Temporal Reasoning about Fuzzy Intervals

Steven Schockaert [*,1], Martine De Cock

Ghent University, Department of Applied Mathematics and Computer Science, Krijgslaan 281 - S9, 9000 Gent, Belgium

Abstract

Traditional approaches to temporal reasoning assume that time periods and time spans of events can be accurately represented as intervals. Real–world time periods and events, on the other hand, are often characterized by vague temporal boundaries, requiring appropriate generalizations of existing formalisms. This paper presents a framework for reasoning about qualitative and metric temporal relations between vague time periods. In particular, we show how several interesting problems, like consistency and entailment checking, can be reduced to reasoning tasks in existing temporal reasoning frameworks. We furthermore demonstrate that all reasoning tasks of interest are NP–complete, which reveals that adding vagueness to temporal reasoning does not increase its computational complexity. To support efficient reasoning, a large tractable subfragment is identified, among others, generalizing the well–known ORD Horn subfragment of the Interval Algebra (extended with metric constraints).

Key words: Temporal Reasoning, Interval Algebra, Fuzzy Set Theory

1 Introduction

Time plays a key role in many application domains, ranging from scheduling and planning [2,17,19] to natural language understanding [29,34], multi–document summarization [6], question answering [39,32,22] and dynamic multimedia presentation [3,10,18]. Starting from Allen's seminal work on qualitative interval relations (e.g., A happened during B, A overlaps with B; [1]), increasingly more expressive formalisms have been proposed to reason about

---

[*] Corresponding author.
  Email addresses: `Steven.Schockaert@UGent.be` (Steven Schockaert),
`Martine.DeCock@UGent.be` (Martine De Cock).
[1] Research Assistant of the Research Foundation – Flanders.

time, among others allowing to specify metric constraints between two time points (e.g., p happened 4 time units before q; [12]), to combine qualitative and metric information [25,31], to specify constraints on the (relative) duration of events [35], and to specify arbitrary disjunctions of temporal constraints [23,27]. Most reasoning tasks of interest in these formalisms are NP–complete. To cope with this, a lot of research efforts have been directed towards identifying subfragments of the various calculi in which reasoning becomes tractable [13,14,28,37], as well as towards deriving efficient solution strategies for NP–complete reasoning problems [8,36,45,46].

Research, however, has largely focused on reasoning about time periods, and time spans of events, which can be accurately represented as an interval. In contrast, many real–world events and time periods are characterized by an inherently gradual or ill–defined beginning and ending. Typical examples are large–scale historical events like the Russian Revolution, the Great Depression, the Second World War, the Cold War, and the Dotcom Bubble, or historical time periods like the Middle Ages, the Renaissance, the Age of Enlightenment, and the Industrial Revolution, but also small-scale events like sleeping and being born. Moreover, in natural language, vague temporal markers are frequently found to convey underspecified temporal information: early summer, during his childhood, in the evening, etc. Note that the vagueness of these events and time periods is fundamentally different from the uncertainty that exists among historians about, for example, the time period during which the Mona Lisa was painted.

A formal definition of the notion of an event is difficult to provide. Clearly, an event is something that happens at a particular time and a particular place (e.g., World War II); it can have parts (e.g., the Battle of the Bulge), it can belong to a certain category (e.g., Military Conflict) and it can have consequences (e.g., the Cold War) [47]. We will, however, abstract away from any particular formalization of events, and focus on their temporal dimension only. As such, we will conceptually make no difference between time periods and events. Vague time periods are naturally represented as fuzzy sets [48]. A vague time period is then represented as a mapping A from the real line $\mathbb{R}$ to the unit interval $[0, 1]$. For a time instant t ($t \in \mathbb{R}$), $A(t)$ expresses to what extent t belongs to the time period A. When A is a crisp time period, for all t in $\mathbb{R}$, $A(t)$ is either 0 (perfect non-membership) or 1 (perfect membership). When A is a vague time period, on the other hand, A will typically be gradually increasing over an interval $[t_1, t_2]$ and gradually decreasing over an interval $[t_3, t_4]$, where $A(t) = 1$ for t in $[t_2, t_3]$ and $A(t) = 0$ for $t < t_1$ and $t > t_4$. As an example, consider Picasso's Blue, Rose and Cubist periods. Regarding the definition of the Rose period, for example, we find [2]

---

[2] http://pablo-picasso.paintings.name/rose-period/, accessed May 21, 2007

2

Fig. 1. Fuzzy sets defining the vague time span of Picasso's Blue, Rose, and Cubist periods.

> So 1904 is a transitional year and belongs neither truly to the blue period, nor to the rose period.

Similarly, the ending of the Rose period, as well as the beginning and ending of the Cubist period are inherently gradual. Figure 1 depicts a possible definition of Picasso's Rose period, as well as the ending of his Blue period and the beginning of his Cubist period. These definitions reflect the gradual transition to the Rose period during 1904, as well as Picasso's experiments with new styles from 1906 and especially from 1907, eventually leading to his Cubist period. Clearly, the definition of a fuzzy set representing a vague time period is to some extent subjective. In fact, there is no real reason why January 1, 1907 should belong to the Rose period to degree 0.8 and not to degree 0.75 or 0.85. What is most important is the qualitative ordering the membership degrees impose, e.g., June 1, 1907 is more compatible with the Rose period than the Cubist period; March 15, 1904 is less compatible with the Rose period than June 1, 1907, etc.

Applications based on classical temporal reasoning algorithms, like temporal question answering or multi–document summarization, fail to work correctly when the events or time periods involved are vague. For example, when extracting information about the life and work of Picasso from web documents, inconsistencies quickly arise:

(1) Bread and Fruit Dish on a Table (1909) marks the beginning of Picasso's "Analytical" Cubism . . . [3]
(2) The first stage of Picasso's cubism is known as analytical cubism. It began in 1908 and ended in 1912, . . . [4]

––––––––

[3] http://www.abcgallery.com/P/picasso/picassobio.html, accessed May 21, 2007

[4] http://www.pokemonultimate.wanadoo.co.uk/picasso.html, accessed May 21, 2007

3

(3) The 'Demoiselles d'Avignon' of 1907 mark the beginning of his [Picasso's] Cubist period in which he exceeded the classical form. [5]

The solution to this problem is not to discard the least reliable sources until the resulting knowledge base is consistent, but to acknowledge that some of the temporal relations expressed in the sentences above are only true to some extent: the beginning of Picasso's Analytical Cubism coincides with the beginning of cubism to some degree λ_1, Picasso's Cubist period began with "Demoiselles d'Avignon" in 1907 to some degree λ_2, Picasso's Analytical Cubism began in 1908 to some degree λ_3, Picasso's Analytical Cubism began with "Bread and Fruit Dish on a Table" in 1909 to some degree λ_4. The aim of this paper is to derive algorithms for reasoning about such fuzzy temporal information, e.g., which values of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ result in a consistent interpretation of the sentences above? What conclusions can we establish given a consistent set of (fuzzy) assertions about (vague) time periods? Our primary objective is to obtain a temporal reasoning framework that is, among others, suitable for natural language applications like multi–document summarization or question answering when some of the time periods and events involved are vague.

The structure of this paper is as follows. In the next section, we review related work on fuzzy temporal information processing, while Section 3 familiarizes the reader with some important preliminaries from fuzzy set theory and temporal reasoning. In Section 4, we introduce our framework for representing fuzzy temporal information. Next, in Section 5, we introduce an algorithm to check the consistency of a set of assertions about fuzzy time periods. The computational complexity of this problem is investigated in Section 6. Section 7 discusses how new information can be derived from given information. Finally, Section 8 presents some concluding remarks and directions for future work.

2   Related work

Although processing fuzzy temporal information is well studied in literature, research has tended to focus on modelling vague temporal information about crisp events (e.g., Picasso died in the early 1970s), rather than on modelling temporal information about vague events. For example, in [16] possibility theory is employed to represent vague dates (e.g., early summer), and vague temporal constraints (e.g., A happened about three months before B). The underlying assumption is that all events have crisp, albeit unknown, temporal boundaries; only our knowledge about these crisp boundaries is vague. Based on this possibilistic approach, [5] introduced the notion of a fuzzy temporal

---

[5] http://www.kettererkunst.com/details-e.php?obnr=410702527&anummer= 315, accessed May 21, 2007

4

constraint network. In this framework, temporal information is represented as fuzzy temporal constraints, i.e., fuzzy restrictions on the possible distances between time points. Sound and complete reasoning procedures were provided in [30]. A generalization in which disjunctions of fuzzy temporal constraints can be expressed has been introduced in [8].

A different line of research has focused on fuzzy extensions of classical calculi for temporal reasoning to encode preferences. For example, [26] discusses a generalization of Temporal Constraint Satisfaction Problems (TCSP) in which a preference value is attached to each temporal constraint. When a given TCSP is inconsistent, the preference values are used to determine which constraints should be ignored. Similarly, [4] introduces the framework IA^{fuz} in which preference values are attached to atomic Allen relations. A relation in IA^{fuz} can thus be regarded as a fuzzy set of atomic Allen relations. Interestingly, all main reasoning tasks are shown to be NP–complete and a maximal tractable subfragment is identified. Fuzzy sets of atomic Allen relations have also been considered in [21], where the adequate modelling of temporal expressions in natural language was the main motivation, rather than encoding preferences.

The need for formalisms dealing with vague events and time periods has been pointed out in various contexts, including semantic web reasoning [9], historical databases and ontologies [33], document retrieval [24], and temporal question answering [40]. Nevertheless, none of the approaches mentioned above is suitable to represent temporal information about events whose boundaries are inherently gradual or ill–defined. Inspired by measures for comparing and ranking fuzzy numbers [7,15], some definitions of fuzzy temporal relations between vague events have already been proposed [33,38,42]. A key problem in generalizing temporal relations to cope with fuzzy time spans is that traditionally, temporal relations have been defined as constraints on boundary points of intervals. Because such well–defined boundary points are absent in fuzzy time intervals, alternative ways of looking at temporal relations are required.

Nagypál and Motik [33] start from the observation that several sets of time points can be associated with each interval $A = [a^-, a^+]$, viz. the semi–intervals $A^{<-} =] - \infty, a^-[$, $A^{\leq-} =] - \infty, a^-]$, $A^{<+} =] - \infty, a^+[$, $A^{\leq+} =] - \infty, a^+]$, $A^{>-} =]a^-, +\infty[$, $A^{\geq-} = [a^-, +\infty[$, $A^{>+} =]a^+, +\infty[$ and $A^{\geq+} = [a^+, +\infty[$. Qualitative constraints on the boundary points of two intervals $A = [a^-, a^+]$ and $B = [b^-, b^+]$ can be translated into set operations on the corresponding semi–intervals. For example, $m(A, B)$ holds iff $a^+ = b^-$, which can be expressed as $A^{>+} \cap B^{<-} = \emptyset \wedge A^{<+} \cap B^{>-} = \emptyset$. To define qualitative temporal relations between fuzzy time spans, Nagypál and Motik define $A^{<-}, A^{\leq-}, A^{<+}, A^{\leq+}, A^{>-}, A^{\geq-}, A^{>+}, A^{\geq+}$ for a fuzzy set A as:

$$A^{>-}(p) = \sup_{q<p} A(q) \qquad\qquad A^{\leq-}(p) = 1 - A^{>-}(p)$$

5

$$A^{\geq-}(p) = \sup_{q \leq p} A(q) \qquad\qquad A^{<-}(p) = 1 - A^{\geq-}(p)$$

$$A^{<+}(p) = \sup_{q > p} A(q) \qquad\qquad A^{\geq+}(p) = 1 - A^{<+}(p)$$

$$A^{\leq+}(p) = \sup_{q \geq p} A(q) \qquad\qquad A^{>+}(p) = 1 - A^{\leq+}(p)$$

The degree to which $m(A, B)$ is satisfied, for instance, is then defined as

$$
\begin{aligned}
m(A, B) &= \min(1 - \sup_{p \in \mathbb{R}} \min(A^{>+}(p), B^{<-}(p)), 1 - \sup_{p \in \mathbb{R}} \min(A^{<+}(p), B^{>-}(p))) \\
&= \min(\inf_{p \in \mathbb{R}} \max(1 - A^{>+}(p), 1 - B^{<-}(p)), \\
&\qquad\quad \inf_{p \in \mathbb{R}} \max(1 - A^{<+}(p), 1 - B^{>-}(p))) \\
&= \min(\inf_{p \in \mathbb{R}} \max(A^{\leq+}(p), B^{\geq-}(p)), \inf_{p \in \mathbb{R}} \max(A^{\geq+}(p), B^{\leq-}(p)))
\end{aligned}
$$

Although this approach has a certain appeal, the resulting fuzzy temporal relations do not always behave intuitively. For example, for crisp intervals the equals relation is reflexive, while starts, finishes and during are irreflexive. Taking into account this intended meaning, we would expect that for fuzzy time spans $e(A, A) = 1$ and $s(A, A) = f(A, A) = d(A, A) = 0$, or at least, that $e(A, A) > \max(s(A, A), f(A, A), d(A, A))$. However, using the definitions proposed by Nagypál and Motik, if A is a continuous fuzzy set, it holds that $e(A, A) = s(A, A) = f(A, A) = d(A, A) = 0.5$. The reason for this anomaly lies in the definition of the fuzzy sets $A^{<-}, A^{\leq-}, \ldots, A^{\geq+}$. While these definitions do correspond to their intended meaning when A is a crisp interval, for a continuous fuzzy set A, we have the undesirable property that $A^{>-} = A^{\geq-}$, $A^{<-} = A^{\leq-}$, $A^{>+} = A^{\geq+}$ and $A^{<+} = A^{\leq+}$.

