Fuzzifying Allen's Temporal Interval Relations

Steven Schockaert, Martine De Cock, and Etienne E. Kerre

***Abstract*—When the time span of an event is imprecise, it can be represented by a fuzzy set, called a fuzzy time interval. In this paper, we propose a framework to represent, compute, and reason about temporal relationships between such events. Since our model is based on fuzzy orderings of time points, it is not only suitable to express precise relationships between imprecise events ("Roosevelt died *before* the beginning of the Cold War") but also imprecise relationships ("Roosevelt died *just* before the beginning of the Cold War"). We show that, unlike previous models, our model is a generalization that preserves many of the properties of the 13 relations Allen introduced for crisp time intervals. Furthermore, we show how our model can be used for efficient fuzzy temporal reasoning by means of a transitivity table. Finally, we illustrate its use in the context of question answering systems.**

***Index Terms*—Fuzzy ordering, fuzzy relation, interval algebra, question answering, temporal reasoning.**

I. Introduction

TEMPORAL representation and reasoning is an important facet in the design of many intelligent systems. For example, question answering systems require at least some basic temporal representation scheme to answer simple temporal questions such as "When was Franklin Roosevelt born?" To enable question answering systems to answer more complex temporal questions, considerable effort has been made to extract temporal information from natural language texts (e.g., [1], [12], [15], [16], and [23]–[25]) and to analyze complex temporal questions (e.g., [22]). However, temporal relationships expressed in natural language are often vague, e.g., "Roosevelt died *just* before the end of the Second World War." Moreover, historic time periods are more often than not characterized by a gradual beginning and/or ending [17]. The traditional temporal reasoning formalisms need to be extended to cope with this kind of vagueness, which is inherently associated with real–world temporal information.

One of those well-known formalisms is Allen's temporal interval algebra [3]. Allen defined a set of 13 qualitative relations that may hold between two compact intervals $A = [a^-, a^+]$ and $B = [b^-, b^+]$. Table I shows how Allen expressed these precise relations by means of constraints on the boundaries of the crisp intervals involved. In this paper, we extend Allen's work to a more general formalism that can handle precise as well as imprecise relationships between crisp and fuzzy intervals.

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TABLE I
Allen's Temporal Interval Relations Between Intervals $A = [a^-, a^+]$ and $B = [b^-, b^+]$

Name	Definition
before	$b(A,B) \equiv a^+ < b^-$
overlaps	$o(A,B) \equiv a^- < b^-$ and $b^- < a^+$ and $a^+ < b^+$
during	$d(A,B) \equiv b^- < a^-$ and $a^+ < b^+$
meets	$m(A,B) \equiv a^+ = b^-$
starts	$s(A,B) \equiv a^- = b^-$ and $a^+ < b^+$
finishes	$f(A,B) \equiv a^+ = b^+$ and $b^- < a^-$
equals	$e(A,B) \equiv a^- = b^-$ and $a^+ = b^+$
after	$bi(A,B) \equiv b(B,A)$
overlapped-by	$oi(A,B) \equiv o(B,A)$
contains	$di(A,B) \equiv d(B,A)$
met–by	$mi(A,B) \equiv m(B,A)$
started–by	$si(A,B) \equiv s(B,A)$
finished–by	$fi(A,B) \equiv f(B,A)$

Our first concern is generalizing the definitions of the qualitative relations of Table I to make them applicable to fuzzy intervals as opposed to only crisp intervals. Indeed, when an event is characterized by a gradual beginning and/or ending, it is natural to represent the corresponding time span as a fuzzy set, which we call a fuzzy (time) interval. Depending on the intended application, this fuzzy set can either be defined by an expert (e.g., [17] and [20]) or constructed automatically (e.g., [27]). Since we cannot refer to the gradual beginning and endings of a fuzzy interval in the same way we refer to the boundaries of a crisp interval, we first have to provide a way to express that, for instance, the beginning of a fuzzy interval A is before the beginning of a fuzzy interval B (as needed in the definition of the qualitative relation "overlaps"). We suggest to do this by measuring the highest extent to which there exists a time point in A that occurs before all the time points in B. In general, in our approach, qualitative relations between fuzzy intervals are defined in terms of the ordering of the gradual beginning and endings of these intervals, which in turn are defined in terms of the ordering of the time points belonging to these intervals. The resulting qualitative relations between the fuzzy intervals are gradual, i.e., they may hold to some degree only; hence the name fuzzy temporal interval relations. When A and B are crisp, our approach reduces to Allen's work.

Our second goal is providing a means to model *imprecise* relations to be able to express that, for instance, event A took place *just* before event B, or that A occurred *long* after B. Although these kind of relations are not considered in Allen's original model, in our approach we arrive at them quite elegantly by using imprecise orderings of time points in the model sketched above. The resulting approach is applicable again to both crisp and fuzzy time intervals.

This paper is organized as follows. In the next section, we review related work concerning fuzzifications of Allen's interval relations. In Section III, we show how imprecise relationships between time points can be modelled by using fuzzy orderings. In Section IV, we rely on relatedness measures for fuzzy sets to lift these imprecise orderings of time points into relationships between fuzzy time intervals [29]. This results in a generalization of Allen's 13 interval relations that are also applicable when the time intervals are fuzzy. Furthermore, this framework is powerful enough to additionally express imprecise relationships that are not considered in Allen's original model. We show that our model preserves many important properties regarding (ir)reflexivity, (a)symmetry, and transitivity, and that our generalized definitions remain mutual exclusive and exhaustive. Moreover, in Section V, we discuss fuzzy temporal reasoning and introduce a transitivity table to derive new temporal knowledge in an efficient way. This transitivity table is a generalization of the transitivity table that was introduced by Freksa in [13], which shows that no transitivity properties are lost in our generalized framework. Section VI illustrates the usefulness of our approach within the context of question answering systems. Sections III and IV contain many new results that require a mathematical proof. To preserve the continuity of the main text, we present these proofs in the Appendixes.

II. Related Work

Most fuzzifications of Allen's interval algebra deal with uncertainty rather than imprecision (e.g., [4], [9]–[11], and [14]). These approaches assume that—in the face of complete knowledge—the time span of an event can always be modelled as a crisp (time) interval. For example, Dubois and Prade [9] represent a time interval as a pair of possibility distributions that define the possible values of the endpoints of the crisp interval. Using possibility theory, the possibility and necessity of each of the interval relations can then be calculated. This approach also allows to model imprecise relations such as "A was *long* before B." In a different approach adopted by Dutta [11], time intervals are abstract entities and the possibility, for each interval i and each event e, that e occurs in i is defined. In [4], uncertainty regarding the temporal relations that hold between crisp time intervals is considered in order to reason with statements such as "the possibility that $m(A, B)$ holds is 0.6." Guesgen *et al.* [14] proposed a similar approach based on the notion of a conceptual neighborhood, a notion originally introduced in [13].

Temporal information is expressed with respect to a certain level of granularity (e.g., years, days, seconds, etc.), which partitions the timeline. In [7], it is argued that the time span of events often skews to the cells of this partitioning. Therefore, a rough set approach is adopted in which the time span of an event is represented by a lower approximation consisting of 1) the cells of the partitioning that are fully included in this time span and 2) an upper approximation consisting of the cells of the partitioning that at least partially overlap with this time span. The temporal interval relations are redefined, using a directed variant of the region connection calculus (RCC) [21], to cope with these "rough time intervals."

In [6], it is suggested to represent time intervals as fuzzy sets, but no definitions of the interval relations are given. Most relevant to our approach are definitely the work of Nagypál and Motik [17] and of Ohlbach [20], which are concerned with generalizing Allen's interval relations when the time span of an event is represented as a fuzzy set. However, these approaches suffer from a number of important disadvantages. For example, the relation "equals" defined in [17] is not reflexive in general; for a continuous fuzzy set A in $\mathbb{R}$, $e(A, A) = s(A, A) = f(A, A) = d(A, A) = 0.5$, while, taking into account Allen's intended meaning of these relations, one would expect $e(A, A) = 1$ and $s(A, A) = f(A, A) = d(A, A) = 0$. Moreover, imprecise temporal relations cannot be expressed. An approach similar to [17] was suggested in [8] within the context of ranking fuzzy numbers. Ohlbach [20] suggests an alternative approach that allows one to model imprecise temporal relations such as "A *more or less* finishes B" based on measures of overlap for fuzzy sets. However, this approach cannot handle imprecise temporal relations such as "A was *long* before B." Moreover, as pointed out in [20], many of the (ir)reflexivity, (a)symmetry, and transitivity properties of the original temporal relations are lost in this approach; hence it is not suitable for temporal reasoning. Imprecise temporal relations are also considered in [10]; however, only crisp intervals are considered in this approach.

III. Fuzzy Ordering of Time Points

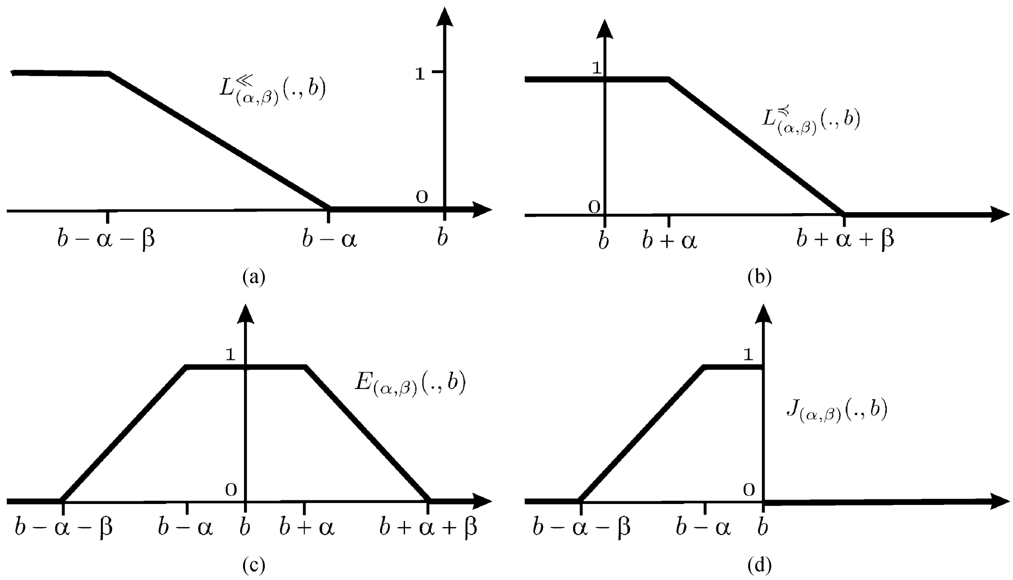
A. *Definitions*

The fuzzy temporal interval relations that we will define in the next section are based on orderings between the time points contained in the intervals. Throughout this paper, we represent time points as real numbers. A real number can, for example, be interpreted as the number of milliseconds since January 1, 1970, or the number of years since 1900. Because we want to model imprecise temporal relations, we need a way to express for two time points a and b that a is *long* before b, that a is *just* before b, and that a and b occur at *approximately* the same time.

Let $\alpha \in \mathbb{R}$ and $\beta \in [0, +\infty[$. Then the extent to which a is long before b (with respect to (α, β)) can be expressed by the fuzzy relation $L^{\ll}_{(\alpha,\beta)}$ in $\mathbb{R}$ defined as [9]

$$L^{\ll}_{(\alpha,\beta)}(a, b) = \begin{cases} 1, & \text{if } b - a > \alpha + \beta \\ 0, & \text{if } b - a \le \alpha \\ \frac{b-a-\alpha}{\beta}, & \text{otherwise} \end{cases} \quad (1)$$

for all a and b in $\mathbb{R}$. The partial mapping $L^{\ll}_{(\alpha,\beta)}(., b)$ is depicted in Fig. 1(a). The parameters α and β define how the concept "long before" should be interpreted. For a time point to be long before b to degree one, the time gap with b should be at least $\alpha + \beta$. If the time gap with b is smaller than α, the time point is long before b to degree zero. In between there is a gradual transition. Although it seems natural to impose that α is positive, for technical reasons we only require $\alpha \in \mathbb{R}$. Moreover, as pointed out by Ohlbach [20], in some applications it may be desirable for some $a > b$ to express that a is (long) before b to a (small) strictly positive degree, which is only possible in our approach if we allow negative values of α. $L^{\ll}_{(\alpha,\beta)}$ is a generalization of the crisp strict ordering relation $<$. Indeed, imposing $\alpha = \beta = 0$,



$L^{\ll}_{(\alpha,\beta)}(a,b)$	a occurs long before b
$L^{\preccurlyeq}_{(\alpha,\beta)}(a,b)$	a occurs before or at approximately the same time as b
$E_{(\alpha,\beta)}(a,b)$	a occurs at approximately the same time as b
$J_{(\alpha,\beta)}(a,b)$	a occurs just before b

(e)

Fig. 1. Fuzzy ordering of time points. (a) $L^{\ll}_{(\alpha,\beta)}(.,b)$: fuzzy set of time points long before b. (b) $L^{\preccurlyeq}_{(\alpha,\beta)}(.,b)$: fuzzy set of time points before or at approximately the same time as b. (c) $E_{(\alpha,\beta)}(.,b)$: fuzzy set of time points at approximately the same time as b. (d) $J_{(\alpha,\beta)}(.,b)$: fuzzy set of time points just before b. (e) Overview of fuzzy relations between time points a and b.

we obtain $L^{\ll}_{(0,0)}(a,b) = 1$ if $a < b$ and $L^{\ll}_{(0,0)}(a,b) = 0$ otherwise.

The fuzzy relation $L^{\preccurlyeq}_{(\alpha,\beta)}$ in $\mathbb{R}$ is defined as [9]

$$L^{\preccurlyeq}_{(\alpha,\beta)}(a,b) = 1 - L^{\ll}_{(\alpha,\beta)}(b,a) \tag{2}$$

for all a and b in $\mathbb{R}$. $L^{\preccurlyeq}_{(\alpha,\beta)}(a,b)$ represents the extent to which b is not "long before" a (with respect to (α,β)), in other words, the extent to which a is before or at approximately the same time as b. It holds that

$$L^{\preccurlyeq}_{(\alpha,\beta)}(a,b) = \begin{cases} 1, & \text{if } b-a \geq -\alpha \\ 0, & \text{if } b-a < -\alpha-\beta \\ \frac{b-a+\alpha+\beta}{\beta}, & \text{otherwise} \end{cases}. \tag{3}$$

Moreover, $L^{\preccurlyeq}_{(0,0)}(a,b) = 1$ if $a \leq b$ and $L^{\preccurlyeq}_{(0,0)}(a,b) = 0$ otherwise, i.e., $L^{\preccurlyeq}_{(\alpha,\beta)}$ is a generalization of the crisp ordering $\leq$.

As will become clear in Section IV, we only need the fuzzy relations $L^{\ll}_{(\alpha,\beta)}$ and $L^{\preccurlyeq}_{(\alpha,\beta)}$ to model imprecise temporal interval relations. The degree $E_{(\alpha,\beta)}(a,b)$ to which a occurs at approximately the same time as b, and the degree $J_{(\alpha,\beta)}(a,b)$ to which a is just before b, can easily be expressed using $L^{\ll}_{(\alpha,\beta)}$ and $L^{\preccurlyeq}_{(\alpha,\beta)}$, i.e.,

$$E_{(\alpha,\beta)}(a,b) = \min(L^{\preccurlyeq}_{(\alpha,\beta)}(a,b), L^{\preccurlyeq}_{(\alpha,\beta)}(b,a)) \tag{4}$$

$$J_{(\alpha,\beta)}(a,b) = \min(E_{(\alpha,\beta)}(a,b), L^{\ll}_{(0,0)}(a,b)). \tag{5}$$

An overview of the four fuzzy relations between time points is given in Fig. 1.

Example 1: Assume that a time point corresponds to the number of years since January 1900. Using $\alpha = 2$ and $\beta = 8$, we obtain, for example

$$L^{\ll}_{(2,8)}(20,23) = 0.125 \quad E_{(2,8)}(20,23) = 0.875$$
$$L^{\preccurlyeq}_{(2,8)}(20,23) = 1 \quad J_{(2,8)}(20,23) = 0.875$$

expressing that 20 occurred long before 23 to a low degree, that 20 occurred just before 23 to a high degree, etc. On the other hand, we also have

$$L^{\ll}_{(2,8)}(23,20) = 0 \quad E_{(2,8)}(23,20) = 0.875$$
$$L^{\preccurlyeq}_{(2,8)}(23,20) = 0.875 \quad J_{(2,8)}(23,20) = 0.$$

In other words, although 23 is not considered to be long before or just before 20, it is still considered to be before or at approximately the same time as 20 to a high degree.

B. Properties

The fuzzy relations $L^{\ll}_{(\alpha,\beta)}$ and $L^{\preccurlyeq}_{(\alpha,\beta)}$ behave as can be intuitively expected from orderings. First recall that, in general, a fuzzy relation R in a universe U is called, for an arbitrary triangular norm T:

1) reflexive iff $R(u,u) = 1$ for all u in U;
2) irreflexive iff $R(u,u) = 0$ for all u in U;
3) symmetric iff $R(u,v) = R(v,u)$ for all u and v in U;
4) T–asymmetric iff $T(R(u,v), R(v,u)) = 0$ for all u and v in U;
5) T–transitive iff $T(R(u,v), R(v,w)) \leq R(u,w)$ for all u, v and w in U.

Furthermore, if a fuzzy relation E in U is reflexive, symmetric, and T–transitive, E is called a fuzzy T–equivalence relation. A fuzzy T–E–ordering relation [8] is then defined as a T–transitive fuzzy relation L in U, which is:

1) E -reflexive, i.e., $E(u,v) \leq L(u,v)$ for all u and v in U;
2) T-E-antisymmetric, i.e., $T(L(u,v),L(v,u)) \leq E(u,v)$ for all u and v in U

Transitivity of the fuzzy orderings is of particular importance for temporal reasoning. Many interesting properties regarding the transitivity of $L^{\ll}_{(\alpha,\beta)}$ and $L^{\preccurlyeq}_{(\alpha,\beta)}$ follow from an important characterization of their composition. Recall that, in general, the sup–T composition of two fuzzy relations R and S in U is the fuzzy relation $R \circ_T S$ in U defined for each u and w in U by

$$(R \circ_T S)(u,w) = \sup_{v \in U} T(R(u,v), S(v,w)). \tag{6}$$

Throughout this paper, we use T_W to denote the Łukasiewicz t–norm, i.e.,

$$T_W(x,y) = \max(0, x+y-1)$$

for all x and y in [0, 1].

Proposition 1 (Composition): Let $\alpha_1, \alpha_2 \in \mathbb{R}$ and $\beta_1, \beta_2 \geq 0$; it holds that

$$L^{\ll}_{\gamma_1} \circ_{T_W} L^{\ll}_{\gamma_2} = L^{\ll}_{(\alpha_1+\alpha_2+\min(\beta_1,\beta_2),\max(\beta_1,\beta_2))} \tag{7}$$
$$L^{\preccurlyeq}_{\gamma_1} \circ_{T_W} L^{\preccurlyeq}_{\gamma_2} = L^{\preccurlyeq}_{(\alpha_1+\alpha_2,\max(\beta_1,\beta_2))} \tag{8}$$
$$L^{\preccurlyeq}_{\gamma_1} \circ_{T_W} L^{\ll}_{\gamma_2} = L^{\ll}_{(\alpha_2-\alpha_1+\min(\beta_1,\beta_2)-\beta_1,\max(\beta_1,\beta_2))} \tag{9}$$
$$L^{\ll}_{\gamma_1} \circ_{T_W} L^{\preccurlyeq}_{\gamma_2} = L^{\ll}_{(\alpha_1-\alpha_2+\min(\beta_1,\beta_2)-\beta_2,\max(\beta_1,\beta_2))} \tag{10}$$

where $\gamma_1 = (\alpha_1, \beta_1)$ and $\gamma_2 = (\alpha_2, \beta_2)$.

For $\alpha = 0$, we obtain the following interesting corollary.

Corollary 1: For $\beta \geq 0$ and a, b, and c in $\mathbb{R}$

$$T_W(L^{\ll}_{(0,\beta)}(a,b), L^{\ll}_{(0,\beta)}(b,c)) \leq L^{\ll}_{(0,\beta)}(a,c) \tag{11}$$
$$T_W(L^{\preccurlyeq}_{(0,\beta)}(a,b), L^{\preccurlyeq}_{(0,\beta)}(b,c)) \leq L^{\preccurlyeq}_{(0,\beta)}(a,c) \tag{12}$$
$$T_W(L^{\preccurlyeq}_{(0,\beta)}(a,b), L^{\ll}_{(0,\beta)}(b,c)) \leq L^{\ll}_{(0,\beta)}(a,c) \tag{13}$$
$$T_W(L^{\ll}_{(0,\beta)}(a,b), L^{\preccurlyeq}_{(0,\beta)}(b,c)) \leq L^{\ll}_{(0,\beta)}(a,c). \tag{14}$$

Equation (11) expresses that $L^{\ll}_{(0,\beta)}$ is T_W–transitive while (12) says that $L^{\preccurlyeq}_{(0,\beta)}$ is T_W–transitive. Furthermore, (13) and (14) express a mixed transitivity between $L^{\ll}_{(0,\beta)}$ and $L^{\preccurlyeq}_{(0,\beta)}$, generalizing that from $a \leq b$ and $b < c$, it follows that $a < c$, and similarly, that from $a < b$ and $b \leq c$, it follows that $a < c$. Corollary 1 and hence Proposition 1 do not hold for an arbitrary triangular norm T in general.

For $\alpha \geq 0$, $L^{\preccurlyeq}_{(\alpha,\beta)}$ is reflexive and $L^{\ll}_{(\alpha,\beta)}$ is irreflexive. The following corollary results from the obvious reflexivity and symmetry of $E_{(\alpha,\beta)}$ and Corollary 1.

Corollary 2: $E_{(0,\beta)}$ is a fuzzy T_W–equivalence relation.

From (4), we obtain the T_W–$E_{(0,\beta)}$–antisymmetry of $L^{\preccurlyeq}_{(0,\beta)}$. Combined with the reflexivity of $E_{(0,\beta)}$, we establish yet another interesting corollary.

TABLE II
RELATION BETWEEN THE BOUNDARIES OF THE CRISP INTERVALS $[a^-, a^+]$ AND $[b^-, b^+]$, AND THE FUZZY INTERVALS A AND B

Crisp	Fuzzy
$a^- < b^-$	$A \circ (L^{\ll}_{(\alpha,\beta)} \rhd B)$
$a^- \leq b^-$	$(A \circ L^{\preccurlyeq}_{(\alpha,\beta)}) \rhd B$
$a^+ < b^+$	$(A \lhd L^{\ll}_{(\alpha,\beta)}) \circ B$
$a^+ \leq b^+$	$A \lhd (L^{\preccurlyeq}_{(\alpha,\beta)} \circ B)$
$a^+ < b^-$	$A \lhd L^{\ll}_{(\alpha,\beta)} \rhd B$
$a^+ \leq b^-$	$A \lhd L^{\preccurlyeq}_{(\alpha,\beta)} \rhd B$
$a^- < b^+$	$A \circ L^{\ll}_{(\alpha,\beta)} \circ B$
$a^- \leq b^+$	$A \circ L^{\preccurlyeq}_{(\alpha,\beta)} \circ B$

Corollary 3: $L^{\preccurlyeq}_{(0,\beta)}$ is a fuzzy T_W–$E_{(0,\beta)}$–ordering.

The following proposition is a generalization of the trichotomy law, stating that if b is long before a, a and b cannot be at approximately the same time and a cannot be before b.

Proposition 2: For $0 \leq \beta_1 \leq \beta_2$ and $\alpha_1 \leq \alpha_2$, it holds that

$$T_W(L^{\preccurlyeq}_{(\alpha_1,\beta_1)}(a,b), L^{\ll}_{(\alpha_2,\beta_2)}(b,a)) = 0 \tag{15}$$

for all a and b in $\mathbb{R}$.

IV. FUZZY TEMPORAL INTERVAL RELATIONS

A. Ordering of Vague Boundaries

We define a *fuzzy time period* as a normalized fuzzy set A in $\mathbb{R}$, which is interpreted as the time span of some event. Recall that a fuzzy set A is called normalized if there exists a u in $\mathbb{R}$ such that $A(u) = 1$. Furthermore, a *fuzzy (time) interval* A is a convex and upper semicontinuous normalized fuzzy set in $\mathbb{R}$, i.e., for each α in]0,1], the α–level set $\{u | A(u) \geq \alpha\}$ is a closed interval.[1] If A and B are fuzzy time intervals, the boundaries of A and B can be gradual. Hence, we cannot refer to these boundaries in the same way we refer to the boundaries of crisp intervals to define temporal relations in the manner of Table I. Nonetheless, we can use the fuzzy orderings between time points defined in the previous section. One possibility to measure, for example, the extent to which the beginning of a fuzzy time interval A is long before the beginning of a fuzzy time interval B is to look at the highest extent to which there exists a time point in A that occurs long before all time points in B. Similarly, for instance, to express the degree to which the beginning of A is before or at the same time as the ending of B, we can use the highest extent to which there exists a time point in A that occurs before or at the same time as some time point in B. This can be accomplished by using relatedness measures, as shown in Table II. For an arbitrary fuzzy relation R in $\mathbb{R}$, and

[1]All the properties of the fuzzy temporal interval relations in this paper are valid for arbitrary fuzzy time periods. Hence from a syntactic point of view, neither convexity nor upper semicontinuity is required. However, from a semantic point of view, it seems natural to consider only temporal interval relations between fuzzy time intervals since the convexity condition is needed to adequately generalize the notion of an interval, while the upper semicontinuity condition reflects the fact that time intervals are closed intervals.

TABLE III
TRANSITIVITY TABLE FOR RELATEDNESS MEASURES

	$B \circ S \circ C$	$B \lhd S \rhd C$	$(B \lhd S) \circ C$
$A \circ R \circ B$	1	$A \circ ((R \circ S) \rhd C)$	$A \circ (R \circ S) \circ C$
$A \lhd R \rhd B$	$(A \lhd (R \circ S)) \circ C$	$A \lhd (R \circ S) \rhd C$	$(A \lhd (R \circ S)) \circ C$
$(A \lhd R) \circ B$	1	$A \lhd (R \circ S) \rhd C$	$(A \lhd (R \circ S)) \circ C$
$A \lhd (R \circ B)$	1	$A \lhd (R \circ S) \rhd C$	$(A \lhd (R \circ S)) \circ C$
$(A \circ R) \rhd B$	$A \circ (R \circ S) \circ C$	$A \circ ((R \circ S) \rhd C)$	$A \circ (R \circ S) \circ C$
$A \circ (R \rhd B)$	$A \circ (R \circ S) \circ C$	$A \circ ((R \circ S) \rhd C)$	$A \circ (R \circ S) \circ C$

	$B \lhd (S \circ C)$	$(B \circ S) \rhd C$	$B \circ (S \rhd C)$
$A \circ R \circ B$	$A \circ (R \circ S) \circ C$	1	1
$A \lhd R \rhd B$	$(A \lhd (R \circ S)) \circ C$	$A \lhd (R \circ S) \rhd C$	$A \lhd (R \circ S) \rhd C$
$(A \lhd R) \circ B$	$(A \lhd (R \circ S)) \circ C$	1	1
$A \lhd (R \circ B)$	$A \lhd ((R \circ S) \circ C)$	1	1
$(A \circ R) \rhd B$	$A \circ (R \circ S) \circ C$	$(A \circ (R \circ S)) \rhd C$	$A \circ ((R \circ S) \rhd C)$
$A \circ (R \rhd B)$	$A \circ (R \circ S) \circ C$	$A \circ ((R \circ S) \rhd C)$	$A \circ ((R \circ S) \rhd C)$

fuzzy sets A and B in $\mathbb{R}$, these relatedness measures are defined as [28]

$$A \circ_T R \circ_T B = \sup_{v \in \mathbb{R}} T(B(v), \sup_{u \in \mathbb{R}} T(A(u), R(u,v))) \quad (16)$$

$$A \lhd_I R \rhd_I B = \inf_{v \in \mathbb{R}} I(B(v), \inf_{u \in \mathbb{R}} I(A(u), R(u,v))) \quad (17)$$

$$(A \lhd_I R) \circ_T B = \sup_{v \in \mathbb{R}} T(B(v), \inf_{u \in \mathbb{R}} I(A(u), R(u,v))) \quad (18)$$

$$A \lhd_I (R \circ_T B) = \inf_{u \in \mathbb{R}} I(A(u), \sup_{v \in \mathbb{R}} T(B(v), R(u,v))) \quad (19)$$

$$(A \circ_T R) \rhd_I B = \inf_{v \in \mathbb{R}} I(B(v), \sup_{u \in \mathbb{R}} T(A(u), R(u,v))) \quad (20)$$

$$A \circ_T (R \rhd_I B) = \sup_{u \in \mathbb{R}} T(A(u), \inf_{v \in \mathbb{R}} I(B(v), R(u,v))) \quad (21)$$

where T is a left-continuous t–norm and I its residual implicator, defined for all x and y in [0, 1] as

$$I(x,y) = \sup\{\lambda \mid T(x,\lambda) \leq y\}.$$

These definitions are closely related to the sup–T composition of fuzzy relations and to the subproduct and superproduct of fuzzy relations [5]. In the remainder of this paper, we will mainly use the Łukasiewicz t–norm T_W and its residual implicator I_W in the definition of the relatedness measures, i.e.,

$$I_W(x,y) = \min(1, 1 - x + y)$$

for all x and y in [0, 1]. When $T = T_W$ and $I = I_W$, we omit the subscripts of $\circ$, $\lhd$, and $\rhd$ in (16)–(21). In the remainder of this section, let R and S be fuzzy relations in $\mathbb{R}$, and A, B, and C fuzzy sets in $\mathbb{R}$. We recall the following three propositions from [28].

Proposition 3 [28]: If A and B are normalized, then

$$A \lhd (R \circ B) \leq A \circ R \circ B \quad (22)$$

$$(A \circ R) \rhd B \leq A \circ R \circ B \quad (23)$$

$$A \lhd R \rhd B \leq (A \lhd R) \circ B \quad (24)$$

$$A \lhd R \rhd B \leq A \circ (R \rhd B) \quad (25)$$

$$(A \lhd R) \circ B \leq A \lhd (R \circ B) \quad (26)$$

$$A \circ (R \rhd B) \leq (A \circ R) \rhd B. \quad (27)$$

Proposition 4 (Reflexivity) [28]: If R is reflexive, then

$$A \lhd (R \circ A) = 1 \quad (28)$$

$$(A \circ R) \rhd A = 1. \quad (29)$$

Proposition 5 (Irreflexivity) [28]: If R is irreflexive, then

$$(A \lhd R) \circ A = 0 \quad (30)$$

$$A \circ (R \rhd A) = 0. \quad (31)$$

Substituting fuzzy time intervals for A and B and either $L^{\preccurlyeq}_{(\alpha,\beta)}$ or $L^{\ll}_{(\alpha,\beta)}$ for R in the propositions above shows that our approach for modeling relations between the vague boundaries of fuzzy time intervals is sound. For example, from (23) and (27), we derive

$$A \circ (L^{\ll}_{(\alpha,\beta)} \rhd B) \leq A \circ L^{\ll}_{(\alpha,\beta)} \circ B.$$

Hence the degree to which the beginning of A is long before the end of B is at least as high as the degree to which the beginning of A is long before the beginning of B. Furthermore, if $\alpha \geq 0$ to ensure the reflexivity of $L^{\preccurlyeq}_{(\alpha,\beta)}$, from (28) and (29), we derive

$$A \lhd (L^{\preccurlyeq}_{(\alpha,\beta)} \circ A) = 1$$

$$(A \circ L^{\preccurlyeq}_{(\alpha,\beta)}) \rhd A = 1.$$

Hence the ending of A is less than or approximately equal to the ending of A to degree one and the beginning of A is less than or approximately equal to the beginning of A to degree one. In the same way, from (30) and (31), we obtain

$$(A \lhd L^{\ll}_{(\alpha,\beta)}) \circ A = 0$$

$$A \circ (L^{\ll}_{(\alpha,\beta)} \rhd A) = 0.$$

In other words, the ending of A is not "long before" the ending of A and the beginning of A is not "long before" the beginning of A. Finally, as a result of the following important proposition, the transitivity behavior of the ordering of the interval boundaries is preserved.

Proposition 6 (Transitivity): For normalized fuzzy sets A, B, and C, the relatedness measures exhibit the transitivity properties displayed in Table III. Let $\mathcal{R}_{(M_1,M_2)}$ be the entry in this table on the row corresponding with the relatedness measure

M_1 and the column corresponding with the relatedness measure M_2. Then it holds that $T_W(M_1, M_2) \leq \mathcal{R}_{(M_1,M_2)}$.

For example, the entry on the sixth line and the third column of Table III should be read as

$$T_W(A \circ (R \rhd B), (B \lhd S) \circ C) \leq A \circ (R \circ S) \circ C.$$

Using Proposition 1, we obtain as a special case

$$T_W(A \circ (L^{\ll}_{(0,0)} \rhd B), (B \lhd L^{\ll}_{(0,0)}) \circ C) \leq A \circ L^{\ll}_{(0,0)} \circ C$$

which generalizes the statement that if the beginning of A is before the beginning of B and the ending of B is before the ending of C, then the beginning of A is before the ending of C. This correspondence with the transitivity behavior of the crisp relations $<$ and $\leq$ also reveals why, for some entries in Table III, we have no information at all, i.e., the entries that equal one. For example, from the fact that the beginning of A is before the ending of B, and the fact that the beginning of B is before the ending of C, we can conclude nothing about the relative positioning of A and C. As a consequence, the entry on the first row, first column equals one.

All results from this section can easily be generalized to an arbitrary universe and an arbitrary left-continuous t–norm. Our commitment to the Łukasiewicz t–norm is mainly motivated by the rich interactions of T_W with $L^{\ll}_{(\alpha,\beta)}$ and $L^{\preccurlyeq}_{(\alpha,\beta)}$, as exemplified by Propositions 1 and 2.

B. Relations Between Fuzzy Time Periods

Using the expressions in Table II, it is straightforward to generalize the temporal interval relations from Table I: using the minimum to generalize the conjunctions in Table I, we obtain the generalized definitions in Table IV. Due to the idempotency of the minimum, using the minimum to combine the different constraints on the vague boundaries in this way seems much more natural than, for example, using the Łukasiewicz t–norm. Moreover, it turns out that this choice of the minimum is a prerequisite for some desirable properties of the fuzzy temporal interval relations, which will be introduced further on in this section.

Note that the definitions in Table IV coincide with Allen's original definitions if each α_i and β_i equals zero and A and B are crisp sets. Quantitative information (A happened at least four years after B) and semiquantitative information (A happened long after B) can be expressed using values α_i or β_i different from zero. The (semi)quantitative information we may have at our disposal about the relative positioning of the beginnings of A and B is independent of the semiquantitative information we may have at our disposal concerning the endings of A and B; hence, the fuzzy relation $d_{\gamma_{12}}(A, B)$ involves two different sets of parameters $\gamma_1 = (\alpha_1, \beta_1)$ and $\gamma_2 = (\alpha_2, \beta_2)$. On the other hand, the two relatedness measures in the definition of $m_{\gamma_1}(A, B)$ together express that the ending of A is approximately equal to the beginning of B; hence the same set of parameters $\gamma_1 = (\alpha_1, \beta_1)$ is used twice. Notice how the notion of approximate equality in the definition of $m_{\gamma_1}(A, B)$ is expressed entirely analogous to the definition of $E_{(\alpha,\beta)}$ in (4) by making use of $L^{\preccurlyeq}_{(\alpha_1,\beta_1)}$.

TABLE IV
FUZZY TEMPORAL INTERVAL RELATIONS. $\gamma_{123} = (\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \gamma_3)$, $\gamma_{12} = (\alpha_1, \beta_1, \alpha_2, \beta_2)$, $\gamma_1 = (\alpha_1, \beta_1)$, $\gamma_2 = (\alpha_2, \beta_2)$, AND $\gamma_3 = (\alpha_3, \beta_3)$

Name	Notation	Definition
before	$b_{\gamma_1}(A, B)$	$A \lhd L^{\ll}_{\gamma_1} \rhd B$
overlaps	$o_{\gamma_{123}}(A, B)$	$\min(A \circ (L^{\ll}_{\gamma_1} \rhd B), B \circ L^{\ll}_{\gamma_2} \circ A, (A \lhd L^{\ll}_{\gamma_3}) \circ B)$
during	$d_{\gamma_{12}}(A, B)$	$\min(B \circ (L^{\ll}_{\gamma_1} \rhd A), (A \lhd L^{\ll}_{\gamma_2}) \circ B)$
meets	$m_{\gamma_1}(A, B)$	$\min(A \lhd L^{\preccurlyeq}_{\gamma_1} \rhd B, B \circ L^{\preccurlyeq}_{\gamma_1} \circ A)$
starts	$s_{\gamma_{12}}(A, B)$	$\min((A \circ L^{\preccurlyeq}_{\gamma_1}) \rhd B, (B \circ L^{\preccurlyeq}_{\gamma_1}) \rhd A, (A \lhd L^{\ll}_{\gamma_2}) \circ B)$
finishes	$f_{\gamma_{12}}(A, B)$	$\min(A \lhd (L^{\preccurlyeq}_{\gamma_1} \circ B), B \lhd (L^{\preccurlyeq}_{\gamma_1} \circ A), B \circ (L^{\ll}_{\gamma_2} \rhd A))$
equals	$e_{\gamma_{12}}(A, B)$	$\min((A \circ L^{\preccurlyeq}_{\gamma_1}) \rhd B, (B \circ L^{\preccurlyeq}_{\gamma_1}) \rhd A, A \lhd (L^{\preccurlyeq}_{\gamma_2} \circ B), B \lhd (L^{\preccurlyeq}_{\gamma_2} \circ A))$

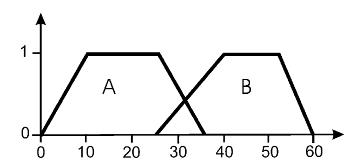


Fig. 2. Fuzzy time intervals A and B.

Example 2: For the fuzzy time intervals A and B displayed in Fig. 2, it holds that $b_{(0,0)}(A, B) = 1$ and $m_{(0,0)}(A, B) = 0$. In other words, A is considered to be fully before B, as the overlap between A and B is too low for $m(A, B)$ to hold to a degree higher than zero. However, it is clear that A also more or less meets B. By increasing the value of α, we apply a stricter definition of "long before" and a more tolerant definition of "approximately at the same time." Hence the degree to which A is long before B decreases and the degree to which A more or less meets B increases. We obtain

$$\begin{aligned} b_{(5,0)}(A, B) &= 0.67 & m_{(5,0)}(A, B) &= 0.33 \\ b_{(10,0)}(A, B) &= 0.33 & m_{(10,0)}(A, B) &= 0.67 \\ b_{(15,0)}(A, B) &= 0 & m_{(15,0)}(A, B) &= 1. \end{aligned}$$

When α is sufficiently large, the end of A is not considered to be long before the beginning of B anymore, hence $b_{(15,0)}(A, B) = 0$. A similar observation can be made when increasing the value of β

$$\begin{aligned} b_{(0,20)}(A, B) &= 0.75 & m_{(0,20)}(A, B) &= 0.25 \\ b_{(0,30)}(A, B) &= 0.5 & m_{(0,30)}(A, B) &= 0.5. \end{aligned}$$

Our generalization preserves several interesting properties of Allen's original algebra, many of which are lost in other approaches. First, Allen's temporal interval relations are jointly exhaustive, which means that between any two time intervals, at least one of the temporal relations holds. For fuzzy time periods we obtain a generalization, using the Łukasiewicz t–conorm S_W defined for all x and y in [0, 1], as

$$S_W(x,y) = \min(x+y,1).$$

Proposition 7 (Exhaustivity): Let A and B be fuzzy time periods. It holds that

$$\begin{aligned} &S_W(b_{\gamma^1}(A,B), bi_{\gamma^1}(A,B), o_{\gamma^3}(A,B), oi_{\gamma^3}(A,B),\\ &\quad d_{\gamma^2}(A,B), di_{\gamma^2}(A,B), m_{\gamma^1}(A,B), mi_{\gamma^1}(A,B),\\ &\quad s_{\gamma^2}(A,B), si_{\gamma^2}(A,B), f_{\gamma^2}(A,B), fi_{\gamma^2}(A,B),\\ &\quad e_{\gamma^2}(A,B)) = 1. \end{aligned} \tag{32}$$

For nondegenerate time intervals, i.e., time intervals $[a^-, a^+]$ with $a^- < a^+$, Allen's relations are mutually exclusive. This means that at most one of the temporal relations holds between two given nondegenerate time intervals, and hence precisely one. We call a fuzzy time period A nondegenerate with respect to (α, β) iff $A \circ L^{\ll}_{(\alpha,\beta)} \circ A = 1$, i.e., if the beginning of A is long before the end of A. Again, we obtain a generalization of this property.

Proposition 8 (Mutual Exclusiveness): Let A and B be nondegenerate fuzzy time periods with respect to $(2\alpha, \beta)$. Moreover, let R and S both be one of the 13 fuzzy temporal relations defined in Table IV, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha \geq 0$ and $\beta_1 = \beta_2 = \beta_3 = \beta \geq 0$. If $R \neq S$, it holds that

$$T_W(R(A,B), S(A,B)) = 0. \tag{33}$$

The condition that A and B should be nondegenerate fuzzy time periods is only needed when R or S is $m_{(\alpha,\beta)}$ or $mi_{(\alpha,\beta)}$. This is not different from the traditional crisp case. For example, using Allen's definitions, we have for two crisp intervals $A = [a, b]$ and $B = [b, c]$ that $m(A, B)$ holds. However, if $a = b$, we also have that $s(A, B)$ holds. Likewise, if $b = c$, we have that $f(A, B)$ holds.

Finally, we obtain generalizations of the (a)symmetry and the (ir)reflexivity properties of Allen's relations.

Proposition 9 [(A)symmetry]: Let γ_1, γ_{12}, and γ_{123} be defined as in Table IV, and let $\alpha_i \geq 0$, $\beta_i \geq 0$ $(i \in \{1,2,3\})$. The relations b_{γ_1}, bi_{γ_1}, $o_{\gamma_{123}}$, $oi_{\gamma_{123}}$, $d_{\gamma_{12}}$, $di_{\gamma_{12}}$, $s_{\gamma_{12}}$, $si_{\gamma_{12}}$, $f_{\gamma_{12}}$, and $fi_{\gamma_{12}}$ are T_W–asymmetric, i.e., let R be one of the aforementioned fuzzy relations and let A and B be fuzzy time periods. It holds that

$$T_W(R(A,B), R(B,A)) = 0. \tag{34}$$

Furthermore, it holds that

$$e_{\gamma_{12}}(A,B) = e_{\gamma_{12}}(B,A). \tag{35}$$

If A and B are nondegenerate fuzzy time periods with respect to (α_1, β_1), it holds that

$$T_W(m_{\gamma_1}(A,B), m_{\gamma_1}(B,A)) = 0 \tag{36}$$

$$T_W(mi_{\gamma_1}(A,B), mi_{\gamma_1}(B,A)) = 0. \tag{37}$$

Proposition 10 [(Ir)reflexivity]: Let γ_1, γ_{12}, and γ_{123} be defined as in Table IV, and let $\alpha_i \geq 0$, $\beta_i \geq 0$ $(i \in \{1,2,3\})$. The relations b_{γ_1}, bi_{γ_1}, $o_{\gamma_{123}}$, $oi_{\gamma_{123}}$, $d_{\gamma_{12}}$, $di_{\gamma_{12}}$, $s_{\gamma_{12}}$, $si_{\gamma_{12}}$, $f_{\gamma_{12}}$, and $fi_{\gamma_{12}}$ are irreflexive, i.e., let R be one of the aforementioned fuzzy relations and let A and B be fuzzy time periods. It holds that

$$R(A,A) = 0. \tag{38}$$

Furthermore, it holds that

$$e_{\gamma_{12}}(A,A) = 1. \tag{39}$$

If A is a nondegenerate fuzzy time period with respect to (α_1, β_1), it holds that

$$m_{\gamma_1}(A,A) = mi_{\gamma_1}(A,A) = 0. \tag{40}$$

In Propositions 7–10, fuzzy relations of the form $L^{\ll}_{(\alpha,\beta)}$ and $L^{\preccurlyeq}_{(\alpha,\beta)}$ are used to express the concepts "long before" and "more or less before." In principle, more general classes of fuzzy relations could be used to this end, i.e., fuzzy relations that cannot be written as either $L^{\ll}_{(\alpha,\beta)}$ or $L^{\preccurlyeq}_{(\alpha,\beta)}$. However, as can easily be seen from their proof in Appendix III, these propositions remain valid for more general classes of fuzzy relations, provided some weak assumptions are satisfied. For example, let R and S be arbitrary fuzzy relations in $\mathbb{R}$ that are used to express the concepts "long before" and "more or less before," respectively. Then, Proposition 7 remains valid if $R(x, y) = 1 - S(y, x)$ for all x and y in $\mathbb{R}$. For Propositions 8–10 to hold, we also have to assume, among others, that R, $R \circ S$, $R \circ S \circ S$, etc., are irreflexive.

However, using fuzzy relations of the form $L^{\ll}_{(\alpha,\beta)}$ and $L^{\preccurlyeq}_{(\alpha,\beta)}$ to express fuzzy orderings of time points has a number of important advantages. As shown in Section III-B, these fuzzy relations satisfy many desirable properties, and their sup–T_W composition can be conveniently characterized (Proposition 1), which is important for reasoning with fuzzy temporal relations. Moreover, in [30], we have shown that this choice allows one to evaluate the fuzzy temporal interval relations in an efficient way for piecewise linear fuzzy intervals, an important prerequisite for most real-world applications.

V. Fuzzy Temporal Reasoning

When $A = [a^-, a^+]$, $B = [b^-, b^+]$, and $C = [c^-, c^+]$ are crisp intervals, using Allen's original definitions, we can deduce, for example, from $d(A, B)$ and $m(B, C)$ that $b(A, C)$ holds. Indeed by $d(A, B)$, we have $a^+ < b^+$, and by $m(B, C)$, we have $b^+ = c^-$; from $a^+ < b^+$ and $b^+ = c^-$, we conclude $a^+ < c^-$, or in other words, $b(A, C)$. When A, B, and C are fuzzy time intervals, we would like to make similar deductions, even when the interval relations are imprecise (i.e., $\alpha_i > 0$ or $\beta_i > 0$). To this end, we use the Łukasiewicz t–norm T_W to generalize such deductions. For example, let A, B, and C be fuzzy time intervals, $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ and $\beta_1, \beta_2, \beta_3 \geq 0$. Furthermore let $\gamma_{12} = (\alpha_1, \beta_1, \alpha_2, \beta_2)$, $\gamma_1 = (\alpha_1, \beta_1)$, $\gamma_2 = (\alpha_2, \beta_2)$, and $\gamma_3 = (\alpha_3, \beta_3)$ as before. We obtain the equation shown at the bottom of the page. Using Table III, i.e., the transitivity table for relatedness measures and (10), we obtain

$$\begin{aligned}&\le \min(1, A \lhd (L^{\ll}_{\gamma_2} \circ L^{\preccurlyeq}_{\gamma_3}) \rhd C)\\&= A \lhd (L^{\ll}_{\gamma_2} \circ L^{\preccurlyeq}_{\gamma_3}) \rhd C\\&= A \lhd L^{\ll}_{\gamma} \rhd C\\&= b_\gamma(A, C)\end{aligned}$$

where $\gamma = (\alpha_2 - \alpha_3 + \min(\beta_2, \beta_3) - \beta_3, \max(\beta_2, \beta_3))$. In particular, when $\alpha_1 = \alpha_2 = \alpha_3 = 0$ and $\beta_1 = \beta_2 = \beta_3 = \beta$, we have $T_W(d_{(0,\beta,0,\beta)}(A, B), m_{(0,\beta)}(B, C)) \le b_{(0,\beta)}(A, C)$, stating how the degree to which A is during B and the degree to which B more or less meets C can be used to compute a lower bound for the degree to which A is long before C. This is a generalization of the statement that if A occurs during B and B meets C, then A occurs before C.

As another example, in the crisp case from $m(A, B)$ and $m(B, C)$, one can conclude that $b(A, C)$ holds, under the assumption that B is a nondegenerate interval [3]. In our generalized approach, we obtain for fuzzy time intervals A, B, and C

$$\begin{aligned}&T_W(m_{(0,\beta)}(A, B), m_{(0,\beta)}(B, C))\\&\quad = T_W(\min(A \lhd L^{\preccurlyeq}_{(0,\beta)} \rhd B, B \circ L^{\preccurlyeq}_{(0,\beta)} \circ A)\\&\qquad \min(B \lhd L^{\preccurlyeq}_{(0,\beta)} \rhd C, C \circ L^{\preccurlyeq}_{(0,\beta)} \circ B))\\&\quad \le T_W(A \lhd L^{\preccurlyeq}_{(0,\beta)} \rhd B, B \lhd L^{\preccurlyeq}_{(0,\beta)} \rhd C).\end{aligned}$$

Assuming that B is nondegenerate with respect to $(0, \beta)$, i.e., that $B \circ L^{\ll}_{(0,\beta)} \circ B = 1$, and using Table III and (9), we obtain

$$\begin{aligned}A \lhd L^{\preccurlyeq}_{(0,\beta)} \rhd B &= T_W(A \lhd L^{\preccurlyeq}_{(0,\beta)} \rhd B, B \circ L^{\ll}_{(0,\beta)} \circ B)\\&\le (A \lhd (L^{\preccurlyeq}_{(0,\beta)} \circ L^{\ll}_{(0,\beta)})) \circ B\\&= (A \lhd L^{\ll}_{(0,\beta)}) \circ B\end{aligned}$$

and thus, using Table III and (10)

$$\begin{aligned}&T_W(A \lhd L^{\preccurlyeq}_{(0,\beta)} \rhd B, B \lhd L^{\preccurlyeq}_{(0,\beta)} \rhd C)\\&\quad \le T_W((A \lhd L^{\ll}_{(0,\beta)}) \circ B, B \lhd L^{\preccurlyeq}_{(0,\beta)} \rhd C)\\&\quad \le A \lhd (L^{\ll}_{(0,\beta)} \circ L^{\preccurlyeq}_{(0,\beta)}) \rhd C\\&\quad = A \lhd L^{\ll}_{(0,\beta)} \rhd C\\&\quad = b_{(0,\beta)}(A, C).\end{aligned}$$

This deduction process can easily be automated, which is what we have done to obtain Table V. In the crisp case, the temporal relation that results from composing two temporal relations is not always fully determined. For example, for crisp intervals A, B, and C such that $m(A, B)$ and $d(B, C)$, we have that $o(A, C)$, $d(A, C)$, or $s(A, C)$ may hold since we can deduce only that $a^+ > c^-$ and $a^+ < c^+$. Freksa [13] defined a set of coarser temporal relations, which he calls conceptual neighborhoods, and provided a transitivity table that is deductively closed for Allen's original relations as well as the conceptual neighborhoods. For example, $o(A, C)$ or $d(A, C)$ or $s(A, C)$ is equivalent to $bc(A, C)$. The definitions of the relevant conceptual neighborhoods are shown in Table VI. Generalizing these definitions to cope with fuzzy time intervals and imprecise temporal relations is straightforward, using again the relatedness measures from Table II. To obtain Table V, we have assumed that A, B, and C are nondegenerate fuzzy time intervals with respect to (0 ,0). One can verify that when A, B, and C are crisp intervals, Table V corresponds to Freksa's transitivity table. As a consequence, by restricting Table V to the first 13 rows and the first 13 columns, we obtain a transitivity table that is a sound generalization of Allen's transitivity table. Note that while Table III serves to derive knowledge about relationships between the gradual boundaries of fuzzy intervals, Table V is used to reason about the relationships between fuzzy intervals themselves.

Table V cannot be used for reasoning with (semi) quantitative temporal information, i.e., when some $\alpha_i \neq 0$ or $\beta_i \neq 0$. It is not feasible to construct a more general transitivity table, which would permit this and which is still deductively closed. Instead, when $\alpha \neq 0$ or $\beta \neq 0$, the transitivity table for relatedness measures can be used to make deductions. For example, it holds that

$$\begin{aligned}&T_W(o_{\gamma_{123}}(A, B), d_{\gamma_{45}}(B, C))\\&\quad \le \min(C \circ L^{\ll}_{\gamma} \circ A, A \circ L^{\ll}_{\gamma'} \circ C, \quad (A \lhd L^{\ll}_{\gamma''}) \circ C)\end{aligned}$$

where γ_{123} is defined as in Table IV and

$$\begin{aligned}\gamma_{45} &= (\alpha_4, \beta_4, \alpha_5, \beta_5)\\\gamma &= (\alpha_2 + \alpha_4 + \min(\beta_2, \beta_4), \max(\beta_2, \beta_4))\\\gamma' &= (\alpha_1 + \alpha_5 + \min(\beta_1, \beta_5), \max(\beta_1, \beta_5))\\\gamma'' &= (\alpha_3 + \alpha_5 + \min(\beta_3, \beta_5), \max(\beta_3, \beta_5)).\end{aligned}$$

This result can only be written as $bc_{\gamma'''}$ for a given γ''' if

$$(A \lhd L^{\ll}_{\gamma''}) \circ C \le A \circ L^{\ll}_{\gamma'} \circ C$$

which does not hold for arbitrary α_i and β_i.

$$\begin{aligned}T_W(d_{\gamma_{12}}(A, B), m_{\gamma_3}(B, C)) &= T_W(\min(B \circ (L^{\ll}_{\gamma_1} \rhd A), (A \lhd L^{\ll}_{\gamma_2}) \circ B), \min(B \lhd L^{\preccurlyeq}_{\gamma_3} \rhd C, C \circ L^{\preccurlyeq}_{\gamma_3} \circ B))\\&\le \min(T_W(B \circ (L^{\ll}_{\gamma_1} \rhd A), B \lhd L^{\preccurlyeq}_{\gamma_3} \rhd C), T_W(B \circ (L^{\ll}_{\gamma_1} \rhd A), C \circ L^{\preccurlyeq}_{\gamma_3} \circ B),\\&\qquad T_W((A \lhd L^{\ll}_{\gamma_2}) \circ B, B \lhd L^{\preccurlyeq}_{\gamma_3} \rhd C), T_W((A \lhd L^{\ll}_{\gamma_2}) \circ B, C \circ L^{\preccurlyeq}_{\gamma_3} \circ B))\\&\le \min(T_W(B \circ (L^{\ll}_{\gamma_1} \rhd A), C \circ L^{\preccurlyeq}_{\gamma_3} \circ B), T_W((A \lhd L^{\ll}_{\gamma_2}) \circ B, B \lhd L^{\preccurlyeq}_{\gamma_3} \rhd C))\end{aligned}$$

TABLE V

TRANSITIVITY TABLE FOR FUZZY TEMPORAL INTERVAL RELATIONS ($\gamma^1 = (0,0)$, $\gamma^2 = (0,0,0,0)$, $\gamma^3 = (0,0,0,0,0,0)$). LET $\mathcal{R}_{(R_1,R_2)}$ BE THE ENTRY IN THIS TABLE ON THE ROW CORRESPONDING WITH THE FUZZY TEMPORAL RELATION R_1 AND THE COLUMN CORRESPONDING WITH THE FUZZY TEMPORAL RELATION R_2. FOR NONDEGENERATE (WITH RESPECT TO $(0,0)$) FUZZY TIME PERIODS A, B, AND C, IT HOLDS THAT $T_W(R_1(A,B), R_2(B,C)) \leq \mathcal{R}_{(R_1,R_2)}(A,C)$

	$b_{\gamma1}$	$m_{\gamma1}$	$o_{\gamma3}$	$fi_{\gamma2}$	$di_{\gamma2}$	$si_{\gamma2}$	$e_{\gamma2}$	$s_{\gamma2}$	$d_{\gamma2}$	$f_{\gamma2}$	$oi_{\gamma3}$	$mi_{\gamma1}$	$bi_{\gamma1}$	$ob_{\gamma2}$
$b_{\gamma1}$	$b_{\gamma1}$	$b_{\gamma1}$	$b_{\gamma1}$	$b_{\gamma1}$	$b_{\gamma1}$	$b_{\gamma1}$	$b_{\gamma1}$	$b_{\gamma1}$	$sb_{\gamma1}$	$sb_{\gamma1}$	$sb_{\gamma1}$	$sb_{\gamma1}$	1	$b_{\gamma1}$
$m_{\gamma1}$	$b_{\gamma1}$	$b_{\gamma1}$	$b_{\gamma1}$	$b_{\gamma1}$	$b_{\gamma1}$	$m_{\gamma1}$	$m_{\gamma1}$	$m_{\gamma1}$	$bc_{\gamma2}$	$bc_{\gamma2}$	$bc_{\gamma2}$	$tt_{\gamma1}$	$sv_{\gamma1}$	$b_{\gamma1}$
$o_{\gamma3}$	$b_{\gamma1}$	$b_{\gamma1}$	$ob_{\gamma2}$	$ob_{\gamma2}$	$ol_{\gamma1}$	$oc_{\gamma2}$	$o_{\gamma3}$	$o_{\gamma3}$	$bc_{\gamma2}$	$bc_{\gamma2}$	$ct_{\gamma2}$	$sc_{\gamma2}$	$sv_{\gamma1}$	$ob_{\gamma2}$
$fi_{\gamma2}$	$b_{\gamma1}$	$m_{\gamma1}$	$o_{\gamma3}$	$fi_{\gamma2}$	$di_{\gamma2}$	$di_{\gamma2}$	$fi_{\gamma2}$	$o_{\gamma3}$	$bc_{\gamma2}$	$tt_{\gamma1}$	$sc_{\gamma2}$	$sc_{\gamma2}$	$sv_{\gamma1}$	$ob_{\gamma2}$
$di_{\gamma2}$	$ol_{\gamma1}$	$oc_{\gamma2}$	$oc_{\gamma2}$	$di_{\gamma2}$	$di_{\gamma2}$	$di_{\gamma2}$	$di_{\gamma2}$	$oc_{\gamma2}$	$ct_{\gamma2}$	$sc_{\gamma2}$	$sc_{\gamma2}$	$sc_{\gamma2}$	$sv_{\gamma1}$	$ol_{\gamma1}$
$si_{\gamma2}$	$ol_{\gamma1}$	$oc_{\gamma2}$	$oc_{\gamma2}$	$di_{\gamma2}$	$di_{\gamma2}$	$si_{\gamma2}$	$si_{\gamma2}$	$hh_{\gamma1}$	$yc_{\gamma2}$	$oi_{\gamma3}$	$oi_{\gamma3}$	$mi_{\gamma1}$	$bi_{\gamma1}$	$ol_{\gamma1}$
$e_{\gamma2}$	$b_{\gamma1}$	$m_{\gamma1}$	$o_{\gamma3}$	$fi_{\gamma2}$	$di_{\gamma2}$	$si_{\gamma2}$	$e_{\gamma2}$	$s_{\gamma2}$	$d_{\gamma2}$	$f_{\gamma2}$	$oi_{\gamma3}$	$mi_{\gamma1}$	$bi_{\gamma1}$	$ob_{\gamma2}$
$s_{\gamma2}$	$b_{\gamma1}$	$b_{\gamma1}$	$ob_{\gamma2}$	$ob_{\gamma2}$	$ol_{\gamma1}$	$hh_{\gamma1}$	$s_{\gamma2}$	$s_{\gamma2}$	$d_{\gamma2}$	$d_{\gamma2}$	$yc_{\gamma2}$	$mi_{\gamma1}$	$bi_{\gamma1}$	$ob_{\gamma2}$
$d_{\gamma2}$	$b_{\gamma1}$	$b_{\gamma1}$	$sb_{\gamma1}$	$sb_{\gamma1}$	1	$yo_{\gamma1}$	$d_{\gamma2}$	$d_{\gamma2}$	$d_{\gamma2}$	$d_{\gamma2}$	$yo_{\gamma1}$	$bi_{\gamma1}$	$bi_{\gamma1}$	$sb_{\gamma1}$
$f_{\gamma2}$	$b_{\gamma1}$	$m_{\gamma1}$	$bc_{\gamma2}$	$tt_{\gamma1}$	$sv_{\gamma1}$	$ys_{\gamma2}$	$f_{\gamma2}$	$d_{\gamma2}$	$d_{\gamma2}$	$f_{\gamma2}$	$ys_{\gamma2}$	$bi_{\gamma1}$	$bi_{\gamma1}$	$sb_{\gamma1}$
$oi_{\gamma3}$	$ol_{\gamma1}$	$oc_{\gamma2}$	$ct_{\gamma2}$	$sc_{\gamma2}$	$sv_{\gamma1}$	$ys_{\gamma2}$	$oi_{\gamma3}$	$yc_{\gamma2}$	$yc_{\gamma2}$	$oi_{\gamma3}$	$ys_{\gamma2}$	$bi_{\gamma1}$	$bi_{\gamma1}$	$bd_{\gamma1}$
$mi_{\gamma1}$	$ol_{\gamma1}$	$hh_{\gamma1}$	$yc_{\gamma2}$	$mi_{\gamma1}$	$bi_{\gamma1}$	$bi_{\gamma1}$	$mi_{\gamma1}$	$yc_{\gamma2}$	$yc_{\gamma2}$	$mi_{\gamma1}$	$bi_{\gamma1}$	$bi_{\gamma1}$	$bi_{\gamma1}$	$bd_{\gamma1}$
$bi_{\gamma1}$	1	$yo_{\gamma1}$	$yo_{\gamma1}$	$bi_{\gamma1}$	$bi_{\gamma1}$	$bi_{\gamma1}$	$bi_{\gamma1}$	$yo_{\gamma1}$	$yo_{\gamma1}$	$bi_{\gamma1}$	$bi_{\gamma1}$	$bi_{\gamma1}$	$bi_{\gamma1}$	1
$ob_{\gamma2}$	$b_{\gamma1}$	$b_{\gamma1}$	$ob_{\gamma2}$	$ob_{\gamma2}$	$ol_{\gamma1}$	$ol_{\gamma1}$	$ob_{\gamma2}$	$ob_{\gamma2}$	$sb_{\gamma1}$	$sb_{\gamma1}$	$bd_{\gamma1}$	$bd_{\gamma1}$	1	$ob_{\gamma2}$
$oc_{\gamma2}$	$ol_{\gamma1}$	$ol_{\gamma1}$	$ol_{\gamma1}$	$ol_{\gamma1}$	$ol_{\gamma1}$	$oc_{\gamma2}$	$oc_{\gamma2}$	$oc_{\gamma2}$	$ct_{\gamma2}$	$ct_{\gamma2}$	$ct_{\gamma2}$	$sc_{\gamma2}$	$sv_{\gamma1}$	$ol_{\gamma1}$
$hh_{\gamma1}$	$ol_{\gamma1}$	$ol_{\gamma1}$	$ol_{\gamma1}$	$ol_{\gamma1}$	$ol_{\gamma1}$	$hh_{\gamma1}$	$hh_{\gamma1}$	$hh_{\gamma1}$	$yc_{\gamma2}$	$yc_{\gamma2}$	$yc_{\gamma2}$	$mi_{\gamma1}$	$bi_{\gamma1}$	$ol_{\gamma1}$
$yc_{\gamma2}$	$ol_{\gamma1}$	$ol_{\gamma1}$	$bd_{\gamma1}$	$bd_{\gamma1}$	1	$yo_{\gamma1}$	$yc_{\gamma2}$	$yc_{\gamma2}$	$yc_{\gamma2}$	$yc_{\gamma2}$	$yo_{\gamma1}$	$bi_{\gamma1}$	$bi_{\gamma1}$	$bd_{\gamma1}$
$bc_{\gamma2}$	$b_{\gamma1}$	$b_{\gamma1}$	$sb_{\gamma1}$	$sb_{\gamma1}$	1	$db_{\gamma1}$	$bc_{\gamma2}$	$bc_{\gamma2}$	$bc_{\gamma2}$	$bc_{\gamma2}$	$db_{\gamma1}$	$sv_{\gamma1}$	$sv_{\gamma1}$	$sb_{\gamma1}$
$tt_{\gamma1}$	$b_{\gamma1}$	$m_{\gamma1}$	$bc_{\gamma2}$	$tt_{\gamma1}$	$sv_{\gamma1}$	$sv_{\gamma1}$	$tt_{\gamma1}$	$bc_{\gamma2}$	$bc_{\gamma2}$	$tt_{\gamma1}$	$sv_{\gamma1}$	$sv_{\gamma1}$	$sv_{\gamma1}$	$sb_{\gamma1}$
$sc_{\gamma2}$	$ol_{\gamma1}$	$oc_{\gamma2}$	$ct_{\gamma2}$	$sc_{\gamma2}$	$sv_{\gamma1}$	$sv_{\gamma1}$	$sc_{\gamma2}$	$ct_{\gamma2}$	$ct_{\gamma2}$	$sc_{\gamma2}$	$sv_{\gamma1}$	$sv_{\gamma1}$	$sv_{\gamma1}$	$bd_{\gamma1}$
$ys_{\gamma2}$	1	$db_{\gamma1}$	$db_{\gamma1}$	$sv_{\gamma1}$	$sv_{\gamma1}$	$ys_{\gamma2}$	$ys_{\gamma2}$	$yo_{\gamma1}$	$yo_{\gamma1}$	$ys_{\gamma2}$	$ys_{\gamma2}$	$bi_{\gamma1}$	$bi_{\gamma1}$	1
$ol_{\gamma1}$	$ol_{\gamma1}$	$ol_{\gamma1}$	$ol_{\gamma1}$	$ol_{\gamma1}$	$ol_{\gamma1}$	$ol_{\gamma1}$	$ol_{\gamma1}$	$ol_{\gamma1}$	$bd_{\gamma1}$	$bd_{\gamma1}$	$bd_{\gamma1}$	$bd_{\gamma1}$	1	$ol_{\gamma1}$
$yo_{\gamma1}$	1	1	1	1	1	$yo_{\gamma1}$	$yo_{\gamma1}$	$yo_{\gamma1}$	$yo_{\gamma1}$	$yo_{\gamma1}$	$yo_{\gamma1}$	$bi_{\gamma1}$	$bi_{\gamma1}$	1
$sb_{\gamma1}$	$b_{\gamma1}$	$b_{\gamma1}$	$sb_{\gamma1}$	$sb_{\gamma1}$	1	1	$sb_{\gamma1}$	$sb_{\gamma1}$	$sb_{\gamma1}$	$sb_{\gamma1}$	1	1	1	$sb_{\gamma1}$
$sv_{\gamma1}$	1	$db_{\gamma1}$	$db_{\gamma1}$	$sv_{\gamma1}$	$sv_{\gamma1}$	$sv_{\gamma1}$	$sv_{\gamma1}$	$db_{\gamma1}$	$db_{\gamma1}$	$sv_{\gamma1}$	$sv_{\gamma1}$	$sv_{\gamma1}$	$sv_{\gamma1}$	1
$ct_{\gamma2}$	$ol_{\gamma1}$	$ol_{\gamma1}$	$bd_{\gamma1}$	$bd_{\gamma1}$	1	$db_{\gamma1}$	$ct_{\gamma2}$	$ct_{\gamma2}$	$ct_{\gamma2}$	$ct_{\gamma2}$	$db_{\gamma1}$	$sv_{\gamma1}$	$sv_{\gamma1}$	$bd_{\gamma1}$
$bd_{\gamma1}$	$ol_{\gamma1}$	$ol_{\gamma1}$	$bd_{\gamma1}$	$bd_{\gamma1}$	1	1	$bd_{\gamma1}$	$bd_{\gamma1}$	$bd_{\gamma1}$	$bd_{\gamma1}$	1	1	1	$bd_{\gamma1}$
$db_{\gamma1}$	1	1	1	1	1	$db_{\gamma1}$	$db_{\gamma1}$	$db_{\gamma1}$	$db_{\gamma1}$	$db_{\gamma1}$	$db_{\gamma1}$	$sv_{\gamma1}$	$sv_{\gamma1}$	1

	$oc_{\gamma2}$	$hh_{\gamma1}$	$yc_{\gamma2}$	$bc_{\gamma2}$	$tt_{\gamma1}$	$sc_{\gamma2}$	$ys_{\gamma2}$	$ol_{\gamma1}$	$yo_{\gamma1}$	$sb_{\gamma1}$	$sv_{\gamma1}$	$ct_{\gamma2}$	$bd_{\gamma1}$	$db_{\gamma1}$
$b_{\gamma1}$	$b_{\gamma1}$	$b_{\gamma1}$	$sb_{\gamma1}$	$sb_{\gamma1}$	$sb_{\gamma1}$	$sb_{\gamma1}$	1	$b_{\gamma1}$	1	$sb_{\gamma1}$	1	$sb_{\gamma1}$	$sb_{\gamma1}$	1
$m_{\gamma1}$	$b_{\gamma1}$	$m_{\gamma1}$	$bc_{\gamma2}$	$sb_{\gamma1}$	$sb_{\gamma1}$	$sb_{\gamma1}$	$db_{\gamma1}$	$b_{\gamma1}$	$db_{\gamma1}$	$sb_{\gamma1}$	1	$sb_{\gamma1}$	$sb_{\gamma1}$	1
$o_{\gamma3}$	$ol_{\gamma1}$	$oc_{\gamma2}$	$ct_{\gamma2}$	$sb_{\gamma1}$	$sb_{\gamma1}$	$bd_{\gamma1}$	$db_{\gamma1}$	$ol_{\gamma1}$	$db_{\gamma1}$	$sb_{\gamma1}$	1	$bd_{\gamma1}$	$bd_{\gamma1}$	1
$fi_{\gamma2}$	$oc_{\gamma2}$	$oc_{\gamma2}$	$ct_{\gamma2}$	$bc_{\gamma2}$	$tt_{\gamma1}$	$sc_{\gamma2}$	$sv_{\gamma1}$	$ol_{\gamma1}$	$db_{\gamma1}$	$sb_{\gamma1}$	$sv_{\gamma1}$	$ct_{\gamma2}$	$bd_{\gamma1}$	$db_{\gamma1}$
$di_{\gamma2}$	$oc_{\gamma2}$	$oc_{\gamma2}$	$ct_{\gamma2}$	$ct_{\gamma2}$	$sc_{\gamma2}$	$sc_{\gamma2}$	$sv_{\gamma1}$	$ol_{\gamma1}$	$db_{\gamma1}$	$bd_{\gamma1}$	$sv_{\gamma1}$	$ct_{\gamma2}$	$bd_{\gamma1}$	$db_{\gamma1}$
$si_{\gamma2}$	$oc_{\gamma2}$	$hh_{\gamma1}$	$yc_{\gamma2}$	$ct_{\gamma2}$	$sc_{\gamma2}$	$sc_{\gamma2}$	$ys_{\gamma2}$	$ol_{\gamma1}$	$yo_{\gamma1}$	$bd_{\gamma1}$	$sv_{\gamma1}$	$ct_{\gamma2}$	$bd_{\gamma1}$	$db_{\gamma1}$
$e_{\gamma2}$	$oc_{\gamma2}$	$hh_{\gamma1}$	$yc_{\gamma2}$	$bc_{\gamma2}$	$tt_{\gamma1}$	$sc_{\gamma2}$	$ys_{\gamma2}$	$ol_{\gamma1}$	$yo_{\gamma1}$	$sb_{\gamma1}$	$sv_{\gamma1}$	$ct_{\gamma2}$	$bd_{\gamma1}$	$db_{\gamma1}$
$s_{\gamma2}$	$ol_{\gamma1}$	$hh_{\gamma1}$	$yc_{\gamma2}$	$sb_{\gamma1}$	$sb_{\gamma1}$	$bd_{\gamma1}$	$yo_{\gamma1}$	$ol_{\gamma1}$	$yo_{\gamma1}$	$sb_{\gamma1}$	1	$bd_{\gamma1}$	$bd_{\gamma1}$	1
$d_{\gamma2}$	1	$yo_{\gamma1}$	$yo_{\gamma1}$	$sb_{\gamma1}$	$sb_{\gamma1}$	1	$yo_{\gamma1}$	1	$yo_{\gamma1}$	$sb_{\gamma1}$	1	1	1	1
$f_{\gamma2}$	$db_{\gamma1}$	$yo_{\gamma1}$	$yo_{\gamma1}$	$bc_{\gamma2}$	$tt_{\gamma1}$	$sv_{\gamma1}$	$ys_{\gamma2}$	1	$yo_{\gamma1}$	$sb_{\gamma1}$	$sv_{\gamma1}$	$db_{\gamma1}$	1	$db_{\gamma1}$
$oi_{\gamma3}$	$db_{\gamma1}$	$yo_{\gamma1}$	$yo_{\gamma1}$	$ct_{\gamma2}$	$sc_{\gamma2}$	$sv_{\gamma1}$	$ys_{\gamma2}$	1	$yo_{\gamma1}$	$bd_{\gamma1}$	$sv_{\gamma1}$	$db_{\gamma1}$	1	$db_{\gamma1}$
$mi_{\gamma1}$	$yo_{\gamma1}$	$yo_{\gamma1}$	$yo_{\gamma1}$	$yc_{\gamma2}$	$mi_{\gamma1}$	$bi_{\gamma1}$	$bi_{\gamma1}$	1	$yo_{\gamma1}$	$bd_{\gamma1}$	$bi_{\gamma1}$	$yo_{\gamma1}$	1	$yo_{\gamma1}$
$bi_{\gamma1}$	$yo_{\gamma1}$	$yo_{\gamma1}$	$yo_{\gamma1}$	$yo_{\gamma1}$	$bi_{\gamma1}$	$bi_{\gamma1}$	$bi_{\gamma1}$	1	$yo_{\gamma1}$	1	$bi_{\gamma1}$	$yo_{\gamma1}$	1	$yo_{\gamma1}$
$ob_{\gamma2}$	$ol_{\gamma1}$	$ol_{\gamma1}$	$bd_{\gamma1}$	$sb_{\gamma1}$	$sb_{\gamma1}$	$bd_{\gamma1}$	1	$ol_{\gamma1}$	1	$sb_{\gamma1}$	1	$bd_{\gamma1}$	$bd_{\gamma1}$	1
$oc_{\gamma2}$	$ol_{\gamma1}$	$oc_{\gamma2}$	$ct_{\gamma2}$	$bd_{\gamma1}$	$bd_{\gamma1}$	$bd_{\gamma1}$	$db_{\gamma1}$	$ol_{\gamma1}$	$db_{\gamma1}$	$bd_{\gamma1}$	1	$bd_{\gamma1}$	$bd_{\gamma1}$	1
$hh_{\gamma1}$	$ol_{\gamma1}$	$hh_{\gamma1}$	$yc_{\gamma2}$	$bd_{\gamma1}$	$bd_{\gamma1}$	$bd_{\gamma1}$	$yo_{\gamma1}$	$ol_{\gamma1}$	$yo_{\gamma1}$	$bd_{\gamma1}$	1	$bd_{\gamma1}$	$bd_{\gamma1}$	1
$yc_{\gamma2}$	1	$yo_{\gamma1}$	$yo_{\gamma1}$	$bd_{\gamma1}$	$bd_{\gamma1}$	1	$yo_{\gamma1}$	1	$yo_{\gamma1}$	$bd_{\gamma1}$	1	1	1	1
$bc_{\gamma2}$	1	$db_{\gamma1}$	$db_{\gamma1}$	$sb_{\gamma1}$	$sb_{\gamma1}$	1	$db_{\gamma1}$	1	$db_{\gamma1}$	$sb_{\gamma1}$	1	1	1	1
$tt_{\gamma1}$	$db_{\gamma1}$	$db_{\gamma1}$	$db_{\gamma1}$	$bc_{\gamma2}$	$tt_{\gamma1}$	$sv_{\gamma1}$	$sv_{\gamma1}$	1	$db_{\gamma1}$	$sb_{\gamma1}$	$sv_{\gamma1}$	$db_{\gamma1}$	1	$db_{\gamma1}$
$sc_{\gamma2}$	$db_{\gamma1}$	$db_{\gamma1}$	$db_{\gamma1}$	$ct_{\gamma2}$	$sc_{\gamma2}$	$sv_{\gamma1}$	$sv_{\gamma1}$	1	$db_{\gamma1}$	$bd_{\gamma1}$	$sv_{\gamma1}$	$db_{\gamma1}$	1	$db_{\gamma1}$
$ys_{\gamma2}$	$db_{\gamma1}$	$yo_{\gamma1}$	$yo_{\gamma1}$	$db_{\gamma1}$	$sv_{\gamma1}$	$sv_{\gamma1}$	$ys_{\gamma2}$	1	$yo_{\gamma1}$	1	$sv_{\gamma1}$	$db_{\gamma1}$	1	$db_{\gamma1}$
$ol_{\gamma1}$	$ol_{\gamma1}$	$ol_{\gamma1}$	$bd_{\gamma1}$	$bd_{\gamma1}$	$bd_{\gamma1}$	$bd_{\gamma1}$	1	$ol_{\gamma1}$	1	$bd_{\gamma1}$	1	$bd_{\gamma1}$	$bd_{\gamma1}$	1
$yo_{\gamma1}$	1	$yo_{\gamma1}$	$yo_{\gamma1}$	1	1	1	$yo_{\gamma1}$	1	$yo_{\gamma1}$	1	1	1	1	1
$sb_{\gamma1}$	1	1	1	$sb_{\gamma1}$	$sb_{\gamma1}$	1	1	1	1	$sb_{\gamma1}$	1	1	1	1
$sv_{\gamma1}$	$db_{\gamma1}$	$db_{\gamma1}$	$db_{\gamma1}$	$db_{\gamma1}$	$sv_{\gamma1}$	$sv_{\gamma1}$	$sv_{\gamma1}$	1	$db_{\gamma1}$	1	$sv_{\gamma1}$	$db_{\gamma1}$	1	$db_{\gamma1}$
$ct_{\gamma2}$	1	$db_{\gamma1}$	$db_{\gamma1}$	$bd_{\gamma1}$	$bd_{\gamma1}$	1	$db_{\gamma1}$	1	$db_{\gamma1}$	$bd_{\gamma1}$	1	1	1	1
$bd_{\gamma1}$	1	1	1	$bd_{\gamma1}$	$bd_{\gamma1}$	1	1	1	1	$bd_{\gamma1}$	1	1	1	1
$db_{\gamma1}$	1	$db_{\gamma1}$	$db_{\gamma1}$	1	1	1	$db_{\gamma1}$	1	$db_{\gamma1}$	1	1	1	1	1

Note that a lot of interesting theoretical results related to temporal reasoning with crisp intervals have already been established (e.g., [18], [31], and [32]). Among others, these results pertain to constructing maximal subalgebras of Allen's original algebra for which reasoning with a composition table is sufficient to detect inconsistencies (i.e., satisfiability) and to find all consequences of an initial set of assumptions (i.e., completeness). Generalizing these results, however, is clearly outside the scope of this paper.

VI. APPLICATION TO QUESTION ANSWERING

The list question "Who was president of the USA during the Cold War?" is problematic for a traditional question and answer (QA)–system, since there is no generally accepted starting

TABLE VI
FREKSA'S CONCEPTUAL NEIGHBORHOODS [13]

Name	Definition
older	$ol(A,B) \equiv a^- < b^-$
head to head with	$hh(A,B) \equiv a^- = b^-$
younger	$yo(A,B) \equiv b^- < a^-$
survived by	$sb(A,B) \equiv a^+ < b^+$
tail to tail with	$tt(A,B) \equiv a^+ = b^+$
survives	$sv(A,B) \equiv b^+ < a^+$
born before death of	$bd(A,B) \equiv a^- < b^+$
contemporary of	$ct(A,B) \equiv a^- < b^+$ and $b^- < a^+$
died after birth of	$db(A,B) \equiv b^- < a^+$
older & survived by	$ob(A,B) \equiv a^- < b^-$ and $a^+ < b^+$
older contemporary of	$oc(A,B) \equiv a^- < b^-$ and $b^- < a^+$
surviving contemporary of	$sc(A,B) \equiv a^- < b^+$ and $b^+ < a^+$
survived by contemporary of	$bc(A,B) \equiv b^- < a^+$ and $a^+ < b^+$
younger contemporary of	$yc(A,B) \equiv b^- < a^-$ and $a^- < b^+$
younger & survives	$ys(A,B) \equiv b^- < a^-$ and $b^+ < a^+$

date and ending date for the Cold War. Assume, however, that, while searching the Web, our QA–system has discovered the following relevant statements.

- Truman took office when President Roosevelt died just before the end of World War II.[2]
- The Cold War began after World War II.[3]
- The Cold War began at the close of World War II and ended with the dissolution of the Soviet Union.[4]
- President (…) George H. W. Bush declared a U.S.-Soviet strategic partnership at the summit of July 1991, decisively marking the end of the Cold War.[5]

For ease of notation, we use the following abbreviations.

FDR	Denotes the life span of Franklin Roosevelt.
TRP	Denotes the time span of the presidency of Harry Truman.
WW2	Denotes the time span of the second World War.
CW	Denotes the time span of the Cold War.
BSP	Denotes the time span of the presidency of George H. W. Bush.

[2]http://www.juntosociety.com/uspresidents/hstruman.html.

[3]http://www.globalsecurity.org/military/ops/cold_war.htm.

[4]http://www.videofact.com/coldwar.htm.

[5]http://www.absoluteastronomy.com/encyclopedia/H/Hi/History_of_the_Soviet_Union_(1985–1991).htm.

Dates are treated as real numbers that express the number of years since January 1, 1900. As temporal information extraction is not the focus of this paper, we assume that we have the following interpretation of the above statements at our disposal:

$$
\begin{aligned}
m_{(0,0)}(\text{FDR,TRP}) &= 1\\
tt_{(0.3,0.5)}(\text{FDR,WW2}) &= 1\\
sb_{(0,0)}(\text{FDR,WW2}) &= 1\\
b_{(0,0)}(\text{WW2,CW}) &= 1\\
m_{(0.3,0.5)}(\text{WW2,CW}) &= 1\\
o_{(0,0)}(\text{CW,BSP}) &= 1.
\end{aligned}
$$

Note that $(\alpha,\beta) = (0.3, 0.5)$ means that two time points within a period of 3.6 months are considered approximately equal to degree one, and two time points within a period of 9.6 months are approximately equal to a degree that is higher than zero. Using Table III and (8), we obtain the equation shown at the bottom of the page. Furthermore, using Table III and (9), we have

$$
\begin{aligned}
&T_W(\text{CW} \circ L^{\eqslantless}_{(0.6,0.5)} \circ \text{FDR}, m_{(0,0)}(\text{FDR,TRP}))\\
&\quad \le T_W(\text{CW} \circ L^{\eqslantless}_{(0.6,0.5)} \circ \text{FDR},\\
&\qquad \min(\text{FDR} \lhd L^{\eqslantless}_{(0,0)} \rhd \text{TRP}, \text{TRP} \circ L^{\eqslantless}_{(0,0)} \circ \text{FDR}))\\
&\quad \le T_W(\text{CW} \circ L^{\eqslantless}_{(0.6,0.5)} \circ \text{FDR}, \text{FDR} \lhd L^{\eqslantless}_{(0,0)} \rhd \text{TRP})\\
&\quad \le \text{CW} \circ ((L^{\eqslantless}_{(0.6,0.5)} \circ L^{\eqslantless}_{(0,0)}) \rhd \text{TRP})\\
&\quad = \text{CW} \circ (L^{\eqslantless}_{(0.6,0.5)} \rhd \text{TRP})
\end{aligned}
$$

and since we can assume that TRP is nondegenerate with respect to (0.6, 0.5) (i.e., Truman was president for more than 13.2 months), we obtain

$$
\begin{aligned}
&= T_W(\text{CW} \circ (L^{\eqslantless}_{(0.6,0.5)} \rhd \text{TRP}), \text{TRP} \circ L^{\ll}_{(0.6,0.5)} \circ \text{TRP})\\
&\le \text{CW} \circ (L^{\eqslantless}_{(0.6,0.5)} \circ L^{\ll}_{(0.6,0.5)}) \circ \text{TRP}\\
&= \text{CW} \circ L^{\ll}_{(0,0.5)} \circ \text{TRP}\\
&= bd_{(0,0.5)}(\text{CW,TRP}).
\end{aligned}
$$

$$
\begin{aligned}
&T_W(tt_{(0.3,0.5)}(\text{FDR,WW2}), m_{(0.3,0.5)}(\text{WW2,CW}))\\
&\quad = T_W(\min(\text{FDR} \lhd (L^{\eqslantless}_{(0.3,0.5)} \circ \text{WW2}), \text{WW2} \lhd (L^{\eqslantless}_{(0.3,0.5)} \circ \text{FDR})),\\
&\qquad \min(\text{WW2} \lhd L^{\eqslantless}_{(0.3,0.5)} \rhd \text{CW}, \text{CW} \circ L^{\eqslantless}_{(0.3,0.5)} \circ \text{WW2}))\\
&\quad \le T_W(\text{WW2} \lhd (L^{\eqslantless}_{(0.3,0.5)} \circ \text{FDR}), \text{CW} \circ L^{\eqslantless}_{(0.3,0.5)} \circ \text{WW2})\\
&\quad \le \text{CW} \circ (L^{\eqslantless}_{(0.3,0.5)} \circ L^{\eqslantless}_{(0.3,0.5)}) \circ \text{FDR}\\
&\quad = \text{CW} \circ L^{\eqslantless}_{(0.6,0.5)} \circ \text{FDR}
\end{aligned}
$$

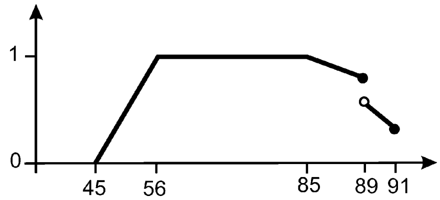


Fig. 3. Modelling the time span of the Cold War.

From Table V, we immediately have

$$T_W(mi_{(0,0)}(\text{TRP,FDR}), sb_{(0,0)}(\text{FDR,WW2})) \leq bd_{(0,0)}(\text{TRP,WW2})$$
$$T_W(bd_{(0,0)}(\text{TRP,WW2}), b_{(0,0)}(\text{WW2,CW})) \leq ol_{(0,0)}(\text{TRP,CW}).$$

Hence we have that when Harry Truman became president, the Cold War had not yet started, i.e., $ol_{(0,0)}$(TRP,CW), while it had already started when his presidency ended, i.e., $bd_{(0,0.5)}$(CW,TRP). In other words, Truman was the first to be president of the United States during the Cold War. Moreover, $o_{(0,0)}$(CW,BSP) means that George H. W. Bush was the last to be president of the United States during the Cold War. To answer the question under consideration, we only have to determine who was president of the United States between Harry Truman and George H. W. Bush, which is a fairly simple task.

Another solution would be to construct a fuzzy time interval CW that represents the time span of the Cold War. Such a fuzzy time interval is illustrated in Fig. 3. For each president X of the United States, $ct_{(0,0)}$(CW,X) expresses the degree to which X has been president during the Cold War. A clear advantage of this method is that we can differentiate between presidents that were definitely president during the Cold War (e.g., Richard Nixon, 1969–1974) and presidents whose presidency was more or less during the Cold War (e.g., Harry Truman, 1945–1953). A disadvantage of this method is that the automatic construction of the fuzzy time interval CW is more time-consuming. Moreover, in some situations, we may lack sufficient information to construct the fuzzy time interval corresponding to a given imprecise event.

In practice, both approaches may be combined, yielding a system that simply evaluates which fuzzy temporal relations hold if a suitable fuzzy time span can be constructed and applies fuzzy temporal reasoning otherwise. We refer to [2] and [26] for more details on the architecture and implementation of such a system. Currently, only qualitative temporal relations are considered in this system. This is due to the fact that interpreting a natural language statement such as "X began just before the end of Y," i.e., providing suitable values for the parameters α and β, is far from trivial.

VII. Conclusions

We have suggested a general approach to represent and compute precise and imprecise temporal relations between crisp as well as fuzzy time intervals. To this end, we have used fuzzy orderings of time points, which are lifted into interval relations through the use of relatedness measures. We have shown that both the fuzzy orderings of time points and the relatedness measures satisfy many desirable properties regarding (ir)reflexivity, (a)symmetry, and transitivity. When considering only precise relations between crisp time intervals, our approach coincides with Allen's temporal interval algebra. However, even in the most general model, i.e., for imprecise relations between fuzzy intervals, unlike in previous approaches, generalizations of all the important properties of Allen's interval relations are valid, in particular those related to exhaustivity, mutual exclusiveness, (ir)reflexivity, and (a)symmetry. Moreover, a sound generalization of Freksa's transitivity table was given that can be used for fuzzy temporal reasoning with qualitative temporal information. In general, fuzzy temporal reasoning can easily be automated by using the transitivity table for relatedness measures which we have introduced in this paper. Finally, we have provided an example that illustrates how fuzzy temporal reasoning could be useful for (temporal) question answering systems, as these systems have to deal with both imprecise events (e.g., the Cold War) and imprecise temporal relations expressed in natural language (e.g., Roosevelt died just before the end of the Second World War).

Appendix I
Proof of Properties Concerning the Fuzzy Orderings $L^{\ll}_{(\alpha,\beta)}$ and $L^{\preccurlyeq}_{(\alpha,\beta)}$

Throughout the appendixes, let T denote a left-continuous t–norm (i.e., a t–norm with left-continuous partial mappings); then T and its residual implicator I satisfy the residuation principle, i.e., for all a, b, and c in [0, 1], it holds that

$$T(a,b) \leq c \Leftrightarrow a \leq I(b,c). \tag{41}$$

Moreover, it can be shown that for all a, b, c, and d in [0, 1], it holds that (see, e.g., [19])

$$T(I(a,b),c) \leq I(a,T(b,c)) \tag{42}$$
$$T(a,I(a,b)) \leq b \tag{43}$$
$$I(T(a,b),c) = I(a,I(b,c)) \tag{44}$$
$$I(a,I(b,c)) = I(b,I(a,c)) \tag{45}$$
$$T(I(a,b),I(c,d)) \leq I(T(a,c),T(b,d)). \tag{46}$$

If J is an arbitrary index set and if $(a_j)_{j\in J}$ and $(b_j)_{j\in J}$ are families in [0, 1], it holds that

$$T(\sup_{j\in J} a_j, b) = \sup_{j\in J} T(a_j,b) \tag{47}$$
$$I(a, \inf_{j\in J} b_j) = \inf_{j\in J} I(a,b_j) \tag{48}$$
$$T(\inf_{j\in J} a_j, b) \leq \inf_{j\in J} T(a_j,b) \tag{49}$$
$$I(a, \sup_{j\in J} b_j) \geq \sup_{j\in J} I(a,b_j). \tag{50}$$

It is easy to see that for an arbitrary t–norm T, it holds that

$$I(1,b) = b. \tag{51}$$

Lemma 1: Let $\alpha_1, \alpha_2 \in \mathbb{R}$ and $\beta \geq 0$; it holds that

$$L^{\ll}_{(\alpha_1+\alpha_2,\beta)}(a,b) = L^{\ll}_{(\alpha_1,\beta)}(a+\alpha_2, b) \quad (52)$$
$$L^{\ll}_{(\alpha_1+\alpha_2,\beta)}(a,b) = L^{\ll}_{(\alpha_1,\beta)}(a, b-\alpha_2) \quad (53)$$
$$L^{\lessapprox}_{(\alpha_1+\alpha_2,\beta)}(a,b) = L^{\lessapprox}_{(\alpha_1,\beta)}(a-\alpha_2, b) \quad (54)$$
$$L^{\lessapprox}_{(\alpha_1+\alpha_2,\beta)}(a,b) = L^{\lessapprox}_{(\alpha_1,\beta)}(a, b+\alpha_2) \quad (55)$$

for all a and b in $\mathbb{R}$.

Lemma 2: Let $\alpha \in \mathbb{R}$ and $\beta_1, \beta_2 \geq 0$; it holds that

$$L^{\ll}_{(\alpha+\beta_2,\beta_1)}(a,b) \leq L^{\ll}_{(\alpha+\min(\beta_1,\beta_2),\max(\beta_1,\beta_2))}(a,b) \quad (56)$$

for all a and b in $\mathbb{R}$.

Proof: If $\beta_1 \geq \beta_2$, then the proof is trivial; therefore assume that $\beta_1 < \beta_2$. For $b-a \leq \alpha+\beta_2$ and for $b-a > \alpha+\beta_1+\beta_2$, (56) holds trivially, since in the former case the left-hand side of (56) equals zero, while in the latter case, the right-hand side equals one. Hence we only need to consider the case where $\beta_1 > 0$ and $\alpha+\beta_2 < b-a \leq \alpha+\beta_1+\beta_2$; it holds that

$$\frac{b-a-\alpha-\beta_2}{\beta_1} \leq \frac{b-a-\alpha-\beta_1}{\beta_2}$$
$$\Leftrightarrow (b-a-\alpha)\beta_2 - \beta_2^2 \leq (b-a-\alpha)\beta_1 - \beta_1^2$$
$$\Leftrightarrow (b-a-\alpha)(\beta_2-\beta_1) \leq \beta_2^2 - \beta_1^2.$$

Since $\beta_2 - \beta_1 > 0$, we obtain

$$\Leftrightarrow b-a \leq \alpha+\beta_1+\beta_2$$

which completes the proof. ∎

Lemma 3 (Transitivity): Let $\alpha_1, \alpha_2 \in \mathbb{R}$ and $\beta_1, \beta_2 \geq 0$; it holds that

$$T_W(L^{\ll}_{(\alpha_1,\beta_1)}(a,b), L^{\ll}_{(\alpha_2,\beta_2)}(b,c)) \leq L^{\ll}_{(\alpha_1+\alpha_2+\min(\beta_1,\beta_2),\max(\beta_1,\beta_2))}(a,c) \quad (57)$$
$$T_W(L^{\lessapprox}_{(\alpha_1,\beta_1)}(a,b), L^{\lessapprox}_{(\alpha_2,\beta_2)}(b,c)) \leq L^{\lessapprox}_{(\alpha_1+\alpha_2,\max(\beta_1,\beta_2))}(a,c) \quad (58)$$
$$T_W(L^{\lessapprox}_{(\alpha_1,\beta_1)}(a,b), L^{\ll}_{(\alpha_2,\beta_2)}(b,c)) \leq L^{\ll}_{(\alpha_2-\alpha_1+\min(\beta_1,\beta_2)-\beta_1,\max(\beta_1,\beta_2))}(a,c) \quad (59)$$
$$T_W(L^{\ll}_{(\alpha_1,\beta_1)}(a,b), L^{\lessapprox}_{(\alpha_2,\beta_2)}(b,c)) \leq L^{\ll}_{(\alpha_1-\alpha_2+\min(\beta_1,\beta_2)-\beta_2,\max(\beta_1,\beta_2))}(a,c) \quad (60)$$

for all a, b, and c in $\mathbb{R}$.

Proof: As an example, we prove (57); the proof of (58)–(60) is analogous. When $b-a \leq \alpha_1$ or $c-b \leq \alpha_2$, (57) obviously holds since the left-hand side equals zero. When $b-a > \alpha_1+\beta_1$, we have

$$T_W(L^{\ll}_{(\alpha_1,\beta_1)}(a,b), L^{\ll}_{(\alpha_2,\beta_2)}(b,c)) = L^{\ll}_{(\alpha_2,\beta_2)}(b,c).$$

Because the first partial mappings of $L^{\ll}_{(\alpha_2,\beta_2)}$ are decreasing, and by using the assumption $b-a > \alpha_1+\beta_1$, we obtain

$$L^{\ll}_{(\alpha_2,\beta_2)}(b,c) \leq L^{\ll}_{(\alpha_2,\beta_2)}(a+\alpha_1+\beta_1, c)$$

and by (52)

$$L^{\ll}_{(\alpha_2,\beta_2)}(a+\alpha_1+\beta_1, c) = L^{\ll}_{(\alpha_1+\alpha_2+\beta_1,\beta_2)}(a,c)$$

and by Lemma 2

$$L^{\ll}_{(\alpha_1+\alpha_2+\beta_1,\beta_2)}(a,c) \leq L^{\ll}_{(\alpha_1+\alpha_2+\min(\beta_1,\beta_2),\max(\beta_1,\beta_2))}(a,c).$$

In the same way, we can prove (57) when $c-b > \alpha_2+\beta_2$. Finally, assume $\alpha_1 < b-a \leq \alpha_1+\beta_1$ and $\alpha_2 < c-b \leq \alpha_2+\beta_2$ (hence $\beta_1 > 0$ and $\beta_2 > 0$). For $\beta_1 \leq \beta_2$, we obtain

$$\begin{aligned} &T_W(L^{\ll}_{(\alpha_1,\beta_1)}(a,b), L^{\ll}_{(\alpha_2,\beta_2)}(b,c)) \\ &= \max(0, L^{\ll}_{(\alpha_1,\beta_1)}(a,b) + L^{\ll}_{(\alpha_2,\beta_2)}(b,c) - 1) \\ &= \max\left(0, \frac{b-a-\alpha_1}{\beta_1} + \frac{c-b-\alpha_2}{\beta_2} - 1\right) \\ &= \max\left(0, \frac{b-a-\alpha_1-\beta_1}{\beta_1} + \frac{c-b-\alpha_2}{\beta_2}\right). \end{aligned}$$

Since $b-a-\alpha_1-\beta_1 \leq 0$ and $\beta_1 \leq \beta_2$, we have

$$\begin{aligned} &\leq \max\left(0, \frac{b-a-\alpha_1-\beta_1}{\beta_2} + \frac{c-b-\alpha_2}{\beta_2}\right) \\ &= \max\left(0, \frac{c-a-\alpha_1-\alpha_2-\beta_1}{\beta_2}\right) \\ &= L^{\ll}_{(\alpha_1+\alpha_2+\beta_1,\beta_2)}(a,c). \end{aligned}$$

For $\beta_1 > \beta_2$, the proof is entirely analogous. ∎

Proof of Proposition 1 (Composition): We prove (7) as an example; the proof of (8)–(10) is analogous. By (57), we already have

$$(L^{\ll}_{(\alpha_1,\beta_1)} \circ_{T_W} L^{\ll}_{(\alpha_2,\beta_2)})(a,c) \leq L^{\ll}_{(\alpha_1+\alpha_2+\min(\beta_1,\beta_2),\max(\beta_1,\beta_2))}(a,c)$$

for arbitrary a and c in $\mathbb{R}$. Conversely, for $\beta_1 \leq \beta_2$, we have

$$\begin{aligned} &\sup_{b\in\mathbb{R}} T_W(L^{\ll}_{(\alpha_1,\beta_1)}(a,b), L^{\ll}_{(\alpha_2,\beta_2)}(b,c)) \\ &\geq \sup_{\varepsilon>0} T_W(L^{\ll}_{(\alpha_1,\beta_1)}(a, a+\alpha_1+\beta_1+\varepsilon), \\ &\qquad L^{\ll}_{(\alpha_2,\beta_2)}(a+\alpha_1+\beta_1+\varepsilon, c)) \\ &= \sup_{\varepsilon>0} T_W(1, L^{\ll}_{(\alpha_2,\beta_2)}(a+\alpha_1+\beta_1+\varepsilon, c)) \\ &= \sup_{\varepsilon>0} L^{\ll}_{(\alpha_2,\beta_2)}(a+\alpha_1+\beta_1+\varepsilon, c). \end{aligned}$$

Taking into account that the first partial mappings of $L^{\ll}_{(\alpha_2,\beta_2)}$ are decreasing and right-continuous, we obtain

$$= L^{\ll}_{(\alpha_2,\beta_2)}(a+\alpha_1+\beta_1, c)$$

and by (52)

$$L^{\ll}_{(\alpha_2,\beta_2)}(a+\alpha_1+\beta_1, c) = L^{\ll}_{(\alpha_1+\alpha_2+\beta_1,\beta_2)}(a,c).$$

For $\beta_1 > \beta_2$, the proof is entirely analogous.

Proof of Proposition 2: For $b-a < -\alpha_1-\beta_1$ or $a-b \leq \alpha_2$, (15) trivially holds. Hence, we can assume that $-\alpha_1-\beta_1 \leq b-a < -\alpha_2$, and thus, using the assumptions $\beta_1 \leq \beta_2$ and

$\alpha_1 \leq \alpha_2, -\alpha_1 - \beta_1 \leq b - a < -\alpha_1$ and $\alpha_2 < a - b \leq \alpha_2 + \beta_2$ (hence $\beta_1 > 0$ and $\beta_2 > 0$), we obtain

$$\begin{aligned} T_W(L^{\preccurlyeq}_{(\alpha_1,\beta_1)}(a,b), L^{\ll}_{(\alpha_2,\beta_2)}(b,a)) &= \max(0, L^{\preccurlyeq}_{(\alpha_1,\beta_1)}(a,b) + L^{\ll}_{(\alpha_2,\beta_2)}(b,a) - 1) \\ &= \max\left(0, \frac{b-a+\alpha_1+\beta_1}{\beta_1} + \frac{a-b-\alpha_2}{\beta_2} - 1\right) \\ &= \max\left(0, \frac{b-a+\alpha_1}{\beta_1} + \frac{a-b-\alpha_2}{\beta_2}\right) \\ &\leq \max\left(0, \frac{b-a+\alpha_1}{\beta_2} + \frac{a-b-\alpha_2}{\beta_2}\right) \end{aligned}$$

since $b - a + \alpha_1 < 0$. Finally, we have

$$\max\left(0, \frac{b-a+\alpha_1}{\beta_2} + \frac{a-b-\alpha_2}{\beta_2}\right) = \max\left(0, \frac{\alpha_1-\alpha_2}{\beta_2}\right)$$

which concludes the proof since $(\alpha_1 - \alpha_2)/\beta_2 \leq 0$.

Appendix II
Proof of the Transitivity Table for Relatedness Measures

In this Appendix, we provide a proof for the inequalities in Table III. Throughout this Appendix, let R and S be fuzzy relations in a universe U, and let A, B, and C be normalized fuzzy sets in U. Recall that the inverse fuzzy relation R^{-1} of a fuzzy relation R in U is the fuzzy relation in U defined by $R^{-1}(v,u) = R(u,v)$ for all u and v in U.

Lemma 4 (Duality):

$$A \circ_T R \circ_T B = B \circ_T R^{-1} \circ_T A \tag{61}$$
$$A \lhd_I R \rhd_I B = B \lhd_I R^{-1} \rhd_I A \tag{62}$$
$$(A \circ_T R) \rhd_I B = B \lhd_I (R^{-1} \circ_T A) \tag{63}$$
$$A \circ_T (R \rhd_I B) = (B \lhd_I R^{-1}) \circ_T A. \tag{64}$$

Proof: Equalities (63) and (64) follow immediately from the definitions of the relatedness measures; (61) follows from (47), while (62) follows from (45) and (48). ■

We will now prove the correctness of the inequalities in Table III. First, note that each inequality corresponding to an entry that equals one trivially holds. A first series of inequalities to prove is

$$T(A \lhd_I R \rhd_I B, B \circ_T S \circ_T C) \leq (A \lhd_I Q) \circ_T C \tag{65}$$
$$T(A \lhd_I R \rhd_I B, (B \lhd_I S) \circ_T C) \leq (A \lhd_I Q) \circ_T C \tag{66}$$
$$T(A \lhd_I R \rhd_I B, B \lhd_I (S \circ_T C)) \leq (A \lhd_I Q) \circ_T C \tag{67}$$
$$T(A \circ_T R \circ_T B, B \lhd_I S \rhd_I C) \leq A \circ_T (Q \rhd_I C) \tag{68}$$
$$T((A \circ_T R) \rhd_I B, B \lhd_I S \rhd_I C) \leq A \circ_T (Q \rhd_I C) \tag{69}$$
$$T(A \circ_T (R \rhd_I B), B \lhd_I S \rhd_I C) \leq A \circ_T (Q \rhd_I C) \tag{70}$$

where $Q = R \circ_T S$.

Proof: By using (47), we obtain

$$\begin{aligned} &T\big(A \lhd_I R \rhd_I B, B \circ_T S \circ_T C\big) \\ &= T\big(\inf_{v' \in U} I(B(v'), \inf_{u \in U} I(A(u), R(u,v'))), \\ &\quad \sup_{w \in U} T(C(w), \sup_{v \in U} T(B(v), S(v,w)))\big) \\ &= \sup_{v \in U} \sup_{w \in U} T\big(\inf_{v' \in U} I(B(v'), \inf_{u \in U} I(A(u), R(u,v'))), \\ &\quad T(C(w), T(B(v), S(v,w)))\big) \\ &\leq \sup_{v \in U} \sup_{w \in U} T\big(I(B(v), \inf_{u \in U} I(A(u), R(u,v))), \\ &\quad T(C(w), T(B(v), S(v,w)))\big) \\ &= \sup_{v \in U} \sup_{w \in U} T\big(T(B(v), I(B(v), \inf_{u \in U} I(A(u), R(u,v)))), \\ &\quad T(C(w), S(v,w))\big). \end{aligned}$$

By (43), we have

$$\begin{aligned} &\leq \sup_{v \in U} \sup_{w \in U} T\big(\inf_{u \in U} I(A(u), R(u,v)), T(C(w), S(v,w))\big) \\ &= \sup_{v \in U} \sup_{w \in U} T\big(T(\inf_{u \in U} I(A(u), R(u,v)), S(v,w)), C(w)\big) \end{aligned}$$

and by (49) and (42)

$$\begin{aligned} &\leq \sup_{v \in U} \sup_{w \in U} T\big(\inf_{u \in U} T(I(A(u), R(u,v)), S(v,w)), C(w)\big) \\ &\leq \sup_{v \in U} \sup_{w \in U} T\big(\inf_{u \in U} I(A(u), T(R(u,v), S(v,w))), C(w)\big) \end{aligned}$$

and finally by (47) and (50)

$$\begin{aligned} &= \sup_{v \in U} T\big(\sup_{w \in U} \inf_{u \in U} I(A(u), T(R(u,v), S(v,w))), C(w)\big) \\ &\leq \sup_{w \in U} T\big(\inf_{u \in U} \sup_{v \in U} I(A(u), T(R(u,v), S(v,w))), C(w)\big) \\ &\leq \sup_{w \in U} T\big(\inf_{u \in U} I(A(u), \sup_{v \in U} T(R(u,v), S(v,w))), C(w)\big). \end{aligned}$$

Hence we have shown that (65) holds; (66) and (67) follow from (65) by using Proposition 3. Furthermore, by using duality, (68) follows from (65) [using (61), (62), and (64)], (69) follows from (67) [using (62)–(64)], and (70) follows from (66) [using (62) and (64)]. ■

Next, we prove the following series of inequalities:

$$T((A \circ_T R) \rhd_I B, B \circ_T S \circ_T C) \leq A \circ_T Q \circ_T C \tag{71}$$
$$T(A \circ_T (R \rhd_I B), B \circ_T S \circ_T C) \leq A \circ_T Q \circ_T C \tag{72}$$
$$T((A \circ_T R) \rhd_I B, (B \lhd_I S) \circ_T C) \leq A \circ_T Q \circ_T C \tag{73}$$
$$T(A \circ_T (R \rhd_I B), (B \lhd_I S) \circ_T C) \leq A \circ_T Q \circ_T C \tag{74}$$
$$T((A \circ_T R) \rhd_I B, B \lhd_I (S \circ_T C)) \leq A \circ_T Q \circ_T C \tag{75}$$
$$T(A \circ_T (R \rhd_I B), B \lhd_I (S \circ_T C)) \leq A \circ_T Q \circ_T C \tag{76}$$
$$T(A \circ_T R \circ_T B, (B \lhd_I S) \circ_T C) \leq A \circ_T Q \circ_T C \tag{77}$$
$$T(A \circ_T R \circ_T B, B \lhd_I (S \circ_T C)) \leq A \circ_T Q \circ_T C \tag{78}$$

where $Q = R \circ_T S$.

Proof: The proof of (71) is analogous to the proof of (65); (72)–(76) follow from (71) by using Proposition 3. Furthermore, by using duality, (77) follows from (72) [using (61) and (64)] and (78) follows from (71) [using (61) and (63)]. ■

A third series of inequalities to prove is

$$T(A \lhd_I R \rhd_I B, B \lhd_I S \rhd_I C) \le A \lhd_I Q \rhd_I C \quad (79)$$
$$T((A \lhd_I R) \circ_T B, B \lhd_I S \rhd_I C) \le A \lhd_I Q \rhd_I C \quad (80)$$
$$T(A \lhd_I (R \circ_T B), B \lhd_I S \rhd_I C) \le A \lhd_I Q \rhd_I C \quad (81)$$
$$T(A \lhd_I R \rhd_I B, (B \circ_T S) \rhd_I C) \le A \lhd_I Q \rhd_I C \quad (82)$$
$$T(A \lhd_I R \rhd_I B, B \circ_T (S \rhd_I C)) \le A \lhd_I Q \rhd_I C \quad (83)$$

where $Q = R \circ_T S$.

Proof: The proof of (81) is analogous to the proof of (65); (79) and (80) follow from (81) by using Proposition 3. Furthermore, by using duality, (82) follows from (81) [using (62) and (63)] and (83) follows from (80) [using (62) and (64)]. ■

Next, we prove the following series of inequalities:

$$T((A \lhd_I R) \circ_T B, (B \lhd_I S) \circ_T C) \le (A \lhd_I Q) \circ_T C \quad (84)$$
$$T(A \lhd_I (R \circ_T B), (B \lhd_I S) \circ_T C) \le (A \lhd_I Q) \circ_T C \quad (85)$$
$$T((A \lhd_I R) \circ_T B, B \lhd_I (S \circ_T C)) \le (A \lhd_I Q) \circ_T C \quad (86)$$
$$T(A \circ_T (R \rhd_I B), B \circ_T (S \rhd_I C)) \le A \circ_T (Q \rhd_I C) \quad (87)$$
$$T((A \circ_T R) \rhd_I B, B \circ_T (S \rhd_I C)) \le A \circ_T (Q \rhd_I C) \quad (88)$$
$$T(A \circ_T (R \rhd_I B), (B \circ_T S) \rhd_I C) \le A \circ_T (Q \rhd_I C) \quad (89)$$

where $Q = R \circ_T S$.

Proof: The proof of (85) is analogous to the proof of (65); the proof of (86) is entirely analogous. Furthermore, (84) follows from (85) by using Proposition 3. Finally, by duality, (87) follows from (84) [using (64)], (88) follows from (86) [using (63) and (64)], and (89) follows from (85) [using (63) and (64)]. ■

Finally, we still need to prove the following inequalities:

$$T(A \lhd_I (R \circ_T B), B \lhd_I (S \circ_T C)) \le A \lhd_I (Q \circ_T C) \quad (90)$$
$$T((A \circ_T R) \rhd_I B, (B \circ_T S) \rhd_I C) \le (A \circ_T Q) \rhd_I C \quad (91)$$

where $Q = R \circ_T S$.

Proof: By (49), we have

$$\begin{aligned} &T\big(A \lhd_I (R \circ_T B), B \lhd_I (S \circ_T C)\big) \\ &= T\big(\inf_{u\in U} I(A(u), \sup_{v\in U} T(B(v), R(u,v))), \\ &\quad \inf_{v\in U} I(B(v), \sup_{w\in U} T(C(w), S(v,w)))\big) \\ &\le \inf_{u\in U} T\big(I(A(u), \sup_{v\in U} T(B(v), R(u,v))), \\ &\quad \inf_{v\in U} I(B(v), \sup_{w\in U} T(C(w), S(v,w)))\big). \end{aligned}$$

By (42) and (47), we obtain

$$\begin{aligned} &\le \inf_{u\in U} I\big(A(u), T(\sup_{v\in U} T(B(v), R(u,v)), \\ &\quad \inf_{v\in U} I(B(v), \sup_{w\in U} T(C(w), S(v,w))))\big) \\ &= \inf_{u\in U} I\big(A(u), \sup_{v\in U} T(T(B(v), R(u,v)), \\ &\quad \inf_{v'\in U} I(B(v'), \sup_{w\in U} T(C(w), S(v',w))))\big) \\ &\le \inf_{u\in U} I\big(A(u), \sup_{v\in U} T(T(B(v), R(u,v)), \\ &\quad I(B(v), \sup_{w\in U} T(C(w), S(v,w))))\big) \end{aligned}$$

and by (43) and (47)

$$\begin{aligned} &\le \inf_{u\in U} I\big(A(u), \sup_{v\in U} T(R(u,v), \sup_{w\in U} T(C(w), S(v,w)))\big) \\ &= \inf_{u\in U} I\big(A(u), \sup_{w\in U} T(C(w), \sup_{v\in U} T(R(u,v), S(v,w)))\big) \end{aligned}$$

which completes the proof of (90). By duality, (91) follows from (90) [using (63)]. ■

Appendix III
Proof of Properties of the Fuzzy Temporal Interval Relations

In this Appendix, we will only use the Łukasiewicz t–norm and implicator. Hence we will omit the subscripts in $\circ_{T_W}$, $\lhd_{I_W}$, and $\rhd_{I_W}$.

Lemma 5: Let A be a fuzzy set in U, B a fuzzy set in V, and R a fuzzy relation from U to V. It holds that

$$1 - A \circ (R \rhd B) = (B \circ coR^{-1}) \rhd A \quad (92)$$
$$1 - (A \lhd R) \circ B = B \lhd (coR^{-1} \circ A) \quad (93)$$
$$1 - A \circ R \circ B = B \lhd coR^{-1} \rhd A. \quad (94)$$

Proof: As an example, we prove (92). We obtain

$$\begin{aligned} &1 - A \circ (R \rhd B) \\ &= 1 - \sup_{u\in U} T_W(A(u), \inf_{v\in V} I_W(B(v), R(u,v))) \\ &= 1 - \sup_{u\in U} \max(0, A(u) \\ &\quad + \inf_{v\in V} \min(1, 1 - B(v) + R(u,v)) - 1) \\ &= 1 - \sup_{u\in U} \max(0, A(u) + \inf_{v\in V} \min(0, -B(v) + R(u,v))) \\ &= \inf_{u\in U} \min(1, 1 - A(u) + \sup_{v\in V} \max(0, B(v) - R(u,v))) \\ &= \inf_{u\in U} \min(1, 1 - A(u) \\ &\quad + \sup_{v\in V} \max(0, B(v) + coR(u,v) - 1)) \\ &= \inf_{u\in U} I_W(A(u), \sup_{v\in V} T_W(B(v), coR(u,v))). \end{aligned}$$

■

Lemma 6: For all a, b and c in [0, 1], it holds that

$$S_W(\min(a,b), \min(a,c)) \ge \min(a, S_W(b,c)). \quad (95)$$

Proof: We have

$$\begin{aligned}&S_W(\min(a,b),\min(a,c))\\&\quad=\min(1,\min(a,b)+\min(a,c))\\&\quad=\min(1,a+a,a+c,a+b,b+c).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\min(a,S_W(b,c))&=\min(a,\min(1,b+c))\\&=\min(1,a,b+c)\end{aligned}$$

which proves (95) since $\min(a+a,a+b,a+c)\geq a$. ■

Lemma 7: For all a, b, and c in [0, 1], it holds that

$$S_W(\min(a,b),c)=\min(S_W(a,c),S_W(b,c)). \tag{96}$$

Proof: We obtain

$$\begin{aligned}&\min(S_W(a,c),S_W(b,c))\\&\quad=\min(\min(1,a+c),\min(1,b+c))\\&\quad=\min(1,a+c,b+c)\\&\quad=\min(1,\min(a,b)+c)\\&\quad=S_W(\min(a,b),c).\end{aligned}$$

■

Proof of Proposition 7 (Exhaustivity): We have

$$\begin{aligned}&S_W(s_{\gamma^2}(A,B),si_{\gamma^2}(A,B),e_{\gamma^2}(A,B))\\&\quad=S_W(\min((A\circ L_{\gamma^1}^{\preceq})\rhd B,(B\circ L_{\gamma^1}^{\preceq})\rhd A,\\&\qquad(A\lhd L_{\gamma^1}^{\ll})\circ B),\\&\qquad\min((B\circ L_{\gamma^1}^{\preceq})\rhd A,(A\circ L_{\gamma^1}^{\preceq})\rhd B,\\&\qquad(B\lhd L_{\gamma^1}^{\ll})\circ A),\\&\qquad\min((A\circ L_{\gamma^1}^{\preceq})\rhd B,(B\circ L_{\gamma^1}^{\preceq})\rhd A,\\&\qquad A\lhd(L_{\gamma^1}^{\preceq}\circ B),B\lhd(L_{\gamma^1}^{\preceq}\circ A))).\end{aligned}$$

By twice applying (95), we obtain that

$$\begin{aligned}&S_W(\min(a,b_1),\min(a,b_2),\min(a,b_3))\\&\qquad\qquad\geq\min(a,S_W(b_1,b_2,b_3))\end{aligned}$$

for all a, b_1, b_2, and b_3 in [0, 1]. Substituting

$$\begin{aligned}a&:=\min((A\circ L_{\gamma^1}^{\preceq})\rhd B,(B\circ L_{\gamma^1}^{\preceq})\rhd A)\\b_1&:=(A\lhd L_{\gamma^1}^{\ll})\circ B\\b_2&:=(B\lhd L_{\gamma^1}^{\ll})\circ A\\b_3&:=\min(A\lhd(L_{\gamma^1}^{\preceq}\circ B),B\lhd(L_{\gamma^1}^{\preceq}\circ A))\end{aligned}$$

we obtain

$$\begin{aligned}&S_W(s_{\gamma^2}(A,B),si_{\gamma^2}(A,B),e_{\gamma^2}(A,B))\\&\quad\geq\min((A\circ L_{\gamma^1}^{\preceq})\rhd B,(B\circ L_{\gamma^1}^{\preceq})\rhd A,\\&\qquad S_W((A\lhd L_{\gamma^1}^{\ll})\circ B,(B\lhd L_{\gamma^1}^{\ll})\circ A,\\&\qquad\min(A\lhd(L_{\gamma^1}^{\preceq}\circ B),\\&\qquad B\lhd(L_{\gamma^1}^{\preceq}\circ A))))\end{aligned}$$

and by (96)

$$\begin{aligned}&=\min((A\circ L_{\gamma^1}^{\preceq})\rhd B,(B\circ L_{\gamma^1}^{\preceq})\rhd A,\\&\quad S_W((A\lhd L_{\gamma^1}^{\ll})\circ B,(B\lhd L_{\gamma^1}^{\ll})\circ A,A\lhd(L_{\gamma^1}^{\preceq}\circ B)),\\&\quad S_W((A\lhd L_{\gamma^1}^{\ll})\circ B,(B\lhd L_{\gamma^1}^{\ll})\circ A,B\lhd(L_{\gamma^1}^{\preceq}\circ A)))\end{aligned}$$

and by (93) and (2)

$$\begin{aligned}&=\min((A\circ L_{\gamma^1}^{\preceq})\rhd B,(B\circ L_{\gamma^1}^{\preceq})\rhd A)\\&=hh_{\gamma^1}(A,B)\end{aligned}$$

since $S_W(a,1-a)=1$ for all a in [0, 1]. We can show analogously that

$$\begin{aligned}&S_W(o_{\gamma^3}(A,B),fi_{\gamma^2}(A,B),di_{\gamma^2}(A,B))\\&\quad\geq\min(A\circ(L_{\gamma^1}^{\ll}\rhd A),A\circ L_{\gamma^1}^{\ll}\circ A)\\&\quad=oc_{\gamma^2}(A,B)\end{aligned}$$

$$\begin{aligned}&S_W(oi_{\gamma^3}(A,B),f_{\gamma^2}(A,B),d_{\gamma^2}(A,B))\\&\quad\geq\min(B\circ(L_{\gamma^1}^{\ll}\rhd B),B\circ L_{\gamma^1}^{\ll}\circ B)\\&\quad=yc_{\gamma^2}(A,B)\end{aligned}$$

$$\begin{aligned}&S_W(b_{\gamma^1}(A,B),m_{\gamma^1}(A,B))\\&\quad\geq A\lhd L_{\gamma^1}^{\preceq}\rhd B\\&\quad=pr_{\gamma^1}(A,B)\end{aligned}$$

$$\begin{aligned}&S_W(bi_{\gamma^1}(A,B),mi_{\gamma^1}(A,B))\\&\quad\geq B\lhd L_{\gamma^1}^{\preceq}\rhd A\\&\quad=sd_{\gamma^1}(A,B)\end{aligned}$$

$$\begin{aligned}&S_W(hh_{\gamma^1}(A,B),yc_{\gamma^2}(A,B),oc_{\gamma^2}(A,B))\\&\quad\geq\min(A\circ L_{\gamma^1}^{\ll}\circ B,B\circ L_{\gamma^1}^{\ll}\circ A)\\&\quad=ct_{\gamma^2}(A,B)\end{aligned}$$

$$S_W(ct_{\gamma^2}(A,B),pr_{\gamma^1}(A,B),sd_{\gamma^1}(A,B))=1$$

which completes the proof. ■

Note that the conceptual neighborhoods pr and sd are not included in Table VI, as they are not relevant to the transitivity table for temporal interval relations. Here the (generalized) conceptual neighborhoods are merely used as a shorthand.

Proof of Proposition 8 (Mutual Exclusiveness): To prove the mutual exclusiveness of the fuzzy temporal relations, 78 cases have to be considered. Here, as an example, we provide a proof for two of these cases. First, we show that

$$T_W(d_{(\alpha,\beta,\alpha,\beta)}(A,B), m_{(\alpha,\beta)}(A,B)) = 0.$$

Using Table III, we obtain

$$\begin{aligned} &T_W(d_{(\alpha,\beta,\alpha,\beta)}(A,B), m_{(\alpha,\beta)}(A,B)) \\ &\quad = T_W(\min(B \circ (L_\gamma^{\ll} \rhd A), (A \lhd L_\gamma^{\ll}) \circ B), \\ &\qquad \min(A \lhd L_\gamma^{\precsim} \rhd B, B \circ L_\gamma^{\precsim} \circ A)) \\ &\quad \leq T_W(B \circ (L_\gamma^{\ll} \rhd A), A \lhd L_\gamma^{\precsim} \rhd B) \\ &\quad \leq B \circ ((L_\gamma^{\ll} \circ L_\gamma^{\precsim}) \rhd B) \end{aligned}$$

where $\gamma = (\alpha, \beta)$. By (10), we obtain

$$= B \circ (L_{(0,\beta)}^{\ll} \rhd B)$$

which equals zero by Proposition 5, since for $\alpha \geq 0$, $L_{(\alpha,\beta)}^{\ll}$ is an irreflexive fuzzy relation.

As a second example, we show that

$$T_W(s_{\gamma^2}(A,B), m_{\gamma^1}(A,B)) = 0.$$

Since A is nondegenerate with respect to $(2\alpha, \beta)$, we obtain using Table III and (9)

$$\begin{aligned} &(B \circ L_\gamma^{\precsim}) \rhd A \\ &\quad = T_W((B \circ L_\gamma^{\precsim}) \rhd A, A \circ L_{(2\alpha,\beta)}^{\ll} \circ A) \\ &\quad \leq B \circ (L_\gamma^{\precsim} \circ L_{(2\alpha,\beta)}^{\ll}) \circ A \\ &\quad = B \circ L_{(\alpha,\beta)}^{\ll} \circ A. \end{aligned}$$

We obtain using Table III and (10)

$$\begin{aligned} &T_W(s_{\gamma^2}(A,B), m_{\gamma^1}(A,B)) \\ &\quad = T_W(\min((A \circ L_\gamma^{\precsim}) \rhd B, (B \circ L_\gamma^{\precsim}) \rhd A, \\ &\qquad (A \lhd L_\gamma^{\ll}) \circ B), \\ &\qquad \min(A \lhd L_\gamma^{\precsim} \rhd B, B \circ L_\gamma^{\precsim} \circ A)) \\ &\quad \leq T_W((B \circ L_\gamma^{\precsim}) \rhd A, A \lhd L_\gamma^{\precsim} \rhd B) \\ &\quad \leq T_W(B \circ L_\gamma^{\ll} \circ A, A \lhd L_\gamma^{\precsim} \rhd B) \\ &\quad \leq B \circ ((L_\gamma^{\ll} \circ L_\gamma^{\precsim}) \rhd B) \\ &\quad = B \circ (L_{(0,\beta)}^{\ll} \rhd B) \end{aligned}$$

which equals zero by Proposition 5, since for $\alpha \geq 0$, $L_{(\alpha,\beta)}^{\ll}$ is an irreflexive fuzzy relation.

Proof of Proposition 9 ((A)symmetry): As an example, we show that

$$T_W(d_{\gamma_{12}}(A,B), d_{\gamma_{12}}(B,A)) = 0.$$

By using Table III and (7), we obtain

$$\begin{aligned} &T_W(d_{\gamma_{12}}(A,B), d_{\gamma_{12}}(B,A)) \\ &\quad = T_W(\min((A \lhd L_{\gamma_2}^{\ll}) \circ B, B \circ (L_{\gamma_1}^{\ll} \rhd A)), \\ &\qquad \min((B \lhd L_{\gamma_2}^{\ll}) \circ A, A \circ (L_{\gamma_1}^{\ll} \rhd B))) \\ &\quad \leq T_W((A \lhd L_{\gamma_2}^{\ll}) \circ B, (B \lhd L_{\gamma_2}^{\ll}) \circ A) \\ &\quad \leq (A \lhd (L_{\gamma_2}^{\ll} \circ L_{\gamma_2}^{\ll})) \circ A \\ &\quad = (A \lhd L_{(2\alpha_2+\beta_2,\beta_2)}^{\ll}) \circ A \end{aligned}$$

which equals zero by Proposition 5, since for $\alpha \geq 0$, $L_{(\alpha,\beta)}^{\ll}$ is an irreflexive fuzzy relation.

As another example, we show that

$$T_W(m_{(\alpha_1,\beta_1)}(A,B), m_{(\alpha_1,\beta_1)}(B,A)) = 0$$

if A and B are nondegenerate fuzzy time periods with respect to (α_1, β_1). We obtain

$$\begin{aligned} &T_W(m_{(\alpha_1,\beta_1)}(A,B), m_{(\alpha_1,\beta_1)}(B,A)) \\ &\quad = T_W(\min(A \lhd L_{\gamma_1}^{\precsim} \rhd B, B \circ L_{\gamma_1}^{\precsim} \circ A), \\ &\qquad \min(B \lhd L_{\gamma_1}^{\precsim} \rhd A, A \circ L_{\gamma_1}^{\precsim} \circ B)) \\ &\quad \leq T_W(A \lhd L_{\gamma_1}^{\precsim} \rhd B, B \lhd L_{\gamma_1}^{\precsim} \rhd A). \end{aligned}$$

Since A is nondegenerate, we have by Table III and (10) that

$$\begin{aligned} &A \lhd L_{\gamma_1}^{\precsim} \rhd B \\ &\quad = T_W(A \circ L_{\gamma_1}^{\ll} \circ A, A \lhd L_{\gamma_1}^{\precsim} \rhd B) \\ &\quad \leq A \circ ((L_{\gamma_1}^{\ll} \circ L_{\gamma_1}^{\precsim}) \rhd B) \\ &\quad = A \circ (L_{(0,\beta_1)}^{\ll} \rhd B). \end{aligned}$$

Analogously, since B is nondegenerate, we obtain

$$B \lhd L_{\gamma_1}^{\precsim} \rhd A \leq B \circ (L_{(0,\beta_1)}^{\ll} \rhd A).$$

Hence we already have

$$\begin{aligned} &T_W(m_{(\alpha_1,\beta_1)}(A,B), m_{(\alpha_1,\beta_1)}(B,A)) \\ &\qquad \leq T_W(A \circ (L_{(0,\beta_1)}^{\ll} \rhd B), B \circ (L_{(0,\beta_1)}^{\ll} \rhd A)) \end{aligned}$$

and by Table III and (7)

$$\begin{aligned} &\leq A \circ ((L_{(0,\beta_1)}^{\ll} \circ L_{(0,\beta_1)}^{\ll}) \rhd A) \\ &= A \circ (L_{(\beta_1,\beta_1)}^{\ll} \rhd A) \end{aligned}$$

which equals zero by Proposition 5, since for $\alpha \geq 0$, $L_{(\alpha,\beta)}^{\ll}$ is an irreflexive fuzzy relation.

Proof of Proposition 10 [(Ir)reflexivity]: We will only prove that $m_{\gamma_1}(A,A) = 0$ when A is a nondegenerate fuzzy time period with respect to (α_1, β_1), as the other equalities follow straightforwardly from Propositions 4 and 5. We obtain

$$\begin{aligned} m_{\gamma_1}(A,A) &= \min(A \lhd L_{\gamma_1}^{\precsim} \rhd A, A \circ L_{\gamma_1}^{\precsim} \circ A) \\ &\leq A \lhd L_{\gamma_1}^{\precsim} \rhd A. \end{aligned}$$

Since A is nondegenerate with respect to (α_1, β_1), we have by Table III and (9)

$$
\begin{aligned}
&= T_W(A \lhd L^{\precsim}_{\gamma_1} \rhd A, A \circ L^{\ll}_{\gamma_1} \circ A) \\
&\leq (A \lhd (L^{\precsim}_{\gamma_1} \circ L^{\ll}_{\gamma_1})) \circ A \\
&= (A \lhd L^{\ll}_{(0,\beta_1)}) \circ A
\end{aligned}
$$

which equals zero by Proposition 5.

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