbb{R}^n$ with center a and radius α_0 and α respectively. Moreover, for λ in $[0,1]$, we let L_a^λ and P_a^λ be the fuzzy sets in $\mathbb{R}^n$ defined as

$$L_a^\lambda(p) = T\big(\lambda, L_a(p)\big)$$

$$P_a^\lambda(p) = T\big(\lambda, P_a(p)\big)$$

for all p in $\mathbb{R}^n$. Let ρ be a sufficiently large, positive real number ($\alpha_0 \ll \rho$) and let f^I be the function from V to normalised, bounded fuzzy sets in $\mathbb{R}^n$, defined for each v_i in V by

$$f^I(v_i) = P_{a_i}$$

where a_i is the point in $\mathbb{R}^n$ whose coordinates are all 0, except for the ith coordinate which is ρ. For example, $f^I(v_1) = (\rho, 0, 0, \dots, 0)$. Note that in this way, an n-dimensional sphere with radius α_0 is considered. As ρ is sufficiently large, each sphere $f^I(v_i)$ is far apart from all the other spheres $f^I(v_j)$ with $j \neq i$. To achieve the required degree of connectedness, the regions corresponding to variables v_i and v_j should be connected to degree λ_{ij}^C (see (A.1)). We therefore enlarge region $f^I(v_i)$ by adding fuzzy sets in $\mathbb{R}^n$ that are sufficiently close to the centers of $f^I(v_j)$ with $j \neq i$. Specifically, from f^I, we define the function f^C for each v_i in V and p in $\mathbb{R}^n$ as

$$f^C(v_i)(p) = \max\Big(f^I(v_i)(p), \max_{\substack{1 \leqslant j \leqslant n \\ j \neq i}} T\big(\lambda_{ij}^C, L_{b_{ij}}(p)\big)\Big) \tag{A.5}$$

where $b_{ij} = a_j + (2\alpha + \alpha_0)\frac{\overrightarrow{a_j a_i}}{\|\overrightarrow{a_j a_i}\|}$. The construction of the points a_i and b_{ij} for the case $n = 3$ is depicted in Fig. A.1. In particular, b_{ij} is on the line from a_i to a_j at a distance of $2\alpha + \alpha_0$ from a_j. Hence, the sphere $L_{b_{ij}}$ with radius α has a center that is relatively close to a_j. We can show that [47]

$$C_\alpha\big(f^C(v_i), f^C(v_j)\big) = \lambda_{ij}^C$$

for all v_i and v_j in V.

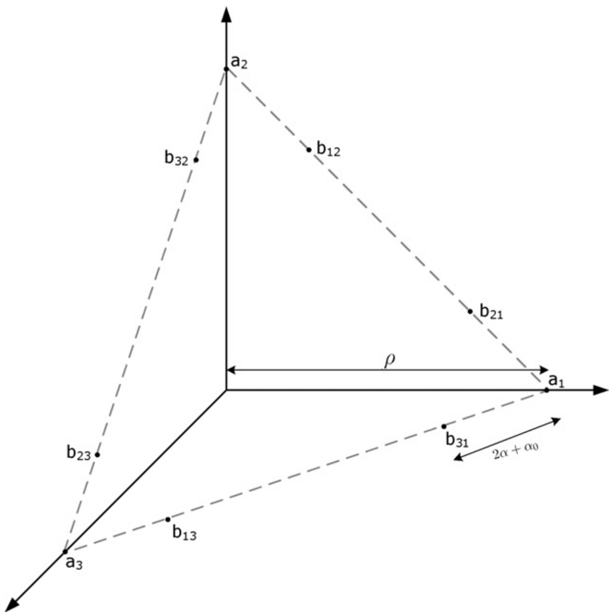

Fig. A.1. Construction of the points a_i and b_{ij} ($1 \leqslant i, j \leqslant 3$, $i \neq j$).

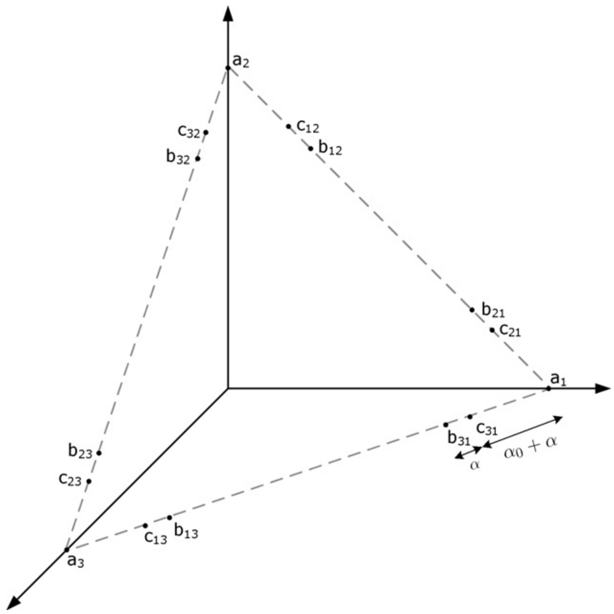

Fig. A.2. Construction of the points a_i, b_{ij} and c_{ij} ($1 \leqslant i, j \leqslant 3$, $i \neq j$).

Next, we define a function f^O satisfying both (A.1) and (A.2) in a similar way. Specifically, for each v_i in V and p in $\mathbb{R}^n$, we define

$$f^O(v_i)(p) = \max\Big(f^C(v_i)(p), \max_{\substack{1\leqslant j\leqslant n\\ j\neq i}} T\big(\lambda_{ij}^O, L_{c_{ij}}(p)\big)\Big) \tag{A.6}$$

where $c_{ij} = a_j + (\alpha_0 + \alpha)\frac{\overrightarrow{a_j a_i}}{\|\overrightarrow{a_j a_i}\|}$. The construction of the points c_{ij} for the case $n = 3$ is depicted in Fig. A.2. We can show that [47]

$$C_\alpha\big(f^O(v_i), f^O(v_j)\big) = \lambda_{ij}^C$$
$$O_\alpha\big(f^O(v_i), f^O(v_j)\big) = \lambda_{ij}^O$$

for all v_i and v_j in V.

We define the function f^P for each v_i in V and p in $\mathbb{R}^n$ as

$$f^P(v_i)(p) = \max_{l=1}^{n} T\big(\lambda_{li}^P, f^O(v_l)(p)\big) \tag{A.7}$$

In other words, the points of the fuzzy set $f^O(v_l)$ are added to the new interpretation $f^P(v_i)$ of v_i to a given extent which depends on λ_{li}^P, i.e., the degree to which v_l should be a part of v_i. Note that $f^P(v_i) \supseteq f^O(v_i)$ for all v_i in V. Indeed, from the fact that Θ satisfies (19), we know that $\lambda_{ii}^P = 1$ for all i in $\{1, 2, \ldots, n\}$. We can show that [47]

$$C_\alpha\big(f^P(v_i), f^P(v_j)\big) = \lambda_{ij}^C$$
$$O_\alpha\big(f^P(v_i), f^P(v_j)\big) = \lambda_{ij}^O$$
$$P_\alpha\big(f^P(v_i), f^P(v_j)\big) = \lambda_{ij}^P$$

for all v_i and v_j in V.

Finally, for m in $\mathbb{N} \setminus \{0\}$, the function f^m is defined as

$$f^m(v_i)(p) = \max\Big(f^{m-1}(v_i)(p), \max_{l=1}^{n} T\big(\lambda_{li}^{NTP}, \big(R_\alpha \uparrow f^{m-1}(v_l)\big)(p)\big)\Big)$$

for all p in $\mathbb{R}^n$ and v_i in V. Note that to obtain $f^m(v_i)(p)$, not only points are added from $f^{m-1}(v_l)$ (to some extent), but also points which are within distance α of the support of $f^{m-1}(v_l)$.

Furthermore, we define $f^0 = f^P$. We can show that there exists an m_0 in $\mathbb{N}$ such that $f^{m_0} = f^l$ for every $l \geqslant m_0$. For this particular m_0, we can moreover show that [47]

$$C_\alpha\big(f^{m_0}(v_i), f^{m_0}(v_j)\big) = \lambda_{ij}^C$$
$$O_\alpha\big(f^{m_0}(v_i), f^{m_0}(v_j)\big) = \lambda_{ij}^O$$
$$P_\alpha\big(f^{m_0}(v_i), f^{m_0}(v_j)\big) = \lambda_{ij}^P$$
$$NTP_\alpha\big(f^{m_0}(v_i), f^{m_0}(v_j)\big) = \lambda_{ij}^{NTP}$$

for all v_i and v_j in V. In other words, we have constructed an $(n;\alpha,0)$-interpretation of Θ.

Appendix B. Proof of Proposition 2

Because only standard sets are considered, occurrences in Θ_1 of formulas involving *NTP* have been introduced as a refinement of formulas involving *NTPP* or *TPP* from Θ_0. We use this observation to generate an alternative refinement Θ of Θ_0 in which formulas of the form $NTP(u,v) \geqslant 1$ are replaced by formulas of the form $NTP(u,v) \geqslant 1-\delta$ for a sufficiently small δ. Subsequently, we show that this refinement Θ satisfies all the conditions of Proposition 1, provided that Θ_1 satisfies (16)–(19) and (23)–(53).

Let $V = \{v_1, v_2, \ldots, v_k\}$ be the set of variables occurring in Θ_0 and let Θ_1 be defined for every v_i and v_j in V by

$$\begin{array}{ll} C(v_i,v_j) \geqslant \lambda_{ij}^C & C(v_i,v_j) \leqslant \lambda_{ij}^C \\ P(v_i,v_j) \geqslant \lambda_{ij}^P & P(v_i,v_j) \leqslant \lambda_{ij}^P \\ O(v_i,v_j) \geqslant \lambda_{ij}^O & O(v_i,v_j) \leqslant \lambda_{ij}^O \\ NTP(v_i,v_j) \geqslant \lambda_{ij}^{NTP} & NTP(v_i,v_j) \leqslant \lambda_{ij}^{NTP} \end{array}$$

for some λ_{ij}^C, λ_{ij}^P, λ_{ij}^O and λ_{ij}^{NTP} in $[0,1]$. Let M be the set of all these membership degrees, i.e.

$$M = \bigcup_{1\leqslant i,j\leqslant k} \big\{\lambda_{ij}^C, \lambda_{ij}^P, \lambda_{ij}^O, \lambda_{ij}^{NTP}\big\}$$

and let $\delta > 0$ be such that

$$\delta < \min\big\{\min\big(T_W(\lambda_1,\lambda_2), 1 - T_W(\lambda_1,\lambda_2)\big) \;\big|\; \lambda_1,\lambda_2 \in M \text{ and } 0 < T_W(\lambda_1,\lambda_2) < 1\big\}$$

provided $M \supset \{0,1\}$; if $M = \{0,1\}$, on the other hand, δ can be an arbitrary value from $]0,1[$. Note that a suitable δ can always be found, since Θ_0, and therefore also Θ_1 and M are finite. Furthermore, by assumption, we have that $\lambda_{ii}^P = 1$ for all i in $\{1,2,\ldots,k\}$, hence it holds that $1 \in M$. This implies that for every m in $M \setminus \{0,1\}$

$$\delta < \min(m, 1-m) \tag{B.1}$$

To prove this proposition, we will construct a normalised set of fuzzy RCC formulas Θ which satisfies the assumptions of Proposition 1, and is, moreover, a refinement of Θ_0. In particular, Θ contains the following formulas

$$\begin{array}{ll} C(v_i,v_j) \geqslant \lambda_{ij}^C & C(v_i,v_j) \leqslant \lambda_{ij}^C \\ P(v_i,v_j) \geqslant \lambda_{ij}^P & P(v_i,v_j) \leqslant \lambda_{ij}^P \\ O(v_i,v_j) \geqslant \lambda_{ij}^O & O(v_i,v_j) \leqslant \lambda_{ij}^O \\ NTP(v_i,v_j) \geqslant \gamma_{ij}^{NTP} & NTP(v_i,v_j) \leqslant \gamma_{ij}^{NTP} \end{array}$$

for every v_i and v_j in V, where

$$\gamma_{ij}^{NTP} = \begin{cases} 1-\delta & \text{if } \lambda_{ij}^{NTP} = 1 \text{ and } \lambda_{ji}^P = 1 \\ \lambda_{ij}^{NTP} & \text{otherwise} \end{cases}$$

Note that $\gamma_{ij}^{NTP} \leqslant \lambda_{ij}^{NTP}$ always holds. To complete the proof, we need to show that $\gamma_{ii}^{NTP} < 1$ for all i in $\{1,2,\ldots,k\}$, that Θ is a refinement of Θ_0, and that Θ satisfies (16)–(19) as well as the transitivity rules for fuzzy topological relations.

By assumption, we know that $\lambda_{ii}^P = 1$ for all i in $\{1,2,\ldots,k\}$. If $\lambda_{ii}^{NTP} = 1$, we therefore have $\gamma_{ii}^{NTP} = 1-\delta < 1$ and if $\lambda_{ii}^{NTP} < 1$, we have $\gamma_{ii}^{NTP} = \lambda_{ii}^{NTP} < 1$. Next, if Θ were not a refinement of Θ_0, there would have to be a (disjunct of a) formula in Θ_0 of one of the following forms

$$NTPP(v_i,v_j) \geqslant \lambda \tag{B.2}$$

$$NTPP(v_i,v_j) \leqslant \lambda \tag{B.3}$$

$$TPP(v_i,v_j) \geqslant \lambda \tag{B.4}$$

$$TPP(v_i,v_j) \leqslant \lambda \tag{B.5}$$

which is violated in models of Θ. Indeed, there would have to exist a model $\mathcal{I}$ of Θ which is not a model of Θ_0. Since every model of Θ_1 is a model of Θ_0, this means that some fuzzy RCC formula in Θ_0 is violated because of the fact that $NTP^{\mathcal{I}}(v_i^{\mathcal{I}}, v_j^{\mathcal{I}})$ is γ_{ij}^{NTP} instead of λ_{ij}^{NTP} for some i and j in $\{1, 2, \ldots, k\}$. However, (B.3) can certainly not be violated by $\mathcal{I}$ as for all v_i and v_j, the value of $NTPP^{\mathcal{I}}(v_i^{\mathcal{I}}, v_j^{\mathcal{I}})$ is not greater in models of Θ than in models of Θ_1. Neither can (B.4) be violated, for the same reason. Furthermore, (B.2) could only be violated by $\mathcal{I}$ if $\gamma_{ij}^{NTP} \neq \lambda_{ij}^{NTP}$, but this implies $\lambda_{ji}^{P} = 1$ and therefore $P^{\mathcal{I}}(v_j^{\mathcal{I}}, v_i^{\mathcal{I}}) = 1$, and hence $NTPP^{\mathcal{I}}(v_i^{\mathcal{I}}, v_j^{\mathcal{I}}) = 0$, in any model of Θ_1. As the value of $NTPP^{\mathcal{I}}(v_i^{\mathcal{I}}, v_j^{\mathcal{I}})$ is not greater in models of Θ than in models of Θ_1, this implies that $NTPP^{\mathcal{I}}(v_i^{\mathcal{I}}, v_j^{\mathcal{I}}) = 0$. Hence, as the value of $NTPP^{\mathcal{I}}(v_i^{\mathcal{I}}, v_j^{\mathcal{I}})$ is the same in models of Θ_1 and in models of Θ, (B.2) cannot be violated. In entirely the same way, we have that (B.5) can never be violated.

As Θ_1 satisfies requirements (16)–(19), we immediately find that Θ satisfies requirements (17)–(19) and, moreover

$$\lambda_{ij}^{NTP} \leqslant \lambda_{ij}^{P} \leqslant \lambda_{ij}^{O} \leqslant \lambda_{ij}^{C}$$

for all i and j in $\{1, 2, \ldots, k\}$. Since $\gamma_{ij}^{NTP} \leqslant \lambda_{ij}^{NTP}$, we also have $\gamma_{ij}^{NTP} \leqslant \lambda_{ij}^{P}$. In other words, Θ also satisfies (16).

Finally, we show that Θ satisfies each of the transitivity rules for fuzzy topological relations, and in particular, each of the 31 transitivity rules introduced above. Note that we only need to check the transitivity rules involving *NTP*; we know that the others are satisfied from the fact that Θ_1 satisfies all transitivity rules. Moreover, most of the remaining transitivity rules are satisfied because $\gamma_{ij}^{NTP} \leqslant \lambda_{ij}^{NTP}$. For this reason, we only need to show that Θ satisfies the following transitivity rules.

$$T_W\big(P(v_i, v_j), NTP(v_j, v_l)\big) \leqslant NTP(v_i, v_l) \tag{B.6}$$

$$T_W\big(NTP(v_i, v_j), NTP(v_j, v_l)\big) \leqslant NTP(v_i, v_l) \tag{B.7}$$

$$T_W\big(P^{-1}(v_i, v_j), NTP^{-1}(v_j, v_l)\big) \leqslant NTP^{-1}(v_i, v_l) \tag{B.8}$$

$$T_W\big(P(v_i, v_j), coNTP^{-1}(v_j, v_l)\big) \leqslant coNTP^{-1}(v_i, v_l) \tag{B.9}$$

$$T_W\big(P^{-1}(v_i, v_j), coNTP(v_j, v_l)\big) \leqslant coNTP(v_i, v_l) \tag{B.10}$$

$$T_W\big(NTP(v_i, v_j), coNTP^{-1}(v_j, v_l)\big) \leqslant coP^{-1}(v_i, v_l) \tag{B.11}$$

$$T_W\big(NTP^{-1}(v_i, v_j), coNTP(v_j, v_l)\big) \leqslant coP(v_i, v_l) \tag{B.12}$$

First, assume that (B.6) were violated, i.e.

$$T_W\big(\lambda_{ij}^{P}, \gamma_{jl}^{NTP}\big) > \gamma_{il}^{NTP} \tag{B.13}$$

Since $T_W(\lambda_{ij}^{P}, \lambda_{jl}^{NTP}) \leqslant \lambda_{il}^{NTP}$, this is only possible if $\gamma_{il}^{NTP} < \lambda_{il}^{NTP}$, which implies $\lambda_{il}^{NTP} = \lambda_{li}^{P} = 1$ and $\gamma_{il}^{NTP} = 1 - \delta$. By definition of δ, $\gamma_{il}^{NTP} = 1 - \delta$ entails, together with assumption (B.13), that $T_W(\lambda_{ij}^{P}, \gamma_{jl}^{NTP}) = 1$ and, in other words, $\lambda_{ij}^{P} = \gamma_{jl}^{NTP} = 1$. From $\lambda_{jl}^{NTP} \geqslant \gamma_{jl}^{NTP}$, we furthermore establish $\lambda_{jl}^{NTP} = 1$. Since Θ_1 satisfies all transitivity rules, we know that

$$\lambda_{lj}^{P} \geqslant T_W\big(\lambda_{li}^{P}, \lambda_{ij}^{P}\big) = 1$$

From $\lambda_{lj}^{P} = 1$ and $\lambda_{jl}^{NTP} = 1$, we finally obtain $\gamma_{jl}^{NTP} = 1 - \delta$, a contradiction. Hence (B.6) is always satisfied. Moreover, since $\gamma_{ij}^{NTP} \leqslant \lambda_{ij}^{NTP} \leqslant \lambda_{ij}^{P}$, this implies that (B.7) is always satisfied as well. In entirely the same fashion, we can show that (B.8) can never be violated by Θ.

Next, assume that (B.9) were violated, i.e.

$$T_W\big(\lambda_{ij}^{P}, 1 - \gamma_{lj}^{NTP}\big) > 1 - \gamma_{li}^{NTP}$$

This is only possible if $\gamma_{lj}^{NTP} < \lambda_{lj}^{NTP}$, in other words, if $\lambda_{lj}^{NTP} = \lambda_{jl}^{P} = 1$ and $\gamma_{lj}^{NTP} = 1 - \delta$. Hence, we have $1 - \gamma_{li}^{NTP} < T_W(\lambda_{ij}^{P}, 1 - \gamma_{lj}^{NTP}) \leqslant \delta$, which implies $\gamma_{li}^{NTP} = 1$ by (B.1), and thus $\lambda_{li}^{NTP} = 1$. This yields $T_W(\lambda_{ij}^{P}, \delta) > 0$, which, in turn, entails $\lambda_{ij}^{P} = 1$, again using (B.1). Because Θ_1 satisfies all transitivity rules, we know that

$$T_W\big(\lambda_{ij}^{P}, \lambda_{jl}^{P}\big) \leqslant \lambda_{il}^{P}$$

As $\lambda_{ij}^{P} = \lambda_{jl}^{P} = 1$, we therefore have $\lambda_{il}^{P} = 1$. As moreover $\lambda_{li}^{NTP} = 1$, we find that $\gamma_{li}^{NTP} = 1 - \delta$, a contradiction. The proof for (B.10) is entirely analogous.

Finally, assume that (B.11) were violated, i.e.

$$T_W\big(\gamma_{ij}^{NTP}, 1 - \gamma_{lj}^{NTP}\big) > 1 - \lambda_{li}^{P}$$

Note that this can only be the case if $\gamma_{lj}^{NTP} < \lambda_{lj}^{NTP}$, or, in other words, $\lambda_{lj}^{NTP} = \lambda_{jl}^{P} = 1$ and $\gamma_{lj}^{NTP} = 1 - \delta$. From $T_W(\gamma_{ij}^{NTP}, \delta) > 0$, we find $\gamma_{ij}^{NTP} = 1$ using (B.1), and thus $\lambda_{ij}^{NTP} = 1$. Furthermore, we have from $1 - \lambda_{li}^{P} < T_W(\gamma_{ij}^{NTP}, 1 - \gamma_{lj}^{NTP}) = \delta$ that $\lambda_{li}^{P} = 1$, again using (B.1). However, from $\lambda_{jl}^{P} = 1$ and $\lambda_{li}^{P} = 1$, we find $\lambda_{ji}^{P} = 1$, which together with $\lambda_{ij}^{NTP} = 1$ implies $\gamma_{ij}^{NTP} = 1 - \delta$, a contradiction. The proof of (B.12) is entirely analogous.

Appendix C. Proof of Proposition 4

First we show the following lemma.

Lemma 9. *Let a, a' and b in $[0,1]$, and ε in $]0,1]$ such that $a < a' + \varepsilon$. It holds that*

$$T_W(a,b) < T_W(a',b) + \varepsilon$$

Proof. If $T_W(a,b) > 0$, we have

$$T_W(a,b) = a + b - 1 < a' + b - 1 + \varepsilon \leqslant \max(a' + b - 1, 0) + \varepsilon = T_W(a',b) + \varepsilon$$

If $T_W(a',b) = 0$, we have

$$T_W(a,b) = 0 < \varepsilon \leqslant T_W(a',b) + \varepsilon \qquad \square$$

From Proposition 1 and Proposition 2, we already know that Θ has an $(n;\alpha,0)$-model $\mathcal{I}'$. To prove the proposition, we show how we can modify $\mathcal{I}'$ to an $(n;\alpha,0)$-model $\mathcal{I}$ which satisfies the additional requirement that all membership degrees belong to $M_{\frac{\Delta}{2}}$. In particular for v in V and p in $\mathbb{R}^n$, $\mathcal{I}$ is defined as

$$v^{\mathcal{I}}(p) = \begin{cases} v^{\mathcal{I}'}(p) & \text{if } v^{\mathcal{I}'}(p) \in M_{\frac{\Delta}{2}} \setminus M_\Delta \\ k\Delta & \text{if } k\Delta - \frac{\Delta}{2} < v^{\mathcal{I}'}(p) < k\Delta + \frac{\Delta}{2}, \text{ for a given } k \text{ in } \mathbb{N} \end{cases}$$

Note that $v^{\mathcal{I}}(p) < v^{\mathcal{I}'}(p) + \frac{\Delta}{2}$ always holds. Clearly, we have that $\mathcal{I}$ is an $(n;\alpha,0)$-interpretation mapping every variable from v to a normalised bounded fuzzy set taking only membership degrees from $M_{\frac{\Delta}{2}}$. Hence, we only need to show that $\mathcal{I}$ satisfies all formulas from Θ. In particular, we will show that every atomic fuzzy RCC formula with an upper or lower bound in M_Δ that is satisfied by $\mathcal{I}'$, is also satisfied by $\mathcal{I}$. This implies that also disjunctive fuzzy RCC formulas remain satisfied; indeed, it follows that all disjuncts that are satisfied by $\mathcal{I}'$ are satisfied by $\mathcal{I}$.

Consider atomic fuzzy RCC formulas of the form $C(a,b) \leqslant \lambda$ and $C(a,b) \geqslant \lambda$. We will first show that for each (p_0,q_0) in $\mathbb{R}^n \times \mathbb{R}^n$ and each λ_0 in M_Δ, it holds that

$$T_W\big(a^{\mathcal{I}'}(p_0), R_\alpha(p_0,q_0), b^{\mathcal{I}'}(q_0)\big) \leqslant \lambda_0 \Rightarrow T_W\big(a^{\mathcal{I}}(p_0), R_\alpha(p_0,q_0), b^{\mathcal{I}}(q_0)\big) \leqslant \lambda_0 \tag{C.1}$$

$$T_W\big(a^{\mathcal{I}'}(p_0), R_\alpha(p_0,q_0), b^{\mathcal{I}'}(q_0)\big) \geqslant \lambda_0 \Rightarrow T_W\big(a^{\mathcal{I}}(p_0), R_\alpha(p_0,q_0), b^{\mathcal{I}}(q_0)\big) \geqslant \lambda_0 \tag{C.2}$$

Since $a^{\mathcal{I}}(p_0)$ and $b^{\mathcal{I}}(q_0)$ are in $M_{\frac{\Delta}{2}}$, we also have

$$T_W\big(a^{\mathcal{I}}(p_0), R_\alpha(p_0,q_0), b^{\mathcal{I}}(q_0)\big) \in M_{\frac{\Delta}{2}}$$

First assume that also the stronger statement

$$T_W\big(a^{\mathcal{I}}(p_0), R_\alpha(p_0,q_0), b^{\mathcal{I}}(q_0)\big) \in M_\Delta$$

is satisfied. As $a^{\mathcal{I}}(p_0) < a^{\mathcal{I}'}(p_0) + \frac{\Delta}{2}$ and $b^{\mathcal{I}}(q_0) < b^{\mathcal{I}'}(q_0) + \frac{\Delta}{2}$, we find using Lemma 9

$$\begin{aligned} T_W\big(a^{\mathcal{I}}(p_0), R_\alpha(p_0,q_0), b^{\mathcal{I}}(q_0)\big) &< T_W\big(a^{\mathcal{I}'}(p_0), R_\alpha(p_0,q_0), b^{\mathcal{I}}(q_0)\big) + \frac{\Delta}{2} \\ &< T_W\big(a^{\mathcal{I}'}(p_0), R_\alpha(p_0,q_0), b^{\mathcal{I}'}(q_0)\big) + \Delta \end{aligned}$$

Using the assumption $T_W(a^{\mathcal{I}}(p_0), R_\alpha(p_0,q_0), b^{\mathcal{I}}(q_0)) \in M_\Delta$, and the fact that Δ and λ_0 are in M_Δ, we find

$$\begin{aligned} &T_W\big(a^{\mathcal{I}'}(p_0), R_\alpha(p_0,q_0), b^{\mathcal{I}'}(q_0)\big) \leqslant \lambda_0 \\ &\quad\Rightarrow T_W\big(a^{\mathcal{I}}(p_0), R_\alpha(p_0,q_0), b^{\mathcal{I}}(q_0)\big) - \Delta < \lambda_0 \\ &\quad\Rightarrow T_W\big(a^{\mathcal{I}}(p_0), R_\alpha(p_0,q_0), b^{\mathcal{I}}(q_0)\big) \leqslant \lambda_0 \end{aligned}$$

showing (C.1). In entirely the same fashion, we find that

$$T_W\big(a^{\mathcal{I}}(p_0), R_\alpha(p_0,q_0), b^{\mathcal{I}}(q_0)\big) > T_W\big(a^{\mathcal{I}'}(p_0), R_\alpha(p_0,q_0), b^{\mathcal{I}'}(q_0)\big) - \Delta$$

from which we obtain (C.2).

Next, assume $T_W(a^{\mathcal{I}}(p_0), R_\alpha(p_0,q_0), b^{\mathcal{I}}(q_0)) \in M_{\frac{\Delta}{2}} \setminus M_\Delta$. This means that either $a^{\mathcal{I}}(p_0) \in M_{\frac{\Delta}{2}} \setminus M_\Delta$ and $b^{\mathcal{I}}(q_0) \in M_\Delta$, or $a^{\mathcal{I}}(p_0) \in M_\Delta$ and $b^{\mathcal{I}}(q_0) \in M_{\frac{\Delta}{2}} \setminus M_\Delta$. Assume, for instance, the former case (the proof for the latter case is entirely analogous). From $a^{\mathcal{I}}(p_0) \in M_{\frac{\Delta}{2}} \setminus M_\Delta$, we have by construction $a^{\mathcal{I}}(p_0) = a^{\mathcal{I}'}(p_0)$. From $b^{\mathcal{I}}(q_0) < b^{\mathcal{I}'}(q_0) + \frac{\Delta}{2}$, we obtain using Lemma 9

$$T_W\big(a^{\mathcal{I}}(p_0), R_\alpha(p_0,q_0), b^{\mathcal{I}}(q_0)\big) = T_W\big(a^{\mathcal{I}'}(p_0), R_\alpha(p_0,q_0), b^{\mathcal{I}}(q_0)\big)$$
$$< T_W\big(a^{\mathcal{I}'}(p_0), R_\alpha(p_0,q_0), b^{\mathcal{I}'}(q_0)\big) + \frac{\Delta}{2}$$

Using $T_W(a^{\mathcal{I}}(p_0), R_\alpha(p_0,q_0), b^{\mathcal{I}}(q_0)) \in M_{\frac{\Delta}{2}}$, this leads to

$$T_W\big(a^{\mathcal{I}'}(p_0), R_\alpha(p_0,q_0), b^{\mathcal{I}'}(q_0)\big) \leqslant \lambda_0$$
$$\Rightarrow T_W\big(a^{\mathcal{I}}(p_0), R_\alpha(p_0,q_0), b^{\mathcal{I}}(q_0)\big) - \frac{\Delta}{2} < \lambda_0$$
$$\Rightarrow T_W\big(a^{\mathcal{I}}(p_0), R_\alpha(p_0,q_0), b^{\mathcal{I}}(q_0)\big) \leqslant \lambda_0$$

showing (C.1). In entirely the same fashion, we find that

$$T_W\big(a^{\mathcal{I}}(p_0), R_\alpha(p_0,q_0), b^{\mathcal{I}}(q_0)\big) > T_W\big(a^{\mathcal{I}'}(p_0), R_\alpha(p_0,q_0), b^{\mathcal{I}'}(q_0)\big) - \frac{\Delta}{2}$$

from which we obtain (C.2).

Hence, we have that (C.1) and (C.2) are always satisfied. To complete the proof, we show that this implies for every λ in M_Δ

$$C_\alpha\big(a^{\mathcal{I}'}, b^{\mathcal{I}'}\big) \leqslant \lambda \Rightarrow C_\alpha\big(a^{\mathcal{I}}, b^{\mathcal{I}}\big) \leqslant \lambda \tag{C.3}$$
$$C_\alpha\big(a^{\mathcal{I}'}, b^{\mathcal{I}'}\big) \geqslant \lambda \Rightarrow C_\alpha\big(a^{\mathcal{I}}, b^{\mathcal{I}}\big) \geqslant \lambda \tag{C.4}$$

If $C_\alpha(a^{\mathcal{I}'}, b^{\mathcal{I}'}) \leqslant \lambda$, we have for every (p,q) in $\mathbb{R}^n \times \mathbb{R}^n$

$$T_W\big(a^{\mathcal{I}'}(p), R_\alpha(p,q), b^{\mathcal{I}'}(q)\big) \leqslant \lambda$$

which implies by (C.1)

$$T_W\big(a^{\mathcal{I}}(p), R_\alpha(p,q), b^{\mathcal{I}}(q)\big) \leqslant \lambda$$

From the monotonicity of the supremum we have (C.3), i.e.

$$C_\alpha\big(a^{\mathcal{I}}, b^{\mathcal{I}}\big) = \sup_{p,q\in\mathbb{R}^n} T_W\big(a^{\mathcal{I}}(p), R_\alpha(p,q), b^{\mathcal{I}}(q)\big) \leqslant \lambda$$

Finally, assume $C_\alpha(a^{\mathcal{I}'}, b^{\mathcal{I}'}) \geqslant \lambda$. Because $a^{\mathcal{I}'}$ and $b^{\mathcal{I}'}$ only take a finite number of different membership degrees, the supremum in $C_\alpha(a^{\mathcal{I}'}, b^{\mathcal{I}'})$ is attained in some (p_0,q_0) in $\mathbb{R}^n \times \mathbb{R}^n$ implying

$$T_W\big(a^{\mathcal{I}'}(p_0), R_\alpha(p_0,q_0), b^{\mathcal{I}'}(q_0)\big) \geqslant \lambda$$

or, by (C.2)

$$T_W\big(a^{\mathcal{I}}(p_0), R_\alpha(p_0,q_0), b^{\mathcal{I}}(q_0)\big) \geqslant \lambda$$

We conclude

$$C_\alpha\big(a^{\mathcal{I}}, b^{\mathcal{I}}\big) = \sup_{p,q\in\mathbb{R}^n} T_W\big(a^{\mathcal{I}}(p), R_\alpha(p,q), b^{\mathcal{I}}(q)\big) \geqslant T_W\big(a^{\mathcal{I}}(p_0), R_\alpha(p_0,q_0), b^{\mathcal{I}}(q_0)\big) \geqslant \lambda$$

The proof for fuzzy RCC formulas involving P, O or NTP is analogous.

Appendix D. Proof of Proposition 5

First we show three technical lemmas.

Lemma 10. *Let λ be in $]0,1]$, $T = T_W$, and let A be a fuzzy set in $\mathbb{R}^n$. It holds that*

$$(R_\alpha{\downarrow}A)_\lambda = R_\alpha{\downarrow}(A_\lambda)$$

Proof. Assume $p \in (R_\alpha{\downarrow}A)_\lambda$, i.e.

$$\inf_{q\in\mathbb{R}^n} I_W\big(R_\alpha(p,q), A(q)\big) \geqslant \lambda$$

This means that for every q satisfying $(p,q) \in R_\alpha$, it holds that $A(q) \geqslant \lambda$, or equivalently $q \in A_\lambda$, i.e.

$$(p,q) \in R_\alpha \Rightarrow q \in A_\lambda \tag{D.1}$$

which entails

$$p \in R_\alpha{\downarrow}(A_\lambda)$$

Conversely, assume $p \in R_\alpha{\downarrow}(A_\lambda)$. This implies for every q in $\mathbb{R}^n$

$$(p,q) \in R_\alpha \Rightarrow q \in A_\lambda$$

or

$$(p,q) \in R_\alpha \Rightarrow A(q) \geqslant \lambda$$

implying

$$I_W\big(R_\alpha(p,q), A_\lambda(q)\big) \geqslant \lambda$$

Hence, by definition of the infimum as greatest lower bound, we have

$$\inf_{q\in\mathbb{R}^n} I_W\big(R_\alpha(p,q), A_\lambda(q)\big) \geqslant \lambda$$

which means $p \in (R_\alpha{\downarrow}A)_\lambda$. $\square$

Lemma 11. *Let λ be in $]0,1]$, $T = T_W$ and let A be a fuzzy set in $\mathbb{R}^n$ which only takes a finite number of different membership degrees. It holds that*

$$(R_\alpha{\uparrow}A)_\lambda = R_\alpha{\uparrow}(A_\lambda)$$

Proof. Assume $p \in (R_\alpha{\uparrow}A)_\lambda$, i.e.

$$\sup_{q\in\mathbb{R}^n} T_W\big(R_\alpha(p,q), A(q)\big) \geqslant \lambda$$

Since A only takes a finite number of different membership degrees, this supremum is attained. Hence, there is a q_0 such that $d(p,q_0) \leqslant \alpha$ and $A(q_0) \geqslant \lambda$. In other words

$$\big(\exists q \in \mathbb{R}^n\big)\big((p,q) \in R_\alpha \wedge q \in A_\lambda\big)$$

or $p \in R_\alpha{\uparrow}(A_\lambda)$. Conversely, assume $p \in R_\alpha{\uparrow}(A_\lambda)$. This means that for some q_0, $d(p,q_0) \leqslant \alpha$ and $A(q_0) \geqslant \lambda$, hence, $T_W(R_\alpha(p,q_0), A(q_0)) \geqslant \lambda$. In particular, this entails

$$\sup_{q\in\mathbb{R}^n} T_W\big(R_\alpha(p,q), A(q)\big) \geqslant \lambda$$

or $p \in (R_\alpha{\uparrow}A)_\lambda$. $\square$

Note that the condition that A only takes a finite number of different membership degrees is not redundant. For example, assume $n = 1$ and let A be defined for q in $\mathbb{R}$ by

$$A(q) = \begin{cases} \frac{\lambda q}{\alpha} & \text{if } q \in [0, \alpha[\\ 0 & \text{otherwise} \end{cases}$$

for a given λ in $]0,1]$ and $\alpha > 0$. Then $A(q) < \lambda$ for all q in $\mathbb{R}$ and therefore $\lim_{q \xrightarrow[>]{} \alpha} A(q) = \lambda$. This means

$$(R_\alpha{\uparrow}A)(0) = \sup_{q\in\mathbb{R}} T_W\big(R_\alpha(0,q), A(q)\big) = \sup\left\{\frac{\lambda q}{\alpha} \;\middle|\; q \in [0,\alpha[\right\} = \lambda$$

and $0 \in (R_\alpha{\uparrow}A)_\lambda$. On the other hand, we have $A_\lambda = \emptyset$, implying $R_\alpha{\uparrow}(A_\lambda) = \emptyset$ and in particular $0 \notin R_\alpha{\uparrow}(A_\lambda)$.

Lemma 12. *Let A and B be crisp, non-empty, bounded subsets of $\mathbb{R}^n$. It holds that $NTP_\alpha(A,B) \equiv NTPP_\alpha(A,B)$ $(\alpha > 0)$.*

Proof. We need to show that $NTP_\alpha(A,B) \Rightarrow \neg P_\alpha(B,A)$. Assume that both $NTP_\alpha(A,B)$ and $P_\alpha(B,A)$ hold. From $NTP_\alpha(A,B)$ we know $R_\alpha{\uparrow}A \subseteq R_\alpha{\downarrow}{\uparrow}B$, while $P_\alpha(B,A)$ entails $R_\alpha{\downarrow}{\uparrow}B \subseteq R_\alpha{\downarrow}{\uparrow}A$. Together, this yields $R_\alpha{\uparrow}A \subseteq R_\alpha{\downarrow}{\uparrow}A$. Since $\alpha > 0$, this is only possible if $A = \emptyset$ or $A = \mathbb{R}^n$, which have both been excluded by assumption. $\square$

To show equivalence (70), we find

$$C_\alpha(A,B) \geqslant \lambda \Leftrightarrow \sup_{p,q\in\mathbb{R}^n} T_W\big(A(p), R_\alpha(p,q), B(q)\big) \geqslant \lambda$$

Since A and B only take a finite number of different membership degrees the supremum is attained and

$$
\begin{aligned}
C_\alpha(A,B) \geqslant \lambda &\Leftrightarrow \big(\exists p,q \in \mathbb{R}^n\big)\big(T_W\big(A(p), R_\alpha(p,q), B(q)\big) \geqslant \lambda\big) \\
&\Leftrightarrow \big(\exists p,q \in \mathbb{R}^n\big)\big(d(p,q) \leqslant \alpha \wedge T_W\big(A(p), R_\alpha(p,q), B(q)\big) \geqslant \lambda\big) \\
&\Leftrightarrow \big(\exists p,q \in \mathbb{R}^n\big)\big(d(p,q) \leqslant \alpha \wedge T_W\big(A(p), B(q)\big) \geqslant \lambda\big)
\end{aligned}
$$

Since A and B only take membership degrees from $M_{\frac{\Delta}{2}}$, we can enumerate the possible values of $A(p)$ and $B(q)$ for which $T_W(A(p), B(q)) \geqslant \lambda$:

$$
\begin{aligned}
&\Leftrightarrow \big(\exists p,q \in \mathbb{R}^n\big)\bigg(d(p,q) \leqslant \alpha \wedge \bigg(\big(A(p) \geqslant 1 \wedge B(q) \geqslant \lambda\big) \vee \bigg(A(p) \geqslant 1-\frac{\Delta}{2} \wedge B(q) \geqslant \lambda+\frac{\Delta}{2}\bigg) \\
&\quad \vee \cdots \vee \big(A(p) \geqslant \lambda \wedge B(q) \geqslant 1\big)\bigg)\bigg) \\
&\Leftrightarrow \big(\exists p,q \in \mathbb{R}^n\big)\big(d(p,q) \leqslant \alpha \wedge \big((p \in A_1 \wedge q \in B_\lambda) \vee (p \in A_{1-\frac{\Delta}{2}} \wedge q \in B_{\lambda+\frac{\Delta}{2}}) \vee \cdots \vee (p \in A_\lambda \wedge q \in B_1)\big)\big) \\
&\Leftrightarrow \big(\exists p,q \in \mathbb{R}^n\big)\big(\big(d(p,q) \leqslant \alpha \wedge p \in A_1 \wedge q \in B_\lambda\big) \vee \big(d(p,q) \leqslant \alpha \wedge p \in A_{1-\frac{\Delta}{2}} \wedge q \in B_{\lambda+\frac{\Delta}{2}}\big) \\
&\quad \vee \cdots \vee \big(d(p,q) \leqslant \alpha \wedge p \in A_\lambda \wedge q \in B_1\big)\big) \\
&\Leftrightarrow \big(\exists p,q \in \mathbb{R}^n\big)\big(d(p,q) \leqslant \alpha \wedge p \in A_1 \wedge q \in B_\lambda\big) \vee \big(\exists p,q \in \mathbb{R}^n\big)\big(d(p,q) \leqslant \alpha \wedge p \in A_{1-\frac{\Delta}{2}} \wedge q \in B_{\lambda+\frac{\Delta}{2}}\big) \\
&\quad \vee \cdots \vee \big(\exists p,q \in \mathbb{R}^n\big)\big(d(p,q) \leqslant \alpha \wedge p \in A_\lambda \wedge q \in B_1\big) \\
&\Leftrightarrow C_\alpha(A_1, B_\lambda) \vee C_\alpha(A_{1-\frac{\Delta}{2}}, B_{\lambda+\frac{\Delta}{2}}) \vee \cdots \vee C_\alpha(A_\lambda, B_1)
\end{aligned}
$$

Next, (71) follows easily from (70):

$$C_\alpha(A,B) \leqslant \lambda' \Leftrightarrow \neg\big(C_\alpha(A,B) > \lambda'\big)$$

Since A and B only take values from $M_{\frac{\Delta}{2}}$, $C_\alpha(A,B)$ can only take values from $M_{\frac{\Delta}{2}}$ either. Hence

$$C_\alpha(A,B) \leqslant \lambda' \Leftrightarrow \neg\bigg(C_\alpha(A,B) \geqslant \lambda' + \frac{\Delta}{2}\bigg)$$

and by (70)

$$
\begin{aligned}
C_\alpha(A,B) \leqslant \lambda' &\Leftrightarrow \neg\big(C_\alpha(A_1, B_{\lambda'+\frac{\Delta}{2}}) \vee C_\alpha(A_{1-\frac{\Delta}{2}}, B_{\lambda'+\Delta}) \vee \cdots \vee C_\alpha(A_{\lambda'+\frac{\Delta}{2}}, B_1)\big) \\
&\Leftrightarrow DC_\alpha(A_1, B_{\lambda'+\frac{\Delta}{2}}) \wedge DC_\alpha(A_{1-\frac{\Delta}{2}}, B_{\lambda'+\Delta}) \wedge \cdots \wedge DC_\alpha(A_{\lambda'+\frac{\Delta}{2}}, B_1)
\end{aligned}
$$

To show (72), we find using Lemma 10 and Lemma 11

$$
\begin{aligned}
O_\alpha(A,B) \geqslant \lambda &\Leftrightarrow \sup_{p \in \mathbb{R}^n} T_W\big(R_\alpha{\downarrow}{\uparrow}A(p), R_\alpha{\downarrow}{\uparrow}B(p)\big) \geqslant \lambda \\
&\Leftrightarrow \big(\exists p \in \mathbb{R}^n\big)\big(T_W\big(R_\alpha{\downarrow}{\uparrow}A(p), R_\alpha{\downarrow}{\uparrow}B(p)\big) \geqslant \lambda\big) \\
&\Leftrightarrow \big(\exists p \in \mathbb{R}^n\big)\bigg(\big(R_\alpha{\downarrow}{\uparrow}A(p) \geqslant 1 \wedge R_\alpha{\downarrow}{\uparrow}B(p) \geqslant \lambda\big) \vee \bigg(R_\alpha{\downarrow}{\uparrow}A(p) \geqslant 1-\frac{\Delta}{2} \wedge R_\alpha{\downarrow}{\uparrow}B(p) \geqslant \lambda+\frac{\Delta}{2}\bigg) \\
&\quad \vee \cdots \vee \big(R_\alpha{\downarrow}{\uparrow}A(p) \geqslant \lambda \wedge R_\alpha{\downarrow}{\uparrow}B(p) \geqslant 1\big)\bigg) \\
&\Leftrightarrow \big(\exists p \in \mathbb{R}^n\big)\big(\big(p \in (R_\alpha{\downarrow}{\uparrow}A)_1 \wedge p \in (R_\alpha{\downarrow}{\uparrow}B)_\lambda\big) \vee \big(p \in (R_\alpha{\downarrow}{\uparrow}A)_{1-\frac{\Delta}{2}} \wedge p \in (R_\alpha{\downarrow}{\uparrow}B)_{\lambda+\frac{\Delta}{2}}\big) \\
&\quad \vee \cdots \vee \big(p \in (R_\alpha{\downarrow}{\uparrow}A)_\lambda \wedge p \in (R_\alpha{\downarrow}{\uparrow}B)_1\big)\big) \\
&\Leftrightarrow \big(\exists p \in \mathbb{R}^n\big)\big(\big(p \in (R_\alpha{\downarrow}{\uparrow}A_1) \wedge p \in (R_\alpha{\downarrow}{\uparrow}B_\lambda)\big) \vee \big(p \in (R_\alpha{\downarrow}{\uparrow}A_{1-\frac{\Delta}{2}}) \wedge p \in (R_\alpha{\downarrow}{\uparrow}B_{\lambda+\frac{\Delta}{2}})\big) \\
&\quad \vee \cdots \vee \big(p \in (R_\alpha{\downarrow}{\uparrow}A_\lambda) \wedge p \in (R_\alpha{\downarrow}{\uparrow}B_1)\big)\big) \\
&\Leftrightarrow \big(\exists p \in \mathbb{R}^n\big)\big(p \in (R_\alpha{\downarrow}{\uparrow}A_1) \wedge p \in (R_\alpha{\downarrow}{\uparrow}B_\lambda)\big) \vee \big(\exists p \in \mathbb{R}^n\big)\big(p \in (R_\alpha{\downarrow}{\uparrow}A_{1-\frac{\Delta}{2}}) \wedge p \in (R_\alpha{\downarrow}{\uparrow}B_{\lambda+\frac{\Delta}{2}})\big) \\
&\quad \vee \cdots \vee \big(\exists p \in \mathbb{R}^n\big)\big(p \in (R_\alpha{\downarrow}{\uparrow}A_\lambda) \wedge p \in (R_\alpha{\downarrow}{\uparrow}B_1)\big) \\
&\Leftrightarrow O_\alpha(A_1, B_\lambda) \vee O_\alpha(A_{1-\frac{\Delta}{2}}, B_{\lambda+\frac{\Delta}{2}}) \vee \cdots \vee O_\alpha(A_\lambda, B_1)
\end{aligned}
$$

Furthermore, (73) follows from (72) in the same way that (71) follows from (70). Next, to show (75), we obtain, again employing Lemma 10 and Lemma 11

$$\begin{aligned}
&P_\alpha(A,B) \leqslant \lambda' \\
&\quad\Leftrightarrow \inf_{p\in\mathbb{R}^n} I_W\big(R_\alpha{\downarrow}{\uparrow}A(p), R_\alpha{\downarrow}{\uparrow}B(p)\big) \leqslant \lambda' \\
&\quad\Leftrightarrow \big(\exists p\in\mathbb{R}^n\big)\big(I_W\big(R_\alpha{\downarrow}{\uparrow}A(p), R_\alpha{\downarrow}{\uparrow}B(p)\big) \leqslant \lambda'\big) \\
&\quad\Leftrightarrow \big(\exists p\in\mathbb{R}^n\big)\bigg(\big(R_\alpha{\downarrow}{\uparrow}A(p) \geqslant 1 \wedge R_\alpha{\downarrow}{\uparrow}B(p) \leqslant \lambda'\big) \vee \bigg(R_\alpha{\downarrow}{\uparrow}A(p) \geqslant 1-\frac{\Delta}{2} \wedge R_\alpha{\downarrow}{\uparrow}B(p) \leqslant \lambda'-\frac{\Delta}{2}\bigg) \\
&\qquad\quad \vee\cdots\vee \big(R_\alpha{\downarrow}{\uparrow}A(p) \geqslant 1-\lambda' \wedge R_\alpha{\downarrow}{\uparrow}B(p) \leqslant 0\big)\bigg) \\
&\quad\Leftrightarrow \big(\exists p\in\mathbb{R}^n\big)\bigg(\big(R_\alpha{\downarrow}{\uparrow}A(p) \geqslant 1 \wedge \neg\big(R_\alpha{\downarrow}{\uparrow}B(p) > \lambda'\big)\big) \vee \bigg(R_\alpha{\downarrow}{\uparrow}A(p) \geqslant 1-\frac{\Delta}{2} \wedge \neg\big(R_\alpha{\downarrow}{\uparrow}B(p) > \lambda'-\frac{\Delta}{2}\big)\bigg) \\
&\qquad\quad \vee\cdots\vee \big(R_\alpha{\downarrow}{\uparrow}A(p) \geqslant 1-\lambda' \wedge \neg\big(R_\alpha{\downarrow}{\uparrow}B(p) > 0\big)\big)\bigg) \\
&\quad\Leftrightarrow \big(\exists p\in\mathbb{R}^n\big)\bigg(\bigg(R_\alpha{\downarrow}{\uparrow}A(p) \geqslant 1 \wedge \neg\bigg(R_\alpha{\downarrow}{\uparrow}B(p) \geqslant \lambda'+\frac{\Delta}{2}\bigg)\bigg) \vee \bigg(R_\alpha{\downarrow}{\uparrow}A(p) \geqslant 1-\frac{\Delta}{2} \wedge \neg\big(R_\alpha{\downarrow}{\uparrow}B(p) \geqslant \lambda'\big)\bigg) \\
&\qquad\quad \vee\cdots\vee \bigg(R_\alpha{\downarrow}{\uparrow}A(p) \geqslant 1-\lambda' \wedge \neg\bigg(R_\alpha{\downarrow}{\uparrow}B(p) \geqslant \frac{\Delta}{2}\bigg)\bigg)\bigg) \\
&\quad\Leftrightarrow \big(\exists p\in\mathbb{R}^n\big)\big(\big(p\in(R_\alpha{\downarrow}{\uparrow}A)_1 \wedge p\notin(R_\alpha{\downarrow}{\uparrow}B)_{\lambda'+\frac{\Delta}{2}}\big) \vee \big(p\in(R_\alpha{\downarrow}{\uparrow}A)_{1-\frac{\Delta}{2}} \wedge p\notin(R_\alpha{\downarrow}{\uparrow}B)_{\lambda'}\big) \\
&\qquad\quad \vee\cdots\vee \big(p\in(R_\alpha{\downarrow}{\uparrow}A)_{1-\lambda'} \wedge p\notin(R_\alpha{\downarrow}{\uparrow}B)_{\frac{\Delta}{2}}\big)\big) \\
&\quad\Leftrightarrow \big(\exists p\in\mathbb{R}^n\big)\big(\big(p\in(R_\alpha{\downarrow}{\uparrow}A_1) \wedge p\notin(R_\alpha{\downarrow}{\uparrow}B_{\lambda'+\frac{\Delta}{2}})\big) \vee \big(p\in(R_\alpha{\downarrow}{\uparrow}A_{1-\frac{\Delta}{2}}) \wedge p\notin(R_\alpha{\downarrow}{\uparrow}B_{\lambda'})\big) \\
&\qquad\quad \vee\cdots\vee \big(p\in(R_\alpha{\downarrow}{\uparrow}A_{1-\lambda'}) \wedge p\notin(R_\alpha{\downarrow}{\uparrow}B_{\frac{\Delta}{2}})\big)\big) \\
&\quad\Leftrightarrow \big(\exists p\in\mathbb{R}^n\big)\big(p\in(R_\alpha{\downarrow}{\uparrow}A_1) \wedge p\notin(R_\alpha{\downarrow}{\uparrow}B_{\lambda'+\frac{\Delta}{2}})\big) \vee \big(\exists p\in\mathbb{R}^n\big)\big(p\in(R_\alpha{\downarrow}{\uparrow}A_{1-\frac{\Delta}{2}}) \wedge p\notin(R_\alpha{\downarrow}{\uparrow}B_{\lambda'})\big) \\
&\qquad\quad \vee\cdots\vee \big(\exists p\in\mathbb{R}^n\big)\big(p\in(R_\alpha{\downarrow}{\uparrow}A_{1-\lambda'}) \wedge p\notin(R_\alpha{\downarrow}{\uparrow}B_{\frac{\Delta}{2}})\big) \\
&\quad\Leftrightarrow \neg\big(\forall p\in\mathbb{R}^n\big)\big(p\notin(R_\alpha{\downarrow}{\uparrow}A_1) \vee p\in(R_\alpha{\downarrow}{\uparrow}B_{\lambda'+\frac{\Delta}{2}})\big) \vee \neg\big(\forall p\in\mathbb{R}^n\big)\big(p\notin(R_\alpha{\downarrow}{\uparrow}A_{1-\frac{\Delta}{2}}) \vee p\in(R_\alpha{\downarrow}{\uparrow}B_{\lambda'})\big) \\
&\qquad\quad \vee\cdots\vee \neg\big(\forall p\in\mathbb{R}^n\big)\big(p\notin(R_\alpha{\downarrow}{\uparrow}A_{1-\lambda'}) \vee p\in(R_\alpha{\downarrow}{\uparrow}B_{\frac{\Delta}{2}})\big) \\
&\quad\Leftrightarrow \neg\big(\forall p\in\mathbb{R}^n\big)\big(p\in(R_\alpha{\downarrow}{\uparrow}A_1) \Rightarrow p\in(R_\alpha{\downarrow}{\uparrow}B_{\lambda'+\frac{\Delta}{2}})\big) \vee \neg\big(\forall p\in\mathbb{R}^n\big)\big(p\in(R_\alpha{\downarrow}{\uparrow}A_{1-\frac{\Delta}{2}}) \Rightarrow p\in(R_\alpha{\downarrow}{\uparrow}B_{\lambda'})\big) \\
&\qquad\quad \vee\cdots\vee \neg\big(\forall p\in\mathbb{R}^n\big)\big(p\in(R_\alpha{\downarrow}{\uparrow}A_{1-\lambda'}) \Rightarrow p\in(R_\alpha{\downarrow}{\uparrow}B_{\frac{\Delta}{2}})\big) \\
&\quad\Leftrightarrow \neg P_\alpha(A_1, B_{\lambda'+\frac{\Delta}{2}}) \vee \neg P_\alpha(A_{1-\frac{\Delta}{2}}, B_{\lambda'}) \vee\cdots\vee \neg P_\alpha(A_{1-\lambda'}, B_{\frac{\Delta}{2}})
\end{aligned}$$

and (74) follows from (75) in the same way that (71) follows from (70). To show (77) we find entirely analogously as for (75) that

$$NTP_\alpha(A,B) \leqslant \lambda' \Leftrightarrow \neg NTP_\alpha(A_1, B_{\lambda'+\frac{\Delta}{2}}) \vee \neg NTP_\alpha(A_{1-\frac{\Delta}{2}}, B_{\lambda'}) \vee\cdots\vee \neg NTP_\alpha(A_{1-\lambda'}, B_{\frac{\Delta}{2}})$$

from which (77) follows by Lemma 12. Finally, (76) follows from (77) in the same way that (71) follows from (70).

Appendix E. Proof of Proposition 7

Lemma 13. *Let $A = [a_1, a_2] \cup [a_3, a_4] \cup \cdots \cup [a_{2n-1}, a_{2n}]$ such that $a_1 < a_2 < a_3 < \cdots < a_{2n}$. For sufficiently small $\gamma > 0$, it holds that $R_\gamma{\downarrow}A = [a_1+\gamma, a_2-\gamma] \cup [a_3+\gamma, a_4-\gamma] \cup \cdots \cup [a_{2n-1}+\gamma, a_{2n}-\gamma]$.*

Proof. Let $\gamma > 0$ be chosen such that

$$\gamma < \min_{i=1}^{n} \frac{a_{2i} - a_{2i-1}}{2}$$

We then find for each p in $\mathbb{R}$

$$\begin{aligned}
p \in R_\gamma{\downarrow}A &\equiv (\forall q\in\mathbb{R})\big((p,q)\in R_\gamma \Rightarrow q\in A\big) \\
&\equiv (\forall q\in\mathbb{R})\big(d(p,q)\leqslant\gamma \Rightarrow q\in A\big) \\
&\equiv p\in[a_1+\gamma, a_2-\gamma]\cup[a_3+\gamma, a_4-\gamma]\cup\cdots\cup[a_{2n-1}+\gamma, a_{2n}-\gamma] \qquad \square
\end{aligned}$$

Lemma 14. *Let $A = [a_1, a_2] \cup [a_3, a_4] \cup \cdots \cup [a_{2n-1}, a_{2n}]$ such that $a_1 < a_2 < a_3 < \cdots < a_{2n}$. For sufficiently small $\gamma > 0$, it holds that $R_\gamma{\uparrow}A = [a_1 - \gamma, a_2 + \gamma] \cup [a_3 - \gamma, a_4 + \gamma] \cup \cdots \cup [a_{2n-1} - \gamma, a_{2n} + \gamma]$ and $a_1 - \gamma < a_2 + \gamma < a_3 - \gamma < \cdots < a_{2n} + \gamma$.*

Proof. Let $\gamma > 0$ be chosen such that

$$\gamma < \min_{i=2}^{n} \frac{a_{2i-1} - a_{2i-2}}{2}$$

We then find for each p in $\mathbb{R}$

$$\begin{aligned} p \in R_\gamma{\uparrow}A &\equiv (\exists q \in \mathbb{R})\big(R_\gamma(p,q) \wedge q \in A\big) \\ &\equiv (\exists q \in \mathbb{R})\big(d(p,q) \leqslant \gamma \wedge q \in A\big) \\ &\equiv p \in [a_1 - \gamma, a_2 + \gamma] \cup [a_3 - \gamma, a_4 + \gamma] \cup \cdots \cup [a_{2n-1} - \gamma, a_{2n} + \gamma] \qquad \square \end{aligned}$$

Corollary 7. *If A is the union of a finite number of closed, non-degenerate intervals, it holds that $R_\gamma{\downarrow}{\uparrow}A = A$ for any sufficiently small $\gamma > 0$.*

Since Θ is satisfiable, there exists an Egg–Yolk model in any dimension, and, in particular, an Egg–Yolk model $\mathcal{I}$ in $\mathbb{R}$. Let V be the set of variables used in Θ. For every v in V, $v^{\mathcal{I}}$ corresponds to an Egg–Yolk region $(v^1, v^2, \ldots, v^k)$ whose k nested sets each are the union of a finite number of closed, non-degenerate intervals in $\mathbb{R}$. Next, we show that there exists a model $\mathcal{I}'$ which also maps variables to Egg–Yolk regions, but which interprets C as in (83). From Section 7, we then know that $\mathcal{I}'$ corresponds to an $(1;\alpha,0)$-model in which each variable v in V is interpreted as a fuzzy set taking only membership degrees from $M_{\frac{1}{k}}$, its α-level sets corresponding to the k nested sets of the Egg–Yolk region $v^{\mathcal{I}'}$. In particular, we define $\mathcal{I}'$ for v in V as the Egg–Yolk region whose ith component v'^i is given by

$$v'^i = R_\gamma{\downarrow}v^i$$

for a given $\gamma > 0$. We show that $\mathcal{I}'$ is a model of Θ, provided γ and the parameter $\alpha > 0$ from (83) are taken sufficiently small. In particular, we show that when A and B are the union of a finite number of closed, non-degenerate intervals in $\mathbb{R}$, it holds that

$$\begin{aligned} &C^1(A,B) \Leftrightarrow C_\alpha(R_\gamma{\downarrow}A, R_\gamma{\downarrow}B) \\ &P^1(A,B) \Leftrightarrow P_\alpha(R_\gamma{\downarrow}A, R_\gamma{\downarrow}B) \\ &O^1(A,B) \Leftrightarrow O_\alpha(R_\gamma{\downarrow}A, R_\gamma{\downarrow}B) \\ &NTP^1(A,B) \Leftrightarrow NTP_\alpha(R_\gamma{\downarrow}A, R_\gamma{\downarrow}B) \end{aligned}$$

from which the proposition follows. Note that when γ is sufficiently small, $R_\gamma{\downarrow}A$ and $R_\gamma{\downarrow}B$ are the union of a finite number of closed, non-degenerate intervals in $\mathbb{R}$ by Lemma 13. First, consider

$$C^1(A,B) \Rightarrow C_\alpha(R_\gamma{\downarrow}A, R_\gamma{\downarrow}B) \tag{E.1}$$

If $C^1(A,B)$ holds, we know that there is some p in $A \cap B$. From Lemma 13 we furthermore know that there is some q_1 in $R_\gamma{\downarrow}A$ and some q_2 in $R_\gamma{\downarrow}B$ such that $d(p,q_1) \leqslant \gamma$ and $d(p,q_2) \leqslant \gamma$, and $d(q_1,q_2) \leqslant 2\gamma$. Hence, (E.1) holds as soon as $\gamma \leqslant \frac{\alpha}{2}$. We show the implication in the opposite direction by contraposition, i.e.:

$$DC^1(A,B) \Rightarrow DC_\alpha(R_\gamma{\downarrow}A, R_\gamma{\downarrow}B) \tag{E.2}$$

If $DC^1(A,B)$, we know that $d = \inf_{p \in A, q \in B} d(p,q) > 0$. Therefore, it suffices to choose $\alpha < d$ to have $DC_\alpha(A,B)$, which entails $DC_\alpha(R_\gamma{\downarrow}A, R_\gamma{\downarrow}B)$ since $R_\gamma{\downarrow}A \subseteq A$ and $R_\gamma{\downarrow}B \subseteq B$.

Next, we consider

$$O^1(A,B) \Rightarrow O_\alpha(R_\gamma{\downarrow}A, R_\gamma{\downarrow}B) \tag{E.3}$$

From $O^1(A,B)$ we know that there is some p in $i(A) \cap i(B)$. Since $p \in i(A) \cap i(B)$ we know that all points in a sufficiently small neighborhood of p are also located in A and in B, in other words, that for sufficiently small γ, we have $p \in R_\gamma{\downarrow}A \cap R_\gamma{\downarrow}B$, which implies $p \in R_\alpha{\downarrow}{\uparrow}(R_\gamma{\downarrow}A) \cap R_\alpha{\downarrow}{\uparrow}(R_\gamma{\downarrow}B)$ and (E.3). Again, we show the implication in the opposite direction by contraposition:

$$DR^1(A,B) \Rightarrow DR_\alpha(R_\gamma{\downarrow}A, R_\gamma{\downarrow}B) \tag{E.4}$$

From $DR^1(A,B)$, we know that for any p in $\mathbb{R}$ either $p \notin i(A)$ or $p \notin i(B)$. We obtain

$$\begin{aligned}
&(\forall p \in \mathbb{R})\big(p \notin i(A) \vee p \notin i(B)\big)\\
&\quad\Leftrightarrow (\forall p \in \mathbb{R})\big(\neg(\exists\varepsilon > 0)(\forall q \in \mathbb{R})\big(d(p,q) \leqslant \varepsilon \Rightarrow q \in A\big)\\
&\qquad \vee \neg(\exists\varepsilon > 0)(\forall q \in \mathbb{R})\big(d(p,q) \leqslant \varepsilon \Rightarrow q \in B\big)\big)\\
&\quad\Leftrightarrow (\forall p \in \mathbb{R})\big((\forall\varepsilon > 0)(p \notin R_\varepsilon{\downarrow}A) \vee (\forall\varepsilon > 0)(p \notin R_\varepsilon{\downarrow}B)\big)\\
&\quad\Rightarrow (\forall p \in \mathbb{R})(p \notin R_\gamma{\downarrow}A \vee p \notin R_\gamma{\downarrow}B)
\end{aligned}$$

By Corollary 7, we can assume that $R_\gamma{\downarrow}A = R_\alpha{\downarrow}{\uparrow}(R_\gamma{\downarrow}A)$ and $R_\gamma{\downarrow}B = R_\alpha{\downarrow}{\uparrow}(R_\gamma{\downarrow}B)$. Furthermore, note that this does not imply that the value of α depends on the value of γ. Indeed, from the proof of Lemma 13 and 14, it is clear that any value of α for which $A = R_\alpha{\downarrow}{\uparrow}A$ also satisfies $R_\gamma{\downarrow}A = R_\alpha{\downarrow}{\uparrow}(R_\gamma{\downarrow}A)$, the former expression being independent of γ. Hence, we have

$$(\forall p \in \mathbb{R})\big(p \notin R_\alpha{\downarrow}{\uparrow}(R_\gamma{\downarrow}A) \vee p \notin R_\alpha{\downarrow}{\uparrow}(R_\gamma{\downarrow}B)\big)$$

or, in other words, $DR_\alpha(R_\gamma{\downarrow}A, R_\gamma{\downarrow}B)$.

To show

$$P^1(A,B) \Rightarrow P_\alpha(R_\gamma{\downarrow}A, R_\gamma{\downarrow}B) \tag{E.5}$$

we find from $P^1(A,B)$ that $A \subseteq B$, and therefore also that $R_\alpha{\downarrow}{\uparrow}(R_\gamma{\downarrow}A) \subseteq R_\alpha{\downarrow}{\uparrow}(R_\gamma{\downarrow}B)$, or, $P_\alpha(R_\gamma{\downarrow}A, R_\gamma{\downarrow}B)$. To show the implication in the opposite direction, we find from $\neg P^1(A,B)$ that there is a p in $\mathbb{R}$ such that $p \in i(A)$ and $p \notin B$. This implies that all points in a sufficiently small neighborhood of p are in A, or, $p \in R_\gamma{\downarrow}A$, provided γ is sufficiently small. From $p \notin B$ we also have $p \notin R_\gamma{\downarrow}B$. Finally, from Corollary 7, we can assume $R_\gamma{\downarrow}A = R_\alpha{\downarrow}{\uparrow}(R_\gamma{\downarrow}A)$ and $R_\gamma{\downarrow}B = R_\alpha{\downarrow}{\uparrow}(R_\gamma{\downarrow}B)$, and thus $\neg P_\alpha(R_\gamma{\downarrow}A, R_\gamma{\downarrow}B)$.

Finally, we consider *NTP*:

$$NTP^1(A,B) \Rightarrow NTP_\alpha(R_\gamma{\downarrow}A, R_\gamma{\downarrow}B) \tag{E.6}$$

From $NTP^1(A,B)$, we know that $A \subseteq i(B)$. Furthermore, as A (resp. B), is the union of a finite number of closed, non-degenerate intervals $A_1, A_2, \ldots, A_s$ (resp. $B_1, B_2, \ldots, B_t$), for each $A_i = [a_i^-, a_i^+]$ there exists a $B_j = [b_j^-, b_j^+]$ such that $[a_i^-, a_i^+] \subseteq]b_j^-, b_j^+[$. This implies that for sufficiently small γ, we also have $[a_i^- + \gamma, a_i^+ - \gamma] \subseteq]b_j^- + \gamma, b_j^+ - \gamma[$, which in turn implies that $[a_i^- + \gamma - \alpha, a_i^+ - \gamma + \alpha] \subseteq [b_j^- + \gamma, b_j^+ - \gamma]$ if $\alpha < \min(b_j^+ - \gamma - a_i^+ + \gamma, a_i^- + \gamma - b_j^- - \gamma) = \min(b_j^+ - a_i^+, a_i^- - b_j^-)$. Using Lemma 13, this implies in particular that $R_\alpha{\uparrow}(R_\gamma{\downarrow}A) \subseteq R_\gamma{\downarrow}B$. Finally, from Corollary 7, we know that for sufficiently small α, $R_\gamma{\downarrow}B = R_\alpha{\downarrow}{\uparrow}(R_\gamma{\downarrow}B)$, and thus $NTP_\alpha(R_\gamma{\downarrow}A, R_\gamma{\downarrow}B)$. To show the implication in the opposite direction, we find from $\neg NTP^1(a,b)$ that $\neg(A \subseteq i(B))$, which for sufficiently small α and γ implies $\neg(R_\alpha{\uparrow}(R_\gamma{\downarrow}A) \subseteq i(R_\alpha{\uparrow}(R_\gamma{\downarrow}B)))$ by Lemma 13 and Lemma 14. In other words, there must exist a p in $\mathbb{R}$ such that $p \in R_\alpha{\uparrow}(R_\gamma{\downarrow}A)$ and $p \notin i(R_\alpha{\uparrow}(R_\gamma{\downarrow}B))$. This means that

$$(\forall\varepsilon > 0)(\exists q \in \mathbb{R})\big(d(p,q) \leqslant \varepsilon \wedge q \notin R_\alpha{\uparrow}(R_\gamma{\downarrow}B)\big)$$

and in particular

$$(\exists q \in \mathbb{R})\big(d(p,q) \leqslant \alpha \wedge q \notin R_\alpha{\uparrow}(R_\gamma{\downarrow}B)\big)$$

or

$$\neg(\forall q \in \mathbb{R})\big(d(p,q) \leqslant \alpha \Rightarrow q \in R_\alpha{\uparrow}(R_\gamma{\downarrow}B)\big)$$

and therefore $p \notin R_\alpha{\downarrow}{\uparrow}(R_\gamma{\downarrow}B)$, while $p \in R_\alpha{\uparrow}(R_\gamma{\downarrow}A)$, which means that $\neg NTP_\alpha(R_\gamma{\downarrow}A, R_\gamma{\downarrow}B)$.

Appendix F. Proof of Proposition 8

Lemma 15. *Let $a < b$, $c < d$ and let H and G be the m-dimensional hypercubes associated with $[a,b]$ and $[c,d]$. It holds that*

$$\inf_{p \in [a,b],\, q \in [c,d]} d(p,q) = \inf_{p \in H,\, q \in Q} d(p,q)$$

where the notation d is both used to refer to the Euclidean distance in $\mathbb{R}$ and $\mathbb{R}^m$.

Proof. Clearly, for every p in $[a,b]$ and q in $[c,d]$, it holds that

$$d(p,q) = d\big((p,0,0,\ldots,0),(q,0,0,\ldots,0)\big)$$

and moreover $(p,0,0,\ldots,0) \in H$ and $(q,0,0,\ldots,0) \in G$. Hence, we already have

$$\inf_{p \in [a,b],\, q \in [c,d]} d(p,q) \geqslant \inf_{p \in H,\, q \in Q} d(p,q)$$

On the other hand, let $(p_1, p_2, \ldots, p_m) \in H$ and $(q_1, q_2, \ldots, q_m) \in G$. We have that

$$\begin{aligned} d\big((p_1, p_2, \ldots, p_m), (q_1, q_2, \ldots, q_m)\big) &= \sqrt{d(p_1, q_1)^2 + d(p_2, q_2)^2 + \cdots + d(p_m, q_m)^2} \\ &\geqslant d(p_1, q_1) \end{aligned}$$

and moreover $p_1 \in [a, b]$ and $q_1 \in [c, d]$, hence

$$\inf_{p \in [a,b],\, q \in [c,d]} d(p, q) \leqslant \inf_{p \in H,\, q \in Q} d(p, q) \qquad \square$$

Lemma 16. *Let $\mathcal{I}$ be an $(m; \alpha, 0)$-model of a set of fuzzy RCC formulas Θ ($\alpha > 0$) and let V be the set of variables occurring in Θ. Furthermore, let $\mathcal{I}'$ be the $(m; \alpha, 0)$-interpretation defined for v in V as*

$$v^{\mathcal{I}'} = R_\alpha {\downarrow}{\uparrow} v^{\mathcal{I}}$$

Then $\mathcal{I}'$ is a model of Θ.

Proof. First note that for all fuzzy sets A and B in $\mathbb{R}^m$, it holds that

$$C_\alpha(A, B) = C_\alpha(R_\alpha {\downarrow}{\uparrow} A, R_\alpha {\downarrow}{\uparrow} B)$$

Since T_W is a continuous operation, it holds that $\sup_{i \in I} T_W(a_i, b) = T_W(\sup_{i \in I} a_i, b)$ for every family $(a_i)_{i \in I}$ in $[0, 1]$ and every b in $[0, 1]$. Together with (1), this yields

$$\begin{aligned} C_\alpha(R_\alpha {\downarrow}{\uparrow} A, R_\alpha {\downarrow}{\uparrow} B) &= \sup_{p, q \in \mathbb{R}^m} T_W\big((R_\alpha {\downarrow}{\uparrow} A)(p), R_\alpha(p, q), (R_\alpha {\downarrow}{\uparrow} B)(q)\big) \\ &= \sup_{q \in \mathbb{R}^m} T_W\Big(\sup_{p \in \mathbb{R}^m} T_W\big((R_\alpha {\downarrow}{\uparrow} A)(p), R_\alpha(p, q)\big), (R_\alpha {\downarrow}{\uparrow} B)(q)\Big) \\ &= \sup_{q \in \mathbb{R}^m} T_W\big((R_\alpha {\uparrow}{\downarrow}{\uparrow} A)(q), (R_\alpha {\downarrow}{\uparrow} B)(q)\big) \\ &= \sup_{q \in \mathbb{R}^m} T_W\big((R_\alpha {\uparrow} A)(q), (R_\alpha {\downarrow}{\uparrow} B)(q)\big) \\ &= \sup_{q \in \mathbb{R}^m} T_W\Big(\sup_{p \in \mathbb{R}^m} T_W\big(R_\alpha(p, q), A(p)\big), (R_\alpha {\downarrow}{\uparrow} B)(q)\Big) \\ &= \sup_{p \in \mathbb{R}^m} T_W\Big(A(p), \sup_{q \in \mathbb{R}^m} T_W\big(R_\alpha(p, q), (R_\alpha {\downarrow}{\uparrow} B)(q)\big)\Big) \\ &= \sup_{p \in \mathbb{R}^m} T_W\big(A(p), (R_\alpha {\uparrow}{\downarrow}{\uparrow} B)(p)\big) \\ &= \sup_{p \in \mathbb{R}^m} T_W\big(A(p), (R_\alpha {\uparrow} B)(p)\big) \\ &= \sup_{p \in \mathbb{R}^m} T_W\Big(A(p), \sup_{q \in \mathbb{R}^m} T_W\big(R_\alpha(p, q), B(q)\big)\Big) \\ &= C_\alpha(A, B) \end{aligned}$$

From (2), we immediately find

$$O_\alpha(A, B) = O_\alpha(R_\alpha {\downarrow}{\uparrow} A, R_\alpha {\downarrow}{\uparrow} B)$$

$$P_\alpha(A, B) = P_\alpha(R_\alpha {\downarrow}{\uparrow} A, R_\alpha {\downarrow}{\uparrow} B)$$

Finally, from (1) and (2), we have

$$NTP_\alpha(A, B) = NTP_\alpha(R_\alpha {\downarrow}{\uparrow} A, R_\alpha {\downarrow}{\uparrow} B) \qquad \square$$

Lemma 17. *Let H be an m-dimensional hypercube and $\alpha > 0$. It holds that*

$$R_\alpha {\downarrow}{\uparrow} H = H$$

Proof. From Lemma 2, we already know

$$R_\alpha {\downarrow}{\uparrow} H \supseteq H$$

If $R_\alpha {\downarrow}{\uparrow} H \supset H$, there would be a p in $\mathbb{R}^m$ such that $p \in R_\alpha {\downarrow}{\uparrow} H$ but $p \notin H$. From $p \notin H$ we derive that there is a q in $\mathbb{R}^m$ such that $d(p, q) = \alpha$ and $d(q, h) > \alpha$ for every h in H. However, since $p \in R_\alpha {\downarrow}{\uparrow} H$ we have $q \in R_\alpha {\uparrow} H$, or $d(q, h) \leqslant \alpha$ for some h in H, a contradiction. $\square$

Lemma 18. *Let $\alpha > 0$, $a < b$, $c < d$ and let H and G be the m-dimensional hypercubes associated with $[a,b]$ and $[c,d]$. It holds that*

$$[a-\alpha, b+\alpha] \subseteq [c,d] \Leftrightarrow R_\alpha \uparrow H \subseteq G$$

Proof. First, we show

$$[a-\alpha, b+\alpha] \subseteq [c,d] \Rightarrow R_\alpha \uparrow H \subseteq G$$

Assume that $[a-\alpha, b+\alpha] \subseteq [c,d]$ and that for some $p = (p_1, \ldots, p_m)$ in $\mathbb{R}^m$, it holds that $p \in R_\alpha \uparrow H$. From $p \in R_\alpha \uparrow H$, we know that for some $h = (h_1, \ldots, h_m)$ in H, it holds that $d(p,h) \leqslant \alpha$, and therefore $|p_1 - h_1| \leqslant \alpha$, implying $p_1 \in [a-\alpha, b+\alpha]$ and in particular $p_1 \in [c,d]$. From $p \in R_\alpha \uparrow H$ we derive $d(p,p') \leqslant \frac{b-a}{2} + \alpha$, where $p' = (p_1, 0, 0, \ldots, 0)$. Furthermore, from $[a-\alpha, b+\alpha] \subseteq [c,d]$, we have $\frac{b-a}{2} + \alpha = \frac{b+\alpha-(a-\alpha)}{2} \leqslant \frac{d-c}{2}$, which entails $p \in G$.

The implication in the opposite direction follows trivially from the fact that for every p in $[a-\alpha, b+\alpha]$, it holds that $(p, 0, 0, \ldots, 0)$ is in $R_\alpha \uparrow H$ and therefore in G, if $R_\alpha \uparrow H \subseteq G$, and that for every point in G, the first coordinate is in $[c,d]$. $\square$

For an arbitrary $\alpha > 0$, we know from the discussion above that Θ has an $(1;\alpha,0)$-model $\mathcal{I}$ in which each variable v is mapped to a fuzzy set in $\mathbb{R}$, taking only membership degrees from $M_{\frac{1}{k}}$. Moreover, this fuzzy set is characterized by k α-level sets, which are all the union of a finite number of closed, non-degenerate intervals. In particular, let the $\frac{i}{k}$-level set of $v^{\mathcal{I}}$ be given by

$$\left[v_{i1}^-, v_{i1}^+\right] \cup \left[v_{i2}^-, v_{i2}^+\right] \cup \left[v_{in_i}^-, v_{n_i}^+\right]$$

Without loss of generality, we can assume that $R_\alpha \downarrow \uparrow v^{\mathcal{I}} = v^{\mathcal{I}}$. Indeed, if this were not the case, we could transform $\mathcal{I}$ as in Lemma 16, yielding $R_\alpha \downarrow \uparrow v^{\mathcal{I}} = v^{\mathcal{I}}$ by (2).

We can now define an $(m,\alpha,0)$-interpretation $\mathcal{I}'$ mapping the variable v to the fuzzy set in $\mathbb{R}^m$ which takes only membership degrees from $M_{\frac{1}{k}}$ and whose $\frac{i}{k}$-level set is defined by the union of a finite number of m-dimensional hypercubes

$$H_1^i \cup H_2^i \cup \ldots, H_{n_i}^i$$

where H_j^i is the hypercube associated with $[v_{ij}^-, v_{ij}^+]$. First, we show $C_\alpha(v^{\mathcal{I}}, u^{\mathcal{I}}) = C_\alpha(v^{\mathcal{I}'}, u^{\mathcal{I}'})$. Since $v^{\mathcal{I}}$ and $u^{\mathcal{I}}$ only take membership degrees from $M_{\frac{1}{k}}$, we know that there is a p and q in $\mathbb{R}$ such that $d(p,q) \leqslant \alpha$ and

$$C_\alpha\left(v^{\mathcal{I}}, u^{\mathcal{I}}\right) = T\left(v^{\mathcal{I}}(p), u^{\mathcal{I}}(q)\right)$$

From Lemma 15, we know that there are p' and q' in $\mathbb{R}^m$ such that $d(p',q') \leqslant \alpha$ and

$$T\left(v^{\mathcal{I}}(p), u^{\mathcal{I}}(q)\right) = T\left(v^{\mathcal{I}'}(p'), u^{\mathcal{I}'}(q')\right) \tag{F.1}$$

which already implies $C_\alpha(v^{\mathcal{I}}, u^{\mathcal{I}}) \leqslant C_\alpha(v^{\mathcal{I}'}, u^{\mathcal{I}'})$.

We also have the converse: there are p' and q' in $\mathbb{R}^m$ satisfying $d(p',q') \leqslant \alpha$ and

$$C_\alpha\left(v^{\mathcal{I}'}, u^{\mathcal{I}'}\right) = T\left(v^{\mathcal{I}'}(p'), u^{\mathcal{I}'}(q')\right)$$

and by Lemma 15 we know that there are p and q in $\mathbb{R}$ satisfying (F.1). We can conclude

$$C_\alpha\left(v^{\mathcal{I}}, u^{\mathcal{I}}\right) = C_\alpha\left(v^{\mathcal{I}'}, u^{\mathcal{I}'}\right) \tag{F.2}$$

Note that from $R_\alpha \downarrow \uparrow v^{\mathcal{I}} = v^{\mathcal{I}}$, we easily find $R_\alpha \downarrow \uparrow v^{\mathcal{I}'} = v^{\mathcal{I}'}$ using Lemma 17. This leads to

$$O_\alpha\left(v^{\mathcal{I}}, u^{\mathcal{I}}\right) = overl\left(v^{\mathcal{I}}, u^{\mathcal{I}}\right)$$
$$O_\alpha\left(v^{\mathcal{I}'}, u^{\mathcal{I}'}\right) = overl\left(v^{\mathcal{I}'}, u^{\mathcal{I}'}\right)$$

and $O_\alpha(v^{\mathcal{I}}, u^{\mathcal{I}}) = O_\alpha(v^{\mathcal{I}'}, u^{\mathcal{I}'})$ follows in the same way as (F.2). Furthermore, $P_\alpha(v^{\mathcal{I}}, u^{\mathcal{I}}) = P_\alpha(v^{\mathcal{I}'}, u^{\mathcal{I}'})$ follows from the fact that $[a,b] \subseteq [c,d]$ iff $H \subseteq G$, where H and G are the associated hypercubes of $[a,b]$ and $[c,d]$ respectively. Finally, $NTP_\alpha(v^{\mathcal{I}}, u^{\mathcal{I}}) = NTP_\alpha(v^{\mathcal{I}'}, u^{\mathcal{I}'})$ follows from Lemma 18.

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