In [38], a fundamentally different approach to modelling temporal relations between fuzzy time spans is taken. The starting point is that even for crisp intervals A and B, relations like before can hold to some degree. For example, if $A = [0, 50]$ and $B = [45, 100]$, we may intuitively think of A as being before B, instead of overlapping with B, because most of A is before the beginning of B. In [38], the degree to which $b(A, B)$ holds is therefore defined based on which fraction of A is before the beginning of B, where A and B may be crisp or fuzzy time spans. When temporal relations are defined in this way, we (deliberately) lose the original meaning of Allen's relations. Although such definitions may definitely be useful in many domains (e.g., querying temporal databases), they are not suitable as a basis for fuzzy temporal reasoning.

To the best of our knowledge, however, the issue of temporal reasoning about fuzzy time intervals has only been addressed in [40], where a sound but incomplete algorithm is introduced to find consequences of a given, restricted set of assertions. Finally, note that this paper is an extended and generalized version of [43]. In addition to providing a more detailed discussion, as well as proofs of all results, we generalize the results from [43], where a purely qualitative

6

approach was adopted, by also considering metric constraints.

3   Preliminaries

3.1   *Fuzzy temporal relations*

A fuzzy set [48] in a universe U is defined as a mapping A from U to $[0, 1]$, representing a vague concept. For u in U, $A(u)$ expresses the degree to which u is compatible with the concept A and is called the membership degree of u in A. In particular, we will use fuzzy sets in $\mathbb{R}$ to represent time spans of vague events. For clarity, traditional sets are sometimes called crisp sets in the context of fuzzy set theory. If A and B are fuzzy sets in the same universe U, A is called a fuzzy subset of B, written $A \subseteq B$, iff $A(u) \leq B(u)$ for all u in U.

For every α in $]0, 1]$, we let A_α denote the crisp subset of U defined by

$$A_\alpha = \{u | u \in U \wedge A(u) \geq \alpha\}$$

A_α is called the α–level set of the fuzzy set A. In particular, A_1 is the set of elements from U that are fully compatible with the vague concept modelled by A. If $A_1 \neq \emptyset$, the fuzzy set A is called normalised and elements from A_1 are called modal values of A. The support $supp(A)$ of A is the (crisp) set of elements from U which belong to A to a strictly positive degree:

$$supp(A) = \{u | u \in U \wedge A(u) > 0\}$$

A fuzzy set A in $\mathbb{R}$ is called convex if for every α in $]0, 1]$, the set A_α is convex (i.e., a singleton or an interval).

To adequately generalize the notion of a time interval, a closed and bounded interval of real numbers, some natural restrictions on the α–level sets are typically imposed.

Definition 1 (Fuzzy time interval). *[42] A fuzzy (time) interval is a normalised fuzzy set in $\mathbb{R}$ with a bounded support, such that for every α in $]0, 1]$, A_α is a closed interval.*

For example, the fuzzy sets corresponding to Picasso's Blue, Rose and Cubist period from Figure 1 are fuzzy time intervals. The condition that all α–level sets of a fuzzy time span be intervals implies that any fuzzy time interval is a convex fuzzy set in $\mathbb{R}$. Hence, fuzzy time intervals always consist of a monotonically increasing part, followed by a monotonically decreasing part. As a consequence, the fuzzy set depicted in Figure 2(a) is a fuzzy time span, whereas the fuzzy set from 2(b) is not, because of the decreasing part between
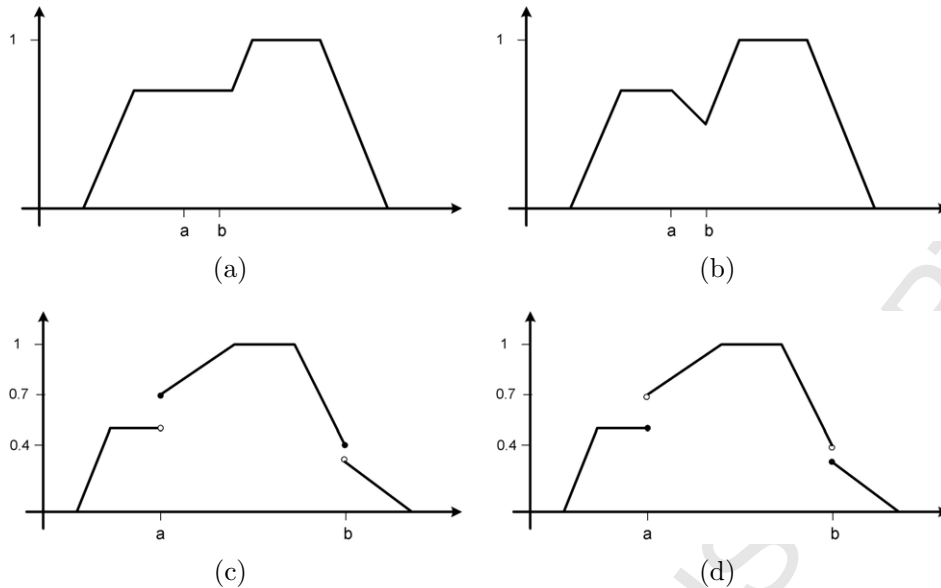
7

Fig. 2. The fuzzy sets of real numbers displayed in (a) and (c) are fuzzy time spans, whereas those in (b) and (d) are not.

a and b. By furthermore requiring that α–level sets be closed sets, we restrict the kind of discontinuities allowed. For example, the fuzzy set from Figure 2(d) is not a fuzzy time interval, since its 0.4– and 0.7–level sets are half open intervals. On the other hand, the fuzzy set shown in Figure 2(c) is a fuzzy time interval. It can be shown that a bounded fuzzy set of real numbers is a fuzzy time interval iff it is convex and upper semi–continuous. For a fuzzy interval A and $\alpha \in]0, 1]$, we let A_α^- and A_α^+ denote the beginning and ending of the interval A_α.

When the time spans of events are vague, also the temporal relations between them are a matter of degree. Traditionally, temporal relations between time intervals have been defined by means of constraints on the boundary points of these intervals. Due to the gradual nature of fuzzy time spans, a different, more general approach has to be adopted here. The definitions of our fuzzy temporal relations are inspired by the fact that temporal relations between time intervals can alternatively be specified using first-order expressions that do not involve any boundary points. For example, for crisp intervals $A = [a^-, a^+]$ and $B = [b^-, b^+]$, it is easy to see that $(d \in \mathbb{R})$

$$a^- < b^- - d \Leftrightarrow (\exists p)(p \in A \land (\forall q)(q \in B \Rightarrow p < q - d)) \qquad (1)$$

To define the degree to which the beginning of a fuzzy time interval A is more than d time units before the beginning of B, written $bb_d^\lll(A, B)$, we generalize the right-hand side of (1) using fuzzy logic connectives. In fuzzy logic, elements from the unit interval $[0, 1]$ represent truth degrees. To generalize logical conjunction to fuzzy logic, a wide class of $[0, 1]^2 - [0, 1]$ mappings,

8

t-norm	residual implicator
$T_M(x, y) = \min(x, y)$	$I_{T_M}(x, y) = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$
$T_P(x, y) = x \cdot y$	$I_{T_P}(x, y) = \begin{cases} 1 & \text{if } x \leq y \\ \frac{y}{x} & \text{otherwise} \end{cases}$
$T_W(x, y) = \max(0, x + y - 1)$	$I_{T_W}(x, y) = \min(1, 1 - x + y)$

Table 1
Commonly used t-norms and their residual implicators.

called t-norms, can be used. The only requirements are that the mapping T being used is symmetric, associative, increasing, and satisfies the boundary condition $T(x, 1) = x$ for all x in $[0, 1]$. Logical implication can be generalized using the residual implicator I_T of a t-norm T, i.e. the $[0, 1]^2 - [0, 1]$ mapping defined for x and y in $[0, 1]$ as

$$I_T(x, y) = \sup\{\lambda | \lambda \in [0, 1] \wedge T(x, \lambda) \leq y\}$$

Some commonly used t-norms are the minimum T_M, the product T_P and the Łukasiewicz t-norm T_W. Their definitions and the corresponding residual implicators are displayed in Table 1. Universal and existential quantification can be generalized using the infimum and supremum respectively.

This leads to the following definition [42]:

$$bb_d^{\ll}(A, B) = \sup_{p \in \mathbb{R}} T(A(p), \inf_{q \in \mathbb{R}} I_T(B(q), L_d^{\ll}(p, q))) \tag{2}$$

where $L_d^{\ll}(p, q) = 1$ iff $p < q - d$ and $L_d^{\ll}(x, y) = 0$ otherwise. In the same way, we can define the degree $ee_d^{\ll}(A, B)$ to which the end of A is more than d time units before the end of B, the degree $be_d^{\ll}(A, B)$ to which the beginning of A is more than d time units before the end of B and the degree $eb^{\ll}(A, B)$ to which the end of A is more than d time units before the beginning of B as [42]:

$$ee_d^{\ll}(A, B) = \sup_{q \in \mathbb{R}} T(B(q), \inf_{p \in \mathbb{R}} I_T(A(p), L_d^{\ll}(p, q))) \tag{3}$$

$$be_d^{\ll}(A, B) = \sup_{p \in \mathbb{R}} T(A(p), \sup_{q \in \mathbb{R}} T(B(q), L_d^{\ll}(p, q))) \tag{4}$$

$$eb_d^{\ll}(A, B) = \inf_{p \in \mathbb{R}} I_T(A(p), \inf_{q \in \mathbb{R}} I_T(B(q), L_d^{\ll}(p, q))) \tag{5}$$

Finally, the degree $bb_d^{\preceq}(A, B)$ to which the beginning of A is at most d time units after the beginning of B is defined by

$$bb_d^{\preceq}(A, B) = 1 - bb_d^{\ll}(B, A) \tag{6}$$

9

Table 2
Characterization of the fuzzy temporal relations when A and B correspond to crisp intervals $[a^-, a^+]$ and $[b^-, b^+]$.

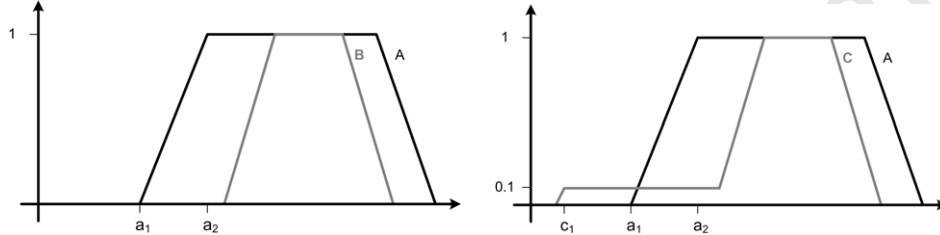| Fuzzy	Crisp	Fuzzy	Crisp
$bb_d^{\ll}(A, B)$	$a^- < b^- - d$	$bb_d^{\preccurlyeq}(A, B)$	$a^- \leq b^- + d$
$ee_d^{\ll}(A, B)$	$a^+ < b^+ - d$	$ee_d^{\preccurlyeq}(A, B)$	$a^+ \leq b^+ + d$
$be_d^{\ll}(A, B)$	$a^- < b^+ - d$	$be_d^{\preccurlyeq}(A, B)$	$a^- \leq b^+ + d$
$eb_d^{\ll}(A, B)$	$a^+ < b^- - d$	$eb_d^{\preccurlyeq}(A, B)$	$a^+ \leq b^- + d$

Fig. 3. Since the definition of the fuzzy time intervals B and C are similar, it is desirable that $|bb^{\ll}(A, B) - bb^{\ll}(A, C)|$ is small.

In the same way, we define $ee_d^{\preccurlyeq}(A, B)$, $be_d^{\preccurlyeq}(A, B)$ and $eb_d^{\preccurlyeq}(A, B)$ as

$$ee_d^{\preccurlyeq}(A, B) = 1 - ee_d^{\ll}(B, A) \tag{7}$$

$$be_d^{\preccurlyeq}(A, B) = 1 - eb_d^{\ll}(B, A) \tag{8}$$

$$eb_d^{\preccurlyeq}(A, B) = 1 - be_d^{\ll}(B, A) \tag{9}$$

For convenience, we will sometimes omit the subscript when $d = 0$ (e.g., $bb_0^{\ll} = bb^{\ll}$). In this case, (2)–(9) express qualitative relations between fuzzy time intervals; e.g., $ee^{\ll}(A, B)$ is the degree to which the end of A is strictly before the end of B, $eb^{\preccurlyeq}(A, B)$ is the degree to which the end of A is before or equal to the beginning of B. Note that it is the vagueness of the events that gives rise to the vagueness of the temporal relations. If A and B correspond to crisp intervals $[a^-, a^+]$ and $[b^-, b^+]$, the fuzzy temporal relations reduce to classical temporal constraints. This is illustrated in Table 2.

There are several reasons why using the Łukasiewicz t-norm T_W in the definitions (2)–(5) is advantageous over using, for example, T_M or T_P. First, it is desirable that small changes in the definitions of the fuzzy time intervals A and B result in small changes of the values of $bb_d^{\ll}(A, B)$, $ee_d^{\ll}(A, B)$, $be_d^{\ll}(A, B)$ and $eb_d^{\ll}(A, B)$. This is particularly true in applications where fuzzy time intervals are constructed automatically from, for example, web documents, as in such applications, small variations in membership degrees may be due to noise (e.g., incorrect information on web pages, errors introduced by the information extraction technique that is used, etc.). Consider the fuzzy time intervals A, B and C depicted in Figure 3. Because B and C are very similar, we would
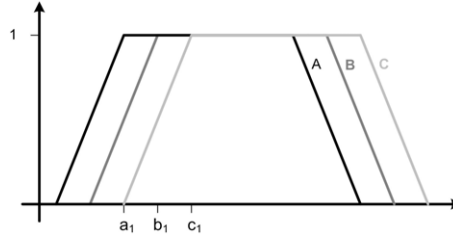
10

Fig. 4. The generalized transitivity rule (11) may be violated when T_M or T_P is used.

like to have that the value of $bb^{\ll}(A, B)$ is close to the value of $bb^{\ll}(A, C)$. Irrespective of the t-norm T being used, it holds that $bb^{\ll}(A, B) = 1$, i.e., the beginning of A is strictly before the beginning of B to degree 1. When using T_M, however, we have that

$$
\begin{aligned}
bb^{\ll}(A, C) &= \sup_{p \in \mathbb{R}} T_M(A(p), \inf_{q \in \mathbb{R}} I_{T_M}(C(q), L^{\ll}(p, q))) \\
&\leq \sup_{p \in \mathbb{R}} T_M(A(p), I_{T_M}(C(c_1), L^{\ll}(p, c_1)))
\end{aligned}
$$

where c_1 is defined as in Figure 3. As for each p in $\mathbb{R}$ either $A(p) = 0$ or $L^{\ll}(p, c_1) = 0$, we establish that $bb^{\ll}(A, C) = 0$. In the same way, we can show that $bb^{\ll}(A, C) = 0$ when using T_P. On the other hand, when $T = T_W$, we can show that

$$
\begin{aligned}
bb^{\ll}(A, C) &= \sup_{p \in \mathbb{R}} T_W(A(p), \inf_{q \in \mathbb{R}} I_{T_W}(C(q), L^{\ll}(p, q))) \\
&= T_W(A(a_2), I_{T_W}(C(c_1), L^{\ll}(a_2, c_1))) \\
&= I_{T_W}(0.1, 0) \\
&= 0.9
\end{aligned}
$$

Another advantage of the Łukasiewicz t-norm is related to transitivity. To ensure that the fuzzy temporal relations (2)–(9) display an intuitive behaviour, it is desirable that they satisfy generalized transitivity rules like

$$
T(bb^{\ll}(A, B), bb^{\ll}(B, C)) \leq bb^{\ll}(A, C) \tag{10}
$$

$$
T(bb^{\prec}(A, B), bb^{\prec}(B, C)) \leq bb^{\prec}(A, C) \tag{11}
$$

expressing that the degree to which the beginning of A is (strictly) before the beginning of C is at least as high as the degree to which both the beginning of A is (strictly) before the beginning of B and the beginning of B is (strictly) before the beginning of C. While it is possible to show that (10) holds for any left-continuous t-norm (i.e., a t-norm whose partial mappings are left-continuous, such as T_M, T_P and T_W), (11) may be violated when T_M or T_P is used. For example, let A, B and C be defined as in Figure 4. Regardless of

11

whether T is T_M, T_P or T_W, it holds that

$$bb^{\ll}(A, B) = bb^{\ll}(B, C) = 0.5$$
$$bb^{\ll}(A, C) = 1$$

This implies

$$T_M(bb^{\preccurlyeq}(C, B), bb^{\preccurlyeq}(B, A))$$
$$= \min(1 - bb^{\ll}(B, C), 1 - bb^{\ll}(A, B))$$
$$= 0.5$$
$$> bb^{\preccurlyeq}(C, A) = 0$$

violating (11). In the same way, we find

$$T_P(bb^{\preccurlyeq}(C, B), bb^{\preccurlyeq}(B, A)) = 0.25 > bb^{\preccurlyeq}(C, A) = 0$$

As shown in [41], all generalized transitivity rules of interest are satisfied when the Łukasiewicz t-norm is used. Although we can use alternative definitions for $bb_d^{\preccurlyeq}$, $ee_d^{\preccurlyeq}$, $be_d^{\preccurlyeq}$ and $eb_d^{\preccurlyeq}$ for which the generalized transitivity rules are satisfied for any left-continuous t-norm, we would then lose the important property that $bb_d^{\ll}(A, B) = 1 - bb_d^{\preccurlyeq}(B, A)$, $ee_d^{\ll}(A, B) = 1 - ee_d^{\preccurlyeq}(B, A)$, etc. We refer to [41] for more details. Henceforth, we will always assume that $T = T_W$. For convenience, we will write I_W instead of I_{T_W}.

Finally, we show two characterizations which will be useful for solving the satisfiability problem in Section 5.

Lemma 1. *It holds that*

$$bb_d^{\ll}(A, B) \geq l \Leftrightarrow (\forall \varepsilon \in]0, l])(\exists \lambda \in]l - \varepsilon, 1])(bb_d^{\ll}(A_\lambda, B_{\lambda+\varepsilon-l})) \tag{12}$$

$$bb_d^{\ll}(A, B) \leq k \Leftrightarrow (\forall \lambda \in]k, 1])(bb_d^{\preccurlyeq}(B_{\lambda-k}, A_\lambda)) \tag{13}$$

$$ee_d^{\ll}(A, B) \geq l \Leftrightarrow (\forall \varepsilon \in]0, l])(\exists \lambda \in]l - \varepsilon, 1])(ee_d^{\ll}(A_{\lambda+\varepsilon-l}, B_\lambda)) \tag{14}$$

$$ee_d^{\ll}(A, B) \leq k \Leftrightarrow (\forall \lambda \in]k, 1])(ee_d^{\preccurlyeq}(B_\lambda, A_{\lambda-k})) \tag{15}$$

$$be_d^{\ll}(A, B) \geq l \Leftrightarrow (\forall \varepsilon \in]0, l[)(\exists \lambda \in [l - \varepsilon, 1])(be_d^{\ll}(A_\lambda, B_{1-\lambda-\varepsilon+l})) \tag{16}$$

$$be_d^{\ll}(A, B) \leq k \Leftrightarrow (\forall \varepsilon \in]0, 1-k[)(\forall \lambda \in [k + \varepsilon, 1])(eb_d^{\preccurlyeq}(B_{1-\lambda+\varepsilon+k}, A_\lambda)) \tag{17}$$

$$eb_d^{\ll}(A, B) \geq l \Leftrightarrow (\forall \varepsilon \in]0, l])(\forall \lambda \in [1 - l + \varepsilon, 1])(eb_d^{\ll}(A_{1-\lambda+\varepsilon+1-l}, B_\lambda)) \tag{18}$$

$$eb_d^{\ll}(A, B) \leq k \Leftrightarrow (\exists \lambda \in [1 - k, 1])(be_d^{\preccurlyeq}(B_\lambda, A_{2-\lambda-k})) \tag{19}$$

Proof. See Appendix A.1. □

Lemma 2. *Let A and B be normalised and convex fuzzy sets in $\mathbb{R}$. Furthermore, let m_a and m_b be arbitrary modal values of A and B respectively. It*

12

holds that

$$bb_d^{\lll}(A, B) = \sup_{p+d<m_b, p\leq m_a} T_W(A(p), 1 - B(p + d)) \tag{20}$$

$$ee_d^{\lll}(A, B) = \sup_{p-d>m_a, p\geq m_b} T_W(B(p), 1 - A(p - d)) \tag{21}$$

Proof. See Appendix A.2. □

3.2 Linear constraints

An atomic linear constraint over a set of variables X is an expression of the form $a_1x_1 + a_2x_2 + \cdots + a_nx_n \triangle b$ where $a_1, a_2, \ldots, a_n, b \in \mathbb{R}$, $x_1, x_2, \ldots, x_n \in X$, and $\triangle$ is $<, \leq, >, \geq, =$ or $\neq$. If $\phi_1, \phi_2, \ldots, \phi_m$ are atomic linear constraints over X, $\phi_1 \vee \phi_2 \vee \cdots \vee \phi_m$ is called a linear constraint over X. If $m > 1$, the linear constraint is called disjunctive. Furthermore, $\phi_1 \vee \phi_2 \vee \cdots \vee \phi_m$ is called a Horn linear constraint if at least $m - 1$ of the disjuncts ϕ_i correspond to $\neq$. In particular, all atomic linear constraints are Horn, as well as, for example, $3x + 4y \leq 6 \vee x \neq 8 \vee y \neq 7 \vee x + 3y \neq 12$. On the other hand, a linear constraint like $3x + 4y \leq 6 \vee x \geq 8$ is not Horn.

The framework of linear constraints subsumes most other frameworks for temporal reasoning, including the Interval Algebra [1] and Temporal Constraint Networks [12], but also, for example, approaches combining qualitative and quantitative information [31] or expressing constraints on the duration of events [35].

A P–interpretation $\mathcal{I}$ over X is a mapping that maps every variable x from X to a real number. For convenience, $\mathcal{I}(x)$ will also we written as $x^{\mathcal{I}}$. An atomic linear constraint $a_1x_1 + a_2x_2 + \cdots + a_nx_n \triangle b$ is satisfied by $\mathcal{I}$ iff $a_1x_1^{\mathcal{I}} + a_2x_2^{\mathcal{I}} + \cdots + a_nx_n^{\mathcal{I}} \triangle b$. A disjunctive linear constraint $\phi_1 \vee \phi_2 \vee \cdots \vee \phi_m$ is satisfied by $\mathcal{I}$ iff $\mathcal{I}$ satisfies at least one of the disjuncts $\phi_1, \phi_2, \ldots, \phi_m$. Let Ψ be a set of linear constraints; $\mathcal{I}$ is called a P–model of Ψ iff $\mathcal{I}$ satisfies every linear constraint in Ψ. If a P–model of Ψ exists, Ψ is called P–satisfiable. It has been shown independently in [23] and [27] that checking the P–satisfiability of a set of Horn linear constraints can be done in polynomial time.

4 Temporal relations between vague events

For A and B fuzzy time intervals, $bb_d^{\lll}(A, B)$, $bb_d^{\prec}(A, B)$, $ee_d^{\lll}(A, B)$, $ee_d^{\prec}(A, B)$, $be_d^{\lll}(A, B)$, $be_d^{\prec}(A, B)$, $eb_d^{\lll}(A, B)$ and $eb_d^{\prec}(A, B)$ take values from $[0, 1]$ ($d \in \mathbb{R}$). The reasoning tasks discussed in this paper are based on upper and lower

13

bounds for these values when the definitions of the fuzzy time intervals A and B are unknown. For example, knowing that x, y, and z are fuzzy time intervals, is it possible that simultaneously $bb_3^{\lll}(x, y) \geq 0.6$, $bb_4^{\lll}(y, z) \leq 0.5$, $be_2^{\lll}(y, z) \leq 0.8$, and $ee_8^{\lll}(x, z) \geq 0.3$? From the available knowledge, what can be said about the possible values of $be_9^{\lll}(x, z)$? Throughout the paper, we will assume that all upper and lower bounds are taken from a fixed, finite set $M = \{0, \Delta, 2\Delta, \ldots, 1\}$, where $\Delta = \frac{1}{\rho}$ for some $\rho \in \mathbb{N} \setminus \{0\}$. For convenience, we write M_0 for $M \setminus \{0\}$ and M_1 for $M \setminus \{1\}$.

Definition 2 (Atomic FI–formula). *An atomic FI–formula over a set of variables X is an expression of the form $r(x, y) \geq l$ or $r(x, y) \leq k$, where $l \in M_0$, $k \in M_1$, $(x, y) \in X^2$ and r is $bb_d^{\lll}$, $ee_d^{\lll}$, $be_d^{\lll}$ or $eb_d^{\lll}$ $(d \in \mathbb{R})$.*

Note that we will not consider atomic FI–formulas like $bb_d^{\vec{\prec}}(x, y) \geq l$ in this paper. Such expressions can be omitted from discussions without loss of generality because of their correspondence to atomic FI–formulas involving $bb_d^{\lll}$, $ee_d^{\lll}$, $be_d^{\lll}$ or $eb_d^{\lll}$. In applications, however, it may be convenient to use $bb_d^{\vec{\prec}}(x, y) \geq l$ as a notational alternative to $bb_d^{\lll}(y, x) \leq 1 - l$.

Definition 3 (FI–formula). *An FI–formula over a set of variables X is an expression of the form*

$$\phi_1 \vee \phi_2 \vee \cdots \vee \phi_n$$

where ϕ_1, ϕ_2, \ldots, ϕ_n are atomic FI–formulas over X. If $n > 1$, the FI–formula is called disjunctive.

Definition 4 (FI–interpretation). *An FI–interpretation over a set of variables X is a mapping that assigns a fuzzy interval to each variable in X. An FI_M–interpretation over X is an FI–interpretation that maps every variable from X to a fuzzy interval which takes only membership degrees from M.*

The interpretation $\mathcal{I}(x)$ of a variable x, corresponding to an FI–interpretation $\mathcal{I}$, will also be written as $x^{\mathcal{I}}$. An FI–interpretation $\mathcal{I}$ over X satisfies the temporal formula $bb_d^{\lll}(x, y) \geq l$ $(x, y \in X, l \in M_0, d \in \mathbb{R})$ iff $bb_d^{\lll}(x^{\mathcal{I}}, y^{\mathcal{I}}) \geq l$, and analogously for other types of atomic FI-formulas. Furthermore, $\mathcal{I}$ satisfies $\phi_1 \vee \phi_2 \vee \cdots \vee \phi_n$ iff $\mathcal{I}$ satisfies ϕ_1 or $\mathcal{I}$ satisfies ϕ_2 or \ldots or $\mathcal{I}$ satisfies ϕ_n.

Definition 5 (FI–satisfiable). *A set Θ of FI–formulas over a set of variables X is said to be FI–satisfiable (resp. FI_M–satisfiable) iff there exists an FI–interpretation (resp. FI_M–interpretation) over X which satisfies every FI–formula in Θ. An FI–interpretation (resp. FI_M–interpretation) meeting this requirement is called an FI–model (resp. FI_M–model) of Θ.*

14

5 FI–satisfiability

One of the most important temporal reasoning tasks consists of checking whether a given knowledge base is consistent. Here, this corresponds to checking the FI–satisfiability of a set of FI–formulas Θ. To solve this problem, we will show how a set Ψ of linear constraints can be constructed which is P–satisfiable iff Θ is FI–satisfiable. In this way, existing, highly optimized reasoners for temporal problems can be reused to reason about vague temporal information. This reduction from FI–satisfiability to P–satisfiability is made possible by virtue of the following proposition, stating that because the upper and lower bounds in a set of FI–formulas are taken from the set M, as defined above, we can restrict ourselves to fuzzy intervals that only take membership degrees from M.

Proposition 1. *Let Θ be a finite set of FI–formulas over X. It holds that Θ is FI–satisfiable iff Θ is FI_M–satisfiable.*

Proof. Clearly, if Θ is FI_M–satisfiable then Θ is also FI–satisfiable. Conversely, we show that given an FI–model $\mathcal{I}$ of Θ, it is always possible to construct an FI_M–model $\mathcal{I}^*$ of Θ.

Let $\mathcal{I}$ be an FI–model of Θ, and let the $[0,1] - [0,1]$ mappings l and u be defined for y_0 in $[0,1]$ as

$$l(y_0) = \max\{y | y \in M \wedge y \leq y_0\}$$
$$u(y_0) = \min\{y | y \in M \wedge y \geq y_0\}$$

From the definition of fuzzy time interval (Definition 1), we establish that for each x in X, there exists an m_x in $\mathbb{R}$ such that $x^{\mathcal{I}}(m_x) = 1$. We now define $\mathcal{I}'$ as a mapping from X to the class of fuzzy sets in $\mathbb{R}$:

$$x^{\mathcal{I}'}(p) = \begin{cases} l(x^{\mathcal{I}}(p)) & \text{if } p \leq m_x \\ u(x^{\mathcal{I}}(p)) & \text{if } p > m_x \end{cases}$$

for all x in X and p in $\mathbb{R}$. Figure 5 depicts the relationship between $\mathcal{I}$ and $\mathcal{I}'$. Although for x in X, the fuzzy set $x^{\mathcal{I}'}$ only takes membership degrees from M, $\mathcal{I}'$ is not an FI_M–interpretation as the α-level sets of $x^{\mathcal{I}'}$ do not necessarily correspond to closed intervals ($\alpha \in]0,1]$, $x \in X$). However, from $\mathcal{I}'$, we can construct an FI_M–interpretation $\mathcal{I}^*$ as follows. Let P be the (finite) set of points in which $x^{\mathcal{I}'}$ is discontinuous for at least one x in X ($P \subseteq \mathbb{R}$), and let P_s^x be the set of points in which $x^{\mathcal{I}'}$ is not upper semi–continuous ($P_s^x \subseteq P$), i.e., P_s^x contains the endpoints of α-level sets of $x^{\mathcal{I}'}$ which do not correspond to closed intervals. Furthermore, let D be the set of distances d occurring in the FI–formulas from Θ ($D \subseteq \mathbb{R}$).

15

(a) $x^{\mathcal{I}}$
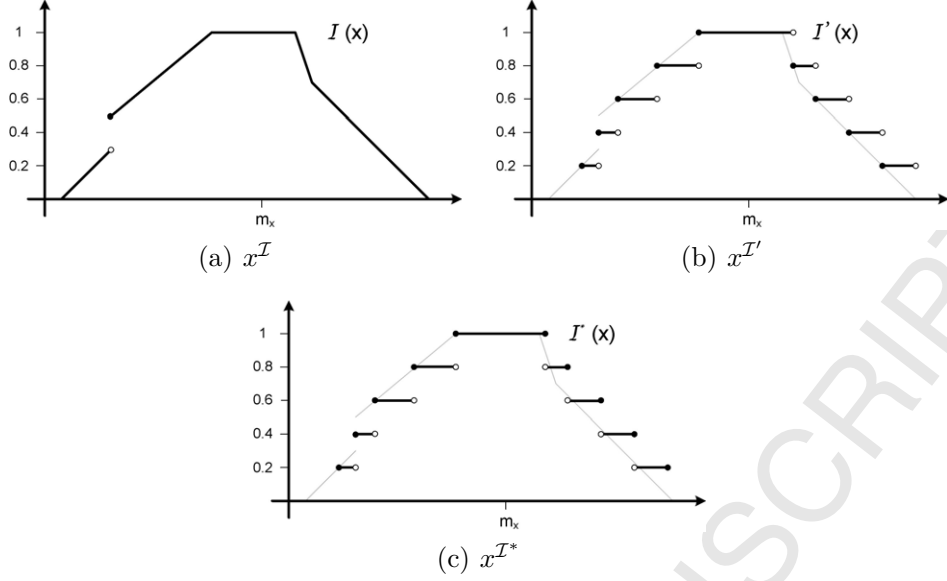
(b) $x^{\mathcal{I}'}$

(c) $x^{\mathcal{I}^*}$

Fig. 5. Relationship between the FI–interpretation $\mathcal{I}$, the mapping $\mathcal{I}'$ and the FI$_M$–interpretation $\mathcal{I}^*$

The FI$_M$–interpretation $\mathcal{I}^*$ is defined as

$$
x^{\mathcal{I}^*}(p) = \begin{cases} \min_{q\in[p,p+\varepsilon[} x^{\mathcal{I}'}(q) & \text{if } (\exists q \in P_s^x)(q \in]p, p+\varepsilon[) \\ x^{\mathcal{I}'}(p) & \text{otherwise} \end{cases}
$$

for all x in X and p in $\mathbb{R}$, where ε denotes an arbitrary, fixed element from $]0, \min\{|p+d-q| : p, q \in P \wedge d \in D \cup \{0\} \wedge p+d \neq q\}[$. The definition of $x^{\mathcal{I}^*}$ is illustrated in Figure 5(c). To prove that $\mathcal{I}^*(x)$ satisfies all FI–formulas from Θ, we first show that for d in $\mathbb{R}$, x and y in X, and m in M, it holds that

$$bb_d^{\ll}(x^{\mathcal{I}}, y^{\mathcal{I}}) \leq m \Rightarrow bb_d^{\ll}(x^{\mathcal{I}'}, y^{\mathcal{I}'}) \leq m \tag{22}$$

$$bb_d^{\ll}(x^{\mathcal{I}}, y^{\mathcal{I}}) \geq m \Rightarrow bb_d^{\ll}(x^{\mathcal{I}'}, y^{\mathcal{I}'}) \geq m \tag{23}$$

$$ee_d^{\ll}(x^{\mathcal{I}}, y^{\mathcal{I}}) \leq m \Rightarrow ee_d^{\ll}(x^{\mathcal{I}'}, y^{\mathcal{I}'}) \leq m \tag{24}$$

$$ee_d^{\ll}(x^{\mathcal{I}}, y^{\mathcal{I}}) \geq m \Rightarrow ee_d^{\ll}(x^{\mathcal{I}'}, y^{\mathcal{I}'}) \geq m \tag{25}$$

$$be_d^{\ll}(x^{\mathcal{I}}, y^{\mathcal{I}}) \leq m \Rightarrow be_d^{\ll}(x^{\mathcal{I}'}, y^{\mathcal{I}'}) \leq m \tag{26}$$

$$be_d^{\ll}(x^{\mathcal{I}}, y^{\mathcal{I}}) \geq m \Rightarrow be_d^{\ll}(x^{\mathcal{I}'}, y^{\mathcal{I}'}) \geq m \tag{27}$$

$$eb_d^{\ll}(x^{\mathcal{I}}, y^{\mathcal{I}}) \leq m \Rightarrow eb_d^{\ll}(x^{\mathcal{I}'}, y^{\mathcal{I}'}) \leq m \tag{28}$$

$$eb_d^{\ll}(x^{\mathcal{I}}, y^{\mathcal{I}}) \geq m \Rightarrow eb_d^{\ll}(x^{\mathcal{I}'}, y^{\mathcal{I}'}) \geq m \tag{29}$$

To show (22) and (23), we obtain by (20)

$$
\begin{aligned}
& bb_d^{\ll}(x^{\mathcal{I}'}, y^{\mathcal{I}'}) \\
& = \sup_{p+d<m_y, p\leq m_x} T_W(x^{\mathcal{I}'}(p), 1 - y^{\mathcal{I}'}(p+d))
\end{aligned}
$$

16

$$= \sup_{p+d<m_y, p\leq m_x} T_W(x^{\mathcal{I}}(p) - (x^{\mathcal{I}}(p) - x^{\mathcal{I}'}(p)),$$
$$1 - y^{\mathcal{I}}(p+d) + (y^{\mathcal{I}}(p+d) - y^{\mathcal{I}'}(p+d)))$$

From the definition of $\mathcal{I}'$, it follows that $(x^{\mathcal{I}}(p) - x^{\mathcal{I}'}(p)) \in [0, \Delta[$ and $(y^{\mathcal{I}}(p+d) - y^{\mathcal{I}'}(p+d)) \in [0, \Delta[$, for $p+d < m_y$ and $p \leq m_x$. Hence, we have

$$\leq \sup_{p+d<m_y, p\leq m_x} T_W(x^{\mathcal{I}}(p), 1 - y^{\mathcal{I}}(p+d) + (y^{\mathcal{I}}(p+d) - y^{\mathcal{I}'}(p+d)))$$
$$\leq \sup_{p+d<m_y, p\leq m_x} T_W(x^{\mathcal{I}}(p), 1 - y^{\mathcal{I}}(p+d)) + (y^{\mathcal{I}}(p+d) - y^{\mathcal{I}'}(p+d))$$
$$< \sup_{p+d<m_y, p\leq m_x} T_W(x^{\mathcal{I}}(p), 1 - y^{\mathcal{I}}(p+d)) + \Delta$$
$$= bb_d^{\lll}(x^{\mathcal{I}}, y^{\mathcal{I}}) + \Delta$$

Similarly, we can show that

$$bb_d^{\lll}(x^{\mathcal{I}'}, y^{\mathcal{I}'}) > bb_d^{\lll}(x^{\mathcal{I}}, y^{\mathcal{I}}) - \Delta$$

Hence

$$bb_d^{\lll}(x^{\mathcal{I}'}, y^{\mathcal{I}'}) - bb_d^{\lll}(x^{\mathcal{I}}, y^{\mathcal{I}}) \in]-\Delta, \Delta[\tag{30}$$

Assume that $bb_d^{\lll}(x^{\mathcal{I}'}, y^{\mathcal{I}'}) > m$ would hold. Since both $bb_d^{\lll}(x^{\mathcal{I}'}, y^{\mathcal{I}'})$ and m are contained in M, this implies that $bb_d^{\lll}(x^{\mathcal{I}'}, y^{\mathcal{I}'}) \geq m + \Delta$. Using (30) we establish that $bb_d^{\lll}(x^{\mathcal{I}}, y^{\mathcal{I}}) > m$ also holds, proving (22) by contraposition. In the same way, we establish from $bb_d^{\lll}(x^{\mathcal{I}'}, y^{\mathcal{I}'}) < m$ that $bb_d^{\lll}(x^{\mathcal{I}'}, y^{\mathcal{I}'}) \leq m - \Delta$ and thus $bb_d^{\lll}(x^{\mathcal{I}}, y^{\mathcal{I}}) < m$, proving (23). The implications (24)–(29) can be shown entirely analogously.

Next, we show that

$$bb_d^{\lll}(x^{\mathcal{I}'}, y^{\mathcal{I}'}) = bb_d^{\lll}(x^{\mathcal{I}^*}, y^{\mathcal{I}^*}) \tag{31}$$
$$ee_d^{\lll}(x^{\mathcal{I}'}, y^{\mathcal{I}'}) = ee_d^{\lll}(x^{\mathcal{I}^*}, y^{\mathcal{I}^*}) \tag{32}$$
$$be_d^{\lll}(x^{\mathcal{I}'}, y^{\mathcal{I}'}) = be_d^{\lll}(x^{\mathcal{I}^*}, y^{\mathcal{I}^*}) \tag{33}$$
$$eb_d^{\lll}(x^{\mathcal{I}'}, y^{\mathcal{I}'}) = eb_d^{\lll}(x^{\mathcal{I}^*}, y^{\mathcal{I}^*}) \tag{34}$$

First note that (31) immediately follows from the definition of $\mathcal{I}^*$ by (20), as for each p satisfying $p \leq m_x$ and $p + d < m_y$, $x^{\mathcal{I}^*}(p) = x^{\mathcal{I}'}(p)$ and $y^{\mathcal{I}^*}(p+d) = y^{\mathcal{I}'}(p+d)$. Turning now to (32), we find using (21) that

$$ee_d^{\lll}(x^{\mathcal{I}'}, y^{\mathcal{I}'}) = \sup_{p-d>m_x, p\geq m_y} T_W(y^{\mathcal{I}'}(p), 1 - x^{\mathcal{I}'}(p-d)) \tag{35}$$
$$ee_d^{\lll}(x^{\mathcal{I}^*}, y^{\mathcal{I}^*}) = \sup_{p-d>m_x, p\geq m_y} T_W(y^{\mathcal{I}^*}(p), 1 - x^{\mathcal{I}^*}(p-d)) \tag{36}$$

17

where m_x and m_y are the smallest modal values of $x^{\mathcal{I}'}$ and $y^{\mathcal{I}'}$, or, equivalently, of $x^{\mathcal{I}^*}$ and $y^{\mathcal{I}^*}$. First, we show that for every p_1 satisfying $p_1 - d > m_x$ and $p_1 \geq m_y$ there exists a p_2 satisfying $p_2 - d > m_x$ and $p_2 \geq m_y$ such that

$$T_W(y^{\mathcal{I}'}(p_1), 1 - x^{\mathcal{I}'}(p_1 - d)) = T_W(y^{\mathcal{I}^*}(p_2), 1 - x^{\mathcal{I}^*}(p_2 - d)) \qquad (37)$$

which already proves $ee_d^{\lll}(x^{\mathcal{I}'}, y^{\mathcal{I}'}) \leq ee_d^{\lll}(x^{\mathcal{I}^*}, y^{\mathcal{I}^*})$. If $y^{\mathcal{I}'}(p_1) = y^{\mathcal{I}^*}(p_1)$ and $x^{\mathcal{I}'}(p_1 - d) = x^{\mathcal{I}^*}(p_1 - d)$ we can choose $p_2 = p_1$. Next, assume that $y^{\mathcal{I}'}(p_1) > y^{\mathcal{I}^*}(p_1)$ and $x^{\mathcal{I}'}(p_1 - d) > x^{\mathcal{I}^*}(p_1 - d)$. This means that there is a q_1 in P_s^y and q_2 in P_s^x such that $q_1 \in]p_1, p_1 + \varepsilon[$ and $q_2 \in]p_1 - d, p_1 - d + \varepsilon[$. The latter implies that $q_2 + d \in]p_1, p_1 + \varepsilon[$ which, combined with the former, yields $|q_2 + d - q_1| < \varepsilon$. By definition of ε, this is only possible if $q_2 = q_1 - d$, since for $q_2 + d \neq q_1$, the definition of ε would imply $\varepsilon < |q_2 + d - q_1|$. We show that (37) is satisfied for $p_2 = q_1 - \varepsilon$. Note that $m_x, m_y \in P$ but $m_x \notin P_s^x$ and $m_y \notin P_s^y$, hence $m_x \neq q_2$ and $m_y \neq q_1$, and even $m_x < q_2$ and $m_y < q_1$. The definition of ε implies that $\varepsilon < q_1 + 0 - m_y$ and $\varepsilon < q_2 + 0 - m_x$. Since $q_2 = q_1 - d$ and $p_2 = q_1 - \varepsilon$, this entails $p_2 \geq m_y$ and $p_2 - d > m_x$. Since $y^{\mathcal{I}'}$ is constant over $[q_1 - \varepsilon, q_1[$, by definition of ε, it holds that $y^{\mathcal{I}^*}(q_1 - \varepsilon) = y^{\mathcal{I}'}(q_1 - \varepsilon)$, or $y^{\mathcal{I}^*}(p_2) = y^{\mathcal{I}'}(p_2)$. As $p_1 \in]q_1 - \varepsilon, q_1[$, it holds that $y^{\mathcal{I}'}$ is constant over $[p_2, p_1]$, hence $y^{\mathcal{I}'}(p_2) = y^{\mathcal{I}'}(p_1)$. In the same way, we establish $x^{\mathcal{I}^*}(p_2 - d) = x^{\mathcal{I}'}(p_1 - d)$.

If $y^{\mathcal{I}'}(p_1) > y^{\mathcal{I}^*}(p_1)$ and $x^{\mathcal{I}'}(p_1 - d) = x^{\mathcal{I}^*}(p_1 - d)$, there is a q_1 in P_s^y such that $q_1 \in]p_1, p_1 + \varepsilon[$. We show that (37) is satisfied for $p_2 = q_1 - \varepsilon$. As in the previous case, the definition of ε implies that $\varepsilon < q_1 + 0 - m_y$, hence $p_2 \geq m_y$. Again, we have that $y^{\mathcal{I}^*}(p_2) = y^{\mathcal{I}^*}(q_1 - \varepsilon) = y^{\mathcal{I}'}(q_1 - \varepsilon) = y^{\mathcal{I}'}(p_2)$, and $y^{\mathcal{I}'}(p_2) = y^{\mathcal{I}'}(p_1)$. Note that $x^{\mathcal{I}'}$ is continuous in $[q_1 - d - \varepsilon, q_1 - d[$. Indeed, if $x^{\mathcal{I}'}$ were discontinuous in a point q_2 in $[q_1 - d - \varepsilon, q_1 - d[$, it would hold that $0 < |q_2 - q_1 + d| \leq \varepsilon$, which is impossible by definition of ε. Since $p_1 - d > m_x$ and there are no discontinuities in $[q_1 - d - \varepsilon, p_1 - d]$, we know that $q_1 - d - \varepsilon > m_x$ (recall that m_x is the smallest modal value of $x^{\mathcal{I}'}$). The continuity of $x^{\mathcal{I}'}$ in $[q_1 - d - \varepsilon, q_1 - d[$ furthermore implies that $x^{\mathcal{I}^*}(q_1 - d - \varepsilon) = x^{\mathcal{I}'}(q_1 - d - \varepsilon)$, and, since $p_1 - d < q_1 - d$, $x^{\mathcal{I}'}(q_1 - d - \varepsilon) = x^{\mathcal{I}'}(p_1 - d)$. From $p_2 = q_1 - \varepsilon$ we can conclude that $x^{\mathcal{I}^*}(p_2 - d) = x^{\mathcal{I}'}(p_1 - d)$

The case where $y^{\mathcal{I}'}(p_1) = y^{\mathcal{I}^*}(p_1)$ and $x^{\mathcal{I}'}(p_1 - d) > x^{\mathcal{I}^*}(p_1 - d)$ is shown entirely analogously.

Conversely, we show that for every p_2 satisfying $p_2 - d > m_x$ and $p_2 \geq m_y$ there exists a p_1 satisfying $p_1 - d > m_x$ and $p_1 \geq m_y$ such that (37) is satisfied. If $y^{\mathcal{I}'}(p_2) = y^{\mathcal{I}^*}(p_2)$ and $x^{\mathcal{I}'}(p_2 - d) = x^{\mathcal{I}^*}(p_2 - d)$ we can choose $p_1 = p_2$. If $y^{\mathcal{I}'}(p_2) > y^{\mathcal{I}^*}(p_2)$ and $x^{\mathcal{I}'}(p_2 - d) > x^{\mathcal{I}^*}(p_2 - d)$ there is a q_1 in P_s^y and q_2 in P_s^x such that $q_1 \in]p_2, p_2 + \varepsilon[$ and $q_2 \in]p_2 - d, p_2 - d + \varepsilon[$. We show that (37) is satisfied for $p_1 = q_1$. Again, this is only possible if $q_2 = q_1 - d$ by definition of ε. First note that $p_1 = q_1$, $q_1 \in]p_2, p_2 + \varepsilon[$ and $p_2 \geq m_y$ entail $p_1 \geq m_y$, while $p_1 = q_1$, $q_1 - d \in]p_2 - d, p_2 + \varepsilon - d[$ and $p_2 - d > m_x$ entail $p_1 - d > m_x$. As $q_1 \in P_s^y$, we know that $y^{\mathcal{I}'}$ is lower semi–continuous in q_1. Since $y^{\mathcal{I}'}$ is decreasing in q_1,

this means that $y^{\mathcal{I}'}$ is right–continuous in q_1. Furthermore, by definition of ε we know that $y^{\mathcal{I}'}$ is continuous over $[q_1 - \varepsilon, q_1[$ and over $]q_1, q_1 + \varepsilon]$. Together with $p_2 \in]q_1 - \varepsilon, q_1[$, this yields $y^{\mathcal{I}^*}(p_2) = y^{\mathcal{I}'}(q_1) = y^{\mathcal{I}'}(p_1)$. Similarly, we can show that $x^{\mathcal{I}^*}(p_2 - d) = x^{\mathcal{I}'}(q_1 - d)$.

If $y^{\mathcal{I}'}(p_2) > y^{\mathcal{I}^*}(p_2)$ and $x^{\mathcal{I}'}(p_2 - d) = x^{\mathcal{I}^*}(p_2 - d)$ there is a q_1 in P_s^y such that $q_1 \in]p_2, p_2 + \varepsilon[$. We show that (37) is satisfied for $p_1 = q_1$. As before, we have that $p_1 \geq m_y$, $p_1 - d > m_x$, and $y^{\mathcal{I}^*}(p_2) = y^{\mathcal{I}'}(q_1)$. Note that $x^{\mathcal{I}'}$ is continuous in $[q_1 - d - \varepsilon, q_1 - d[\cup]q_1 - d, q_1 - d + \varepsilon]$. Indeed, for every q_2 in $[q_1 - d - \varepsilon, q_1 - d + \varepsilon] \setminus \{q_1 - d\}$, it holds that $0 < |q_2 - q_1 + d| \leq \varepsilon$, which implies $q_2 \notin P$, using the definition of ε. If $x^{\mathcal{I}'}$ is also continuous in $q_1 - d$, then obviously $x^{\mathcal{I}^*}(p_2 - d) = x^{\mathcal{I}'}(q_1 - d)$. If $x^{\mathcal{I}'}$ is upper semi–continuous in $q_1 - d$ then $x^{\mathcal{I}'}$ is also left–continuous (as $x^{\mathcal{I}'}$ is decreasing in $q_1 - d$), hence $x^{\mathcal{I}'}(q_1 - d) = x^{\mathcal{I}'}(p_2 - d) = x^{\mathcal{I}^*}(p_2 - d)$. Finally, we show that $x^{\mathcal{I}'}$ cannot be lower semi–continuous in $q_1 - d$. Indeed, this would imply that $q_1 - d \in P_s^x$ and thus $x^{\mathcal{I}'}(p_2 - d) > x^{\mathcal{I}^*}(p_2 - d)$, which contradicts the assumption that $x^{\mathcal{I}'}(p_2 - d) = x^{\mathcal{I}^*}(p_2 - d)$.

The case where $y^{\mathcal{I}'}(p_1) = y^{\mathcal{I}^*}(p_1)$ and $x^{\mathcal{I}'}(p_1 - d) > x^{\mathcal{I}^*}(p_1 - d)$ is shown in the same way.

Finally, (33) and (34) can be shown analogously. $\qquad\square$

From Lemma 1, we already know that upper and lower bounds of fuzzy temporal relations can be characterized by crisp temporal relations between the α–level sets of the fuzzy time intervals involved. As the next proposition shows, when these fuzzy time intervals only take membership degrees from M, only a finite number of α–level sets needs to be considered. The intuition behind this proposition is that a fuzzy interval A taking only membership degrees from M is completely characterized by the set of crisp intervals $\{A_\Delta, A_{2\Delta}, \ldots, A_1\}$, which is in turn completely characterized by the set of instants (real numbers) $\{A_\Delta^-, A_{2\Delta}^-, \ldots, A_1^-, A_1^+, \ldots, A_{2\Delta}^+, A_\Delta^+\}$.

Proposition 2. *Let A and B be fuzzy intervals that only take membership degrees from M, $k \in M_1$ and $l \in M_0$. It holds that:*

$$bb_d^{\lll}(A, B) \geq l \Leftrightarrow A_l^- < B_\Delta^- - d \vee A_{l+\Delta}^- < B_{2\Delta}^- - d$$
$$\vee \cdots \vee A_1^- < B_{1-l+\Delta}^- - d \tag{38}$$
$$bb_d^{\lll}(A, B) \leq k \Leftrightarrow B_\Delta^- \leq A_{k+\Delta}^- + d \wedge B_{2\Delta}^- \leq A_{k+2\Delta}^- + d$$
$$\wedge \cdots \wedge B_{1-k}^- \leq A_1^- + d \tag{39}$$
$$ee_d^{\lll}(A, B) \geq l \Leftrightarrow A_\Delta^+ < B_l^+ - d \vee A_{2\Delta}^+ < B_{l+\Delta}^+ - d$$
$$\vee \cdots \vee A_{1-l+\Delta}^+ < B_1^+ - d \tag{40}$$
$$ee_d^{\lll}(A, B) \leq k \Leftrightarrow B_{k+\Delta}^+ \leq A_\Delta^+ + d \wedge B_{k+2\Delta}^+ \leq A_{2\Delta}^+ + d$$
$$\wedge \cdots \wedge B_1^+ \leq A_{1-k}^+ + d \tag{41}$$
$$be_d^{\lll}(A, B) \geq l \Leftrightarrow A_l^- < B_1^+ - d \vee A_{l+\Delta}^- < B_{1-\Delta}^+ - d$$

$$\vee \cdots \vee A_1^- < B_l^+ - d \tag{42}$$

$$be_d^{\lll}(A, B) \leq k \Leftrightarrow B_1^+ \leq A_{k+\Delta}^- + d \wedge B_{1-\Delta}^+ \leq A_{k+2\Delta}^- + d$$
$$\wedge \cdots \wedge B_{k+\Delta}^+ \leq A_1^- + d \tag{43}$$

$$eb_d^{\lll}(A, B) \geq l \Leftrightarrow A_1^+ < B_{1-l+\Delta}^- - d \wedge A_{1-\Delta}^+ < B_{1-l+2\Delta}^- - d$$
$$\wedge \cdots \wedge A_{1-l+\Delta}^+ < B_1^- - d \tag{44}$$

$$eb_d^{\lll}(A, B) \leq k \Leftrightarrow B_{1-k}^- \leq A_1^+ + d \vee B_{1-k+\Delta}^- \leq A_{1-\Delta}^+ + d$$
$$\vee \cdots \vee B_1^- \leq A_{1-k}^+ + d \tag{45}$$

Proof. As an example, we show (38). From Lemma 1 we already know that

$$bb_d^{\lll}(A, B) \geq l \Leftrightarrow (\forall \varepsilon \in]0, l])(\exists \lambda \in]l - \varepsilon, 1])(bb_d^{\lll}(A_\lambda, B_{\lambda+\varepsilon-l}))$$

First note that if $\varepsilon_1 < \varepsilon_2$, $B_{\lambda+\varepsilon_1-l}^- \leq B_{\lambda+\varepsilon_2-l}^-$ and therefore $bb_d^{\lll}(A_\lambda, B_{\lambda+\varepsilon_1-l}) \Rightarrow bb_d^{\lll}(A_\lambda, B_{\lambda+\varepsilon_2-l})$. We thus obtain

$$bb_d^{\lll}(A, B) \geq l \Leftrightarrow (\forall \varepsilon \in]0, \Delta[)(\exists \lambda \in]l - \varepsilon, 1])(bb_d^{\lll}(A_\lambda, B_{\lambda+\varepsilon-l}))$$

Let $\varepsilon \in]0, \Delta[$ and let λ in $]l - \varepsilon, 1]$ be such that $bb_d^{\lll}(A_\lambda, B_{\lambda+\varepsilon-l})$. We first show that there must exist some λ' in $\{l, l + \Delta, \ldots, 1\}$ such that $bb_d^{\lll}(A_{\lambda'}, B_{\lambda'+\varepsilon-l})$. If $\lambda \in]l - \varepsilon, l[$, then $A_\lambda = A_l$ as A only takes membership degrees from M. Moreover, $B_{\lambda+\varepsilon-l}^- \leq B_{l+\varepsilon-l}^-$, hence from $bb_d^{\lll}(A_\lambda, B_{\lambda+\varepsilon-l})$ we establish $bb_d^{\lll}(A_l, B_{l+\varepsilon-l})$, i.e., we can choose $\lambda' = l$. Similarly, if $\lambda \in]i\Delta, (i + 1)\Delta[$ ($i \in \mathbb{N}$, $l \leq i\Delta < (i + 1)\Delta \leq 1$), we have that $bb_d^{\lll}(A_\lambda, B_{\lambda+\varepsilon-l})$ implies $bb_d^{\lll}(A_{(i+1)\Delta}, B_{(i+1)\Delta+\varepsilon-l})$, and we can choose $\lambda' = (i + 1)\Delta$. This yields

$$bb_d^{\lll}(A, B) \geq l \Leftrightarrow (\forall \varepsilon \in]0, \Delta[)(\exists \lambda \in \{l, l + \Delta, \ldots, 1\})(bb_d^{\lll}(A_\lambda, B_{\lambda+\varepsilon-l}))$$

Since now $\lambda \in M$ and $\varepsilon \in]0, \Delta[$, we have that $B_{\lambda+\varepsilon-l} = B_{\lambda+\Delta-l}$:

$$bb_d^{\lll}(A, B) \geq l \Leftrightarrow (\forall \varepsilon \in]0, \Delta[)(\exists \lambda \in \{l, l + \Delta, \ldots, 1\})(bb_d^{\lll}(A_\lambda, B_{\lambda+\Delta-l}))$$
$$\Leftrightarrow (\exists \lambda \in \{l, l + \Delta, \ldots, 1\})(bb_d^{\lll}(A_\lambda, B_{\lambda+\Delta-l}))$$

proving (38). $\qquad\square$

Given a set of atomic FI–formulas Θ over a set of variables X, we construct a set of variables X' and a set of linear constraints Ψ over X' such that Θ is FI–satisfiable iff Ψ is P–satisfiable. From Proposition 1, we know that we can restrict ourselves to fuzzy intervals that only take membership degrees from M. Proposition 2 furthermore reveals that checking whether an FI_M interpretatation satisfies an FI–formula can be done by evaluating a constant number of linear inequalities. This suggests the following procedure for constructing X' and Ψ.

Let X' and Ψ initially be the empty set. For each variable x in X, we add the new variables $x_\Delta^-, x_{2\Delta}^-, \ldots, x_1^-, x_1^+, \ldots, x_{2\Delta}^+$ and x_Δ^+ to X'. Intuitively, these

20

new variables correspond to the beginning and ending points of α–level sets of the fuzzy interval corresponding with x. By adding the following linear constraints to Ψ, for each m in $M_1 \setminus \{0\}$, we ensure that in every P–model $\mathcal{I}$ of Ψ, these new variables can indeed be interpreted as beginning and ending points of α-level sets of a fuzzy interval:

$$x_1^- \leq x_1^+ \tag{46}$$
$$x_m^- \leq x_{m+\Delta}^- \tag{47}$$
$$x_{m+\Delta}^+ \leq x_m^+ \tag{48}$$

In this way, every P–model $\mathcal{I}$ of Ψ corresponds to an FI_M–interpretation $\mathcal{I}'$ of Θ in which $x^{\mathcal{I}'}$ is the fuzzy interval taking only membership degrees from M, defined through its α–level sets by $(\mathcal{I}'(x))_m = [\mathcal{I}(x_m^-), \mathcal{I}(x_m^+)]$ for all $m \in M_0$.

Finally, for each FI–formula in Θ, we add a particular set of linear constraints to Ψ, based on the equivalences of Proposition 2. For example, if Θ contains the FI–formula $bb_d^{\lll}(x, y) \leq k$, we add the following set of linear constraints:

$$\{y_\Delta^- \leq x_{k+\Delta}^- + d, y_{2\Delta}^- \leq x_{k+2\Delta}^- + d, \ldots, y_{1-k}^- \leq x_1^- + d\} \tag{49}$$

Similarly, if Θ contains the FI–formula $bb_d^{\lll}(x, y) \geq l$, we add the following linear constraint:

$$x_l^- < y_\Delta^- - d \vee x_{l+\Delta}^- < y_{2\Delta}^- - d \vee \cdots \vee x_1^- < y_{1-l+\Delta}^- - d \tag{50}$$

Clearly, $\Theta \cup \{r_1 \vee r_2\}$ is FI–satisfiable iff either $\Theta \cup \{r_1\}$ is FI–satisfiable or $\Theta \cup \{r_2\}$ is FI–satisfiable. Therefore, we only needed to consider sets of atomic FI–formulas in the procedure described above. Nonetheless, this procedure is not inherently restricted to sets of atomic FI–formulas, as disjunctive FI–formulas correspond to (sets of) linear constraints as well. However, the number of linear constraints can be exponential in the number of disjuncts in the FI–formulas.

As expressed by the following proposition, it holds that the set of linear constraints Ψ is P–satisfiable iff Θ is FI–satisfiable.

Proposition 3. *Let Θ be a finite set of atomic FI–formulas over X, and let Ψ be the corresponding set of linear constraints over X', obtained by the procedure outlined above. It holds that Θ is FI–satisfiable iff Ψ is P–satisfiable.*

Proof. Assume that Θ is FI–satisfiable. Then there exists an FI_M–model $\mathcal{I}$ of Θ by Proposition 1. We define the P–interpretation $\mathcal{I}'$ for all variables $x_{i\Delta}^-$ and $x_{i\Delta}^+$ as ($i \in \mathbb{N}$, $\Delta \leq i\Delta \leq 1$):

$$\mathcal{I}'(x_{i\Delta}^-) = (x^{\mathcal{I}})_{i\Delta}^- \tag{51}$$
$$\mathcal{I}'(x_{i\Delta}^+) = (x^{\mathcal{I}})_{i\Delta}^+ \tag{52}$$

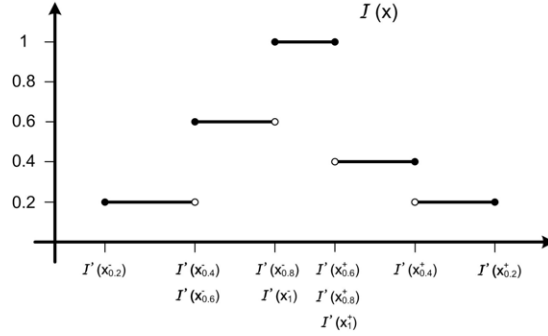Fig. 6. There exists an FI_M–interpretation $\mathcal{I}$ for a set of FI-formulas Θ iff there exists a P–interpretation $\mathcal{I}'$ for the corresponding set Ψ of linear constraints.

In other words, $\mathcal{I}'(x_{i\Delta}^-)$ and $\mathcal{I}'(x_{i\Delta}^+)$ correspond to the beginning and ending of the $i\Delta$–level set of the fuzzy time interval $x^{\mathcal{I}}$. Figure 6 illustrates the relationship between $\mathcal{I}$ and $\mathcal{I}'$. Clearly, $\mathcal{I}'$ satisfies (46)–(48). By Proposition 2, we also have that all (sets of) linear constraints like (49) and (50) are satisfied. Hence, $\mathcal{I}$ is a P–model of Ψ.

Conversely, assume that Ψ is P–satisfiable. Then there exists a P–model $\mathcal{I}'$ of Ψ. We define the FI_M–interpretation $\mathcal{I}$ from $\mathcal{I}'$ as

$$
x^{\mathcal{I}}(r) = \begin{cases} \max\{\lambda | \lambda \in M \wedge r \in [\mathcal{I}'(x_\lambda^-), \mathcal{I}'(x_\lambda^+)]\} & \text{if } r \in [\mathcal{I}'(x_\Delta)^-, \mathcal{I}'(x_\Delta)^+] \\ 0 & \text{otherwise} \end{cases}
$$
(53)

for each x in X and r in $\mathbb{R}$. By construction of Ψ, we have that $x^{\mathcal{I}}$ is a fuzzy time interval. Moreover, by Proposition 2, we establish that $\mathcal{I}$ satisfies every FI-formula in Θ. $\qquad\square$

Interestingly, by reducing FI–satisfiability to P–satisfiability of a set of linear constraints, we can impose additional constraints on the variables involved. For example, we can express that a given variable x corresponds to a crisp interval, rather than a fuzzy interval, by adding the linear constraints $\{x_\Delta^- = x_{2\Delta}^-, x_{2\Delta}^- = x_{3\Delta}^-, \ldots, x_{1-\Delta}^- = x_1^-, x_1^+ = x_{1-\Delta}^+, \ldots, x_{2\Delta}^+ = x_\Delta^+\}$ to Ψ. By additionally adding $x_1^- = x_1^+$, we can even ensure that x is always interpreted as an instant (time point). Similarly, we can add $x_1^- < x_1^+$ to express that x should never be interpreted as a time point. If we know that the beginning of x is inherently gradual, we can even impose $\{x_\Delta^- < x_{2\Delta}^-, x_{2\Delta}^- < x_{3\Delta}^-, \ldots, x_{1-\Delta}^- < x_1^-\}$. Such additional constraints can be very useful if it is *a priori* known which variables correspond to (possibly) vague events, crisp events, and instants. Moreover, adding such constraints does not change the computational complexity of the algorithm.

Another important advantage of the reduction to P–satisfiability is that exist-

22

ing, optimized algorithms for reasoning about linear constraints can be used. Existing algorithms can not only be used for checking FI–satisfiability, but also to find FI–models (or consistent scenarios) of FI–satisfiable sets of FI–formulas. This technique is illustrated in the following example.

Example 1. Let $\Theta = \{bb_{10}^{\lll}(a, b) \geq 0.5, eb_5^{\lll}(c, a) \geq 0.5, eb_5^{\lll}(b, c) \geq 0.75\}$. We can choose $\Delta = 0.25$, and thus $M = \{0, 0.25, 0.5, 0.75, 1\}$. The linear constraints of the form (46)–(48) are given by

$$\begin{aligned}
\Psi_1 = \{ & a_{0.25}^- \leq a_{0.5}^-, a_{0.5}^- \leq a_{0.75}^-, a_{0.75}^- \leq a_1^-, a_1^- \leq a_1^+, \\
& a_1^+ \leq a_{0.75}^+, a_{0.75}^+ \leq a_{0.5}^+, a_{0.5}^+ \leq a_{0.25}^+, \\
& b_{0.25}^- \leq b_{0.5}^-, b_{0.5}^- \leq b_{0.75}^-, b_{0.75}^- \leq b_1^-, b_1^- \leq b_1^+, \\
& b_1^+ \leq b_{0.75}^+, b_{0.75}^+ \leq b_{0.5}^+, b_{0.5}^+ \leq b_{0.25}^+, \\
& c_{0.25}^- \leq c_{0.5}^-, c_{0.5}^- \leq c_{0.75}^-, c_{0.75}^- \leq c_1^-, c_1^- \leq c_1^+, \\
& c_1^+ \leq c_{0.75}^+, c_{0.75}^+ \leq c_{0.5}^+, c_{0.5}^+ \leq c_{0.25}^+ \}
\end{aligned}$$

The additional linear constraints corresponding to the FI–formulas in Θ are given by

$$\begin{aligned}
\Psi_2 = \{ & a_{0.5}^- < b_{0.25}^- - 10 \vee a_{0.75}^- < b_{0.5}^- - 10 \vee a_1^- < b_{0.75}^- - 10, \\
& c_1^+ < a_{0.75}^- - 5, c_{0.75}^+ < a_1^- - 5, \\
& b_1^+ < c_{0.5}^- - 5, b_{0.75}^+ < c_{0.75}^- - 5, b_{0.5}^+ < c_1^- - 5 \}
\end{aligned}$$

The set Ψ of all linear constraints corresponding with Θ is then given by $\Psi_1 \cup \Psi_2$. It holds that Ψ can be satisfied by choosing the first disjunct $a_{0.5}^- < b_{0.25}^- - 10$ in the disjunctive linear constraint $a_{0.5}^- < b_{0.25}^- - 10 \vee a_{0.75}^- < b_{0.5}^- - 10 \vee a_1^- < b_{0.75}^- - 10$. In [20], an algorithm is presented to find a solution of a set of atomic linear constraints, i.e., a P–interpretation $\mathcal{I}$ satisfying Ψ. One possible solution of Ψ is defined by

$$\begin{aligned}
& \mathcal{I}(a_{0.25}^-) = \mathcal{I}(a_{0.5}^-) = \mathcal{I}(c_{0.25}^-) = 0 \\
& \mathcal{I}(a_{0.75}^-) = \mathcal{I}(a_1^-) = \mathcal{I}(a_1^+) = \mathcal{I}(a_{0.75}^+) = \mathcal{I}(a_{0.5}^+) = \mathcal{I}(a_{0.25}^+) = 22 \\
& \mathcal{I}(b_{0.25}^-) = \mathcal{I}(b_{0.5}^-) = \mathcal{I}(b_{0.75}^-) = \mathcal{I}(b_1^-) = 11 \\
& \mathcal{I}(b_1^+) = \mathcal{I}(b_{0.75}^+) = \mathcal{I}(b_{0.5}^+) = \mathcal{I}(b_{0.25}^+) = 11 \\
& \mathcal{I}(c_{0.5}^-) = \mathcal{I}(c_{0.75}^-) = \mathcal{I}(c_1^-) = \mathcal{I}(c_1^+) = \mathcal{I}(c_{0.75}^+) = \mathcal{I}(c_{0.5}^+) = \mathcal{I}(c_{0.25}^+) = 16
\end{aligned}$$

As explained above, the P–model $\mathcal{I}$ of Ψ defines an FI_M–model $\mathcal{I}'$ of Θ.

In applications, we usually need to find an FI–satisfiable set of FI–formulas, corresponding to some given (natural language) description, rather than checking the FI–satisfiablity of a given set of FI–formulas. Typically, in this context, the information provided may be inconsistent when interpreted as classical, crisp temporal relations. The goal is then to weaken information such as *A happened before B* to *A happened before B at least to degree 0.8*. The various lower and upper bounds introduced in this way (e.g., 0.8) should be the

23

strongest possible, w.r.t. a given precision Δ. The next example illustrates this process.

Example 2. Consider again the example about Picasso's work from the introduction. To allow for a concise description, we use the following abbreviations to refer to the relevant events and periods:

BFT	Picasso creates Bread and Fruit Dish on a Table
DMA	Picasso creates the Demoiselles d'Avignon
AC	Picasso's Analytical Cubism period
C	Picasso's Cubism period

The information that Bread and Fruit Dish on a Table marks the beginning of Picasso's Analytical Cubism can be represented as

$$bb^{\preccurlyeq}(BFT, AC) \geq \lambda_1 \tag{54}$$

$$bb^{\preccurlyeq}(AC, BFT) \geq \lambda_2 \tag{55}$$

$$ee^{\ll}(BFT, AC) \geq \lambda_3 \tag{56}$$

where initially λ_1, λ_2 and λ_3 are assumed to be 1. Values lower than 1 are only considered when inconsistencies arise. Similarly, the information that the Demoiselles d'Avignon marks the beginning of Picasso's Cubist period can be represented as

$$bb^{\preccurlyeq}(DMA, C) \geq \lambda_4 \tag{57}$$

$$bb^{\preccurlyeq}(C, DMA) \geq \lambda_5 \tag{58}$$

$$ee^{\ll}(DMA, C) \geq \lambda_6 \tag{59}$$

Next, the information that Analytical Cubism is the first stage of Picasso's Cubism can be represented by

$$bb^{\preccurlyeq}(AC, C) \geq \lambda_7 \tag{60}$$

$$bb^{\preccurlyeq}(C, AC) \geq \lambda_8 \tag{61}$$

$$ee^{\ll}(AC, C) \geq \lambda_9 \tag{62}$$

In addition to this qualitative description, we also have some quantitative information. In particular, we know that Bread and Fruit Dish on a Table was created in 1909, the Demoiselles d'Avignon was created in 1907 and Analytical Cubism lasted from somewhere in 1908 to somewhere in 1912. We can encode this information using metric constraints by referring to an artificial time point Z, for instance, corresponding to the beginning of the year 1900:

$$bb_8^{\preccurlyeq}(Z, AC) \geq \lambda_{10} \qquad\qquad bb_{-9}^{\ll}(AC, Z) \geq \lambda_{11} \tag{63}$$

$$ee_{12}^{\preccurlyeq}(Z, AC) \geq \lambda_{12} \qquad\qquad ee_{-13}^{\ll}(AC, Z) \geq \lambda_{13} \tag{64}$$

$$bb_9^{\preccurlyeq}(Z, BFT) \geq \lambda_{14} \qquad\qquad ee_{-10}^{\ll}(BFT, Z) \geq \lambda_{15} \tag{65}$$

$$bb_7^{\preccurlyeq}(Z, DMA) \geq \lambda_{16} \qquad\qquad ee_{-8}^{\ll}(DMA, Z) \geq \lambda_{17} \tag{66}$$

24

When checking the FI–satisfiability of this representation for various values of the lower bounds λ_i, we need to ensure that Z is a time point. As discussed above, this can be done by adding the constraint $Z_\Delta^- = Z_{2\Delta}^- = \cdots = Z_1^- = Z_1^+ = \cdots = Z_\Delta^+$ to the corresponding set of linear constraints. From available domain knowledge, we may moreover find out that creating a painting is a crisp event [6], and therefore impose that DMA and BFT are crisp intervals in a similar way.

For $\lambda_1 = \lambda_2 = \cdots = \lambda_{17} = 1$, the description above is not FI–satisfiable. Hence, we need to weaken one or more of the lower bounds, i.e., we let some of the λ_i correspond to values from M lower than 1. Different sets of lower bounds may be weakened to obtain an FI–satisfiable representation. Moreover, the actual strategy adopted to decide how to arrive at such a representation may differ from application to application, as well as depend on additional background information (e.g., degrees of confidence in each of the original natural language statements). In the example at hand, we may impose that $\lambda_{14} = \lambda_{15} = \lambda_{16} = \lambda_{17} = 1$, as Z, BFT and DMA all refer to crisp events. Furthermore, we may initially require that $\lambda_1 = \lambda_2 = \cdots = \lambda_{13}$, as we lack any further background knowledge for differentiating between the FI–formulas. Assuming $\Delta = 0.25$, we first try $\lambda_1 = \cdots = \lambda_{13} = 0.75$, which is not FI–satisfiable, and next $\lambda_1 = \cdots = \lambda_{13} = 0.5$, which turns out to be FI–satisfiable. Although we have now arrived at an FI–satisfiable interpretation of the natural language statements, it is not necessarily *maximally* FI–satisfiable, i.e., it may be the case that not all of the λ_i's ($i \in \{1, 2, \ldots, 13\}$) need to be weakened to 0.5. Therefore, we subsequently try to strengthen the λ_i's again, one by one. For example, when $\lambda_2 = 0.75$ or even $\lambda_2 = 1$, the resulting representation remains FI–satisfiable. On the other hand, strengthening λ_1 to 0.75 leads to a representation which is not FI–satisfiable anymore (even when $\lambda_2 = 0.5$). Thus, after a linear number of FI–satisfiability checks, we obtain the following maximally FI–satisfiable representation:

$$\lambda_{14} = \lambda_{15} = \lambda_{16} = \lambda_{17} = 1$$
$$\lambda_2 = \lambda_3 = \lambda_4 = \lambda_6 = \lambda_7 = \lambda_8 = \lambda_9 = \lambda_{10} = \lambda_{12} = \lambda_{13} = 1$$
$$\lambda_1 = \lambda_5 = \lambda_{11} = 0.5$$

A corresponding FI–interpretation is depicted in Figure 7, illustrating that the inconsistencies in the original natural language statements are caused by the vagueness of the Analytical Cubism and Cubism periods. In this FI–interpretation, both periods are assumed to have started in 1907 to degree 0.5,

---

[6] Although the assumption made in this example is reasonable in most contexts, creating a painting could be seen as a vague event as well, assuming, for instance, that related studies and sketches made prior to the actual painting belong to the creation to varying degrees.
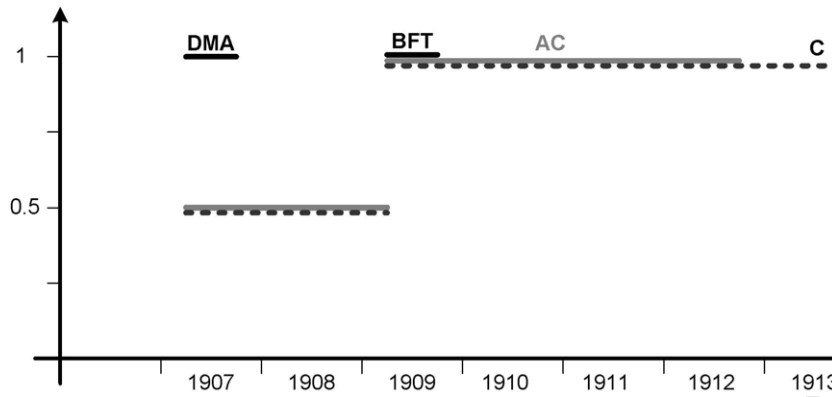
Fig. 7. FI–interpretation of events corresponding to the creation of Bread and Fruit Dish on a Table (BFT) and the Demoiselles d'Avignon (DMA), as well as Picasso's Analytical Cubism (AC) and Cubism (C) periods.

and to have started completely in 1909.

6 Computational complexity

Let $\mathcal{A}$ be a subset of $\mathcal{F}_X$, the set of all FI–formulas over a set of variables X. In the following discussion, we assume that X contains a sufficiently large, or infinite number of different variables. We call FISAT($\mathcal{A}$) the problem of deciding whether a finite set of FI–formulas from $\mathcal{A}$ is FI–satisfiable. Deciding the P–satisfiability of an arbitrary set of linear constraints is NP–complete [44]. To decide whether a set Θ of FI–formulas is FI–satisfiable, we can guess which disjuncts can be satisfied for all disjunctive FI–formulas, resulting in a set of atomic FI–formulas Θ'. Checking if Θ' is FI–satisfiable can be polyomially reduced to checking the P–satisfiability of a set of linear constraints, as explained above. We thus find that FISAT($\mathcal{A}$) is in NP for every $\mathcal{A} \subseteq \mathcal{F}_X$. As will become clear below, FISAT($\mathcal{F}_X$) is also NP–hard and thereby NP–complete. However, checking the P–satisfiability of a set of linear constraints without disjunctions is tractable [23,27]. From Proposition 2, it follows that a significant subset of the FI–formulas do not lead to disjunctive linear constraints. We will refer to this subset as $\mathcal{F}_X^t$:

$$\mathcal{F}_X^t = \bigcup_{(x,y)\in X^2} \bigcup_{d\in\mathbb{R}} \Big(\{bb_d^{\lll}(x,y) \leq k | k \in M_1\} \cup \{ee_d^{\lll}(x,y) \leq k | k \in M_1\}$$
$$\cup \{be_d^{\lll}(x,y) \leq k | k \in M_1\} \cup \{eb_d^{\lll}(x,y) \geq l | l \in M_0\}$$
$$\cup \{bb_d^{\lll}(x,y) \geq 1, ee_d^{\lll}(x,y) \geq 1, be_d^{\lll}(x,y) \geq 1, eb_d^{\lll}(x,y) \leq 0\} \Big)$$

Clearly FISAT($\mathcal{F}_X^t$) is tractable. Note, however, that the procedure described above for deciding FISAT($\mathcal{F}_X^t$) is only weakly polynomial, as it depends on the value of $\frac{1}{\Delta} = \rho$.

26

To support efficient reasoning, it is of interest to identify maximally tractable subsets of $\mathcal{F}_X$, i.e., sets of FI–formulas $\mathcal{A} \subseteq \mathcal{F}_X$ such that FISAT($\mathcal{A}$) is tractable and for any proper superset $\mathcal{A}'$ of $\mathcal{A}$, it holds that FISAT($\mathcal{A}'$) is NP–complete [7] . As we show in the following two propositions, when extending $\mathcal{F}_X^t$ with FI–formulas, it is not possible to keep tractability without putting restrictions on the variables.

Proposition 4. *Let $k \in M_1 \setminus \{0\}$ and $d \in \mathbb{R}$. FISAT(A) is NP-complete if $\mathcal{A}$ contains any of the following sets of FI-formulas:*

$$\mathcal{F}_X^t \cup \bigcup_{(x,y) \in X^2} \{bb_d^{\ll}(x,y) \geq k\} \tag{67}$$

$$\mathcal{F}_X^t \cup \bigcup_{(x,y) \in X^2} \{ee_d^{\ll}(x,y) \geq k\} \tag{68}$$

$$\mathcal{F}_X^t \cup \bigcup_{(x,y) \in X^2} \{be_d^{\ll}(x,y) \geq k\} \tag{69}$$

$$\mathcal{F}_X^t \cup \bigcup_{(x,y) \in X^2} \{eb_d^{\ll}(x,y) \leq k\} \tag{70}$$

Proof. As an example, we show (67) for $d = 0$. The proof for (68)–(70) and $d \neq 0$ is entirely analogous.

Since FISAT($\mathcal{F}_X$) is in NP, we already have that FISAT($\mathcal{A}$) is in NP. To establish the NP-hardness of FISAT($\mathcal{A}$), we will show that 3SAT can be polynomially reduced to it. The proof is inspired by [37], where a similar reduction is made to prove NP-hardness for the satisfiability problem in a subfragment of the Interval Algebra.

Let $\mathcal{D} = \{C_1, C_2, \ldots, C_n\}$, where C_i denotes a clause of the form $l_{i1} \vee l_{i2} \vee l_{i3}$, containing exactly three disjuncts. Each literal l_{ij} is either an atomic proposition or the negation of an atomic proposition. 3SAT is the problem of deciding whether $\mathcal{D}$ is satisfiable, i.e., deciding if there exists a truth assignment of the atomic propositions that makes all clauses from $\mathcal{D}$ true. To prove (67), we will construct a set Θ of FI-formulas from $\mathcal{A}$ which is FI-satisfiable iff $\mathcal{D}$ is satisfiable, thereby reducing 3SAT to FISAT($\mathcal{A}$).

For each i in $\{1, \ldots, n\}$ and j in $\{1, 2, 3\}$, we add the following FI–formulas to Θ:

$$bb^{\ll}(a_{ij}, b_{ij}) \geq k \tag{71}$$
$$bb^{\ll}(c_{ij}, b_{ij}) \leq k - \Delta \tag{72}$$

where a_{ij}, b_{ij} and c_{ij} are different variables from X.

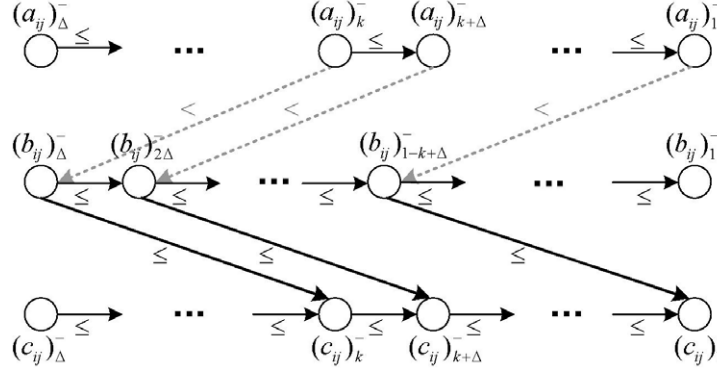[7] Throughout the paper, we assume P $\neq$ NP.

27

Fig. 8. Linear constraints (73)–(74).

These FI-formulas correspond to the following linear constraints:

$$(a_{ij})_k^- < (b_{ij})_\Delta^- \vee (a_{ij})_{k+\Delta}^- < (b_{ij})_{2\Delta}^- \vee \cdots \vee (a_{ij})_1^- < (b_{ij})_{1-k+\Delta}^- \qquad (73)$$

$$\{(b_{ij})_\Delta^- \leq (c_{ij})_k^-, (b_{ij})_{2\Delta}^- \leq (c_{ij})_{k+\Delta}^-, \ldots, (b_{ij})_{1-k+\Delta}^- \leq (c_{ij})_1^-\} \qquad (74)$$

Linear constraints can be depicted as a graph in which nodes correspond to variables, and edges labeled with $<$ or $\leq$ are added between two nodes if $<$ or $\leq$ is imposed on the corresponding variables. Figure 8 shows the graph corresponding to (73)–(74). Linear constraints with disjunctions are displayed as dotted lines, as only one of several possible edges needs to be satisfied in this case. Furthermore, we add the following FI–formulas to Θ:

$$bb^\ll(c_{i1}, d_{i1}) \geq 1 \qquad\qquad bb^\ll(a_{i2}, d_{i1}) \leq 1 - \Delta$$
$$bb^\ll(c_{i2}, d_{i2}) \geq 1 \qquad\qquad bb^\ll(a_{i3}, d_{i2}) \leq 1 - \Delta$$
$$bb^\ll(c_{i3}, d_{i3}) \geq 1 \qquad\qquad bb^\ll(a_{i1}, d_{i3}) \leq 1 - \Delta$$

The corresponding linear constraints are given by

$$(c_{i1})_1^- < (d_{i1})_\Delta^- \qquad\qquad\qquad (d_{i1})_\Delta^- \leq (a_{i2})_1^- \qquad (75)$$
$$(c_{i2})_1^- < (d_{i2})_\Delta^- \qquad\qquad\qquad (d_{i2})_\Delta^- \leq (a_{i3})_1^- \qquad (76)$$
$$(c_{i3})_1^- < (d_{i3})_\Delta^- \qquad\qquad\qquad (d_{i3})_\Delta^- \leq (a_{i1})_1^- \qquad (77)$$

Figure 9 contains a graph corresponding to Figure 8 for a_{i1}, b_{i1} and c_{i1}, as well as the graph for a_{i2}, b_{i2} and c_{i2}, and the graph for a_{i3}, b_{i3} and c_{i3}. For clarity, the nodes for the b_{ij}–variables are omitted. Furthermore, these three subgraphs are linked together by the constraints (75)–(77). If $(a_{i1})_1^- < (c_{i1})_1^-$, $(a_{i2})_1^- < (c_{i2})_1^-$ and $(a_{i3})_1^- < (c_{i3})_1^-$ would hold, we obtain $(a_{i1})_1^- < (c_{i1})_1^- < (a_{i2})_1^- < (c_{i2})_1^- < (a_{i3})_1^- < (c_{i3})_1^- < (a_{i1})_1^-$, and thus $(a_{i1})_1^- < (a_{i1})_1^-$ which cannot be satisfied. Hence, every FI–model of Θ corresponds to a P–model in which at least one of $(a_{i1})_1^- \geq (c_{i1})_1^-$, $(a_{i2})_1^- \geq (c_{i2})_1^-$ and $(a_{i3})_1^- \geq (c_{i3})_1^-$ holds. A truth assignment that makes l_{ij} true if $(a_{ij})_1^- \geq (c_{ij})_1^-$ will therefore make all clauses in $\mathcal{D}$ true. To ensure that such a truth assignment indeed exists, what remains is to make sure that an atomic proposition l_{ij} and its negation,
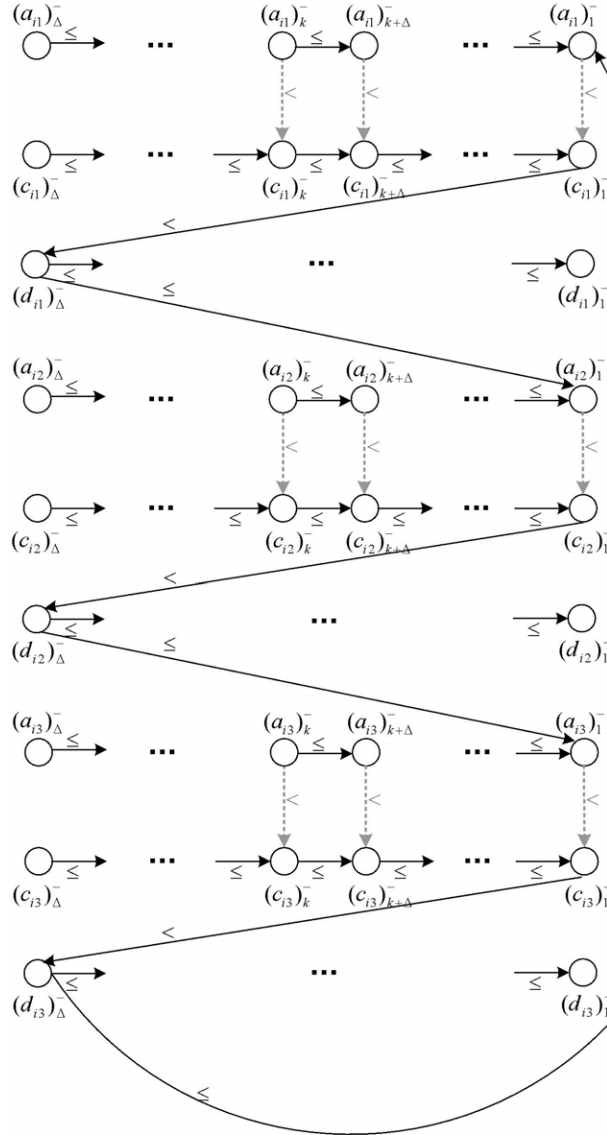
28

Fig. 9. Linear constraints (75)–(77).

denoted below by l_{rs}, are not made true simultaneously. If we want to define a correspondence between Θ and $\mathcal{D}$, we therefore need to encode that one of $(a_{ij})_1^- < (c_{ij})_1^-$ or $(a_{rs})_1^- < (c_{rs})_1^-$ must hold. This can be accomplished by adding the following FI–formulas to Θ:

$$bb^{\ll}(e_{ijrs}, c_{ij}) \leq \Delta \tag{78}$$

$$bb^{\ll}(e_{ijrs}, f_{ijrs}) \geq 1 \tag{79}$$

$$bb^{\ll}(a_{rs}, f_{ijrs}) \leq k - \Delta \tag{80}$$

$$bb^{\ll}(e_{rsij}, c_{rs}) \leq \Delta \tag{81}$$

$$bb^{\ll}(e_{rsij}, f_{rsij}) \geq 1 \tag{82}$$

29

$$bb^{\ll}(a_{ij}, f_{rsij}) \leq k - \Delta \tag{83}$$

which correspond to the following (sets of) linear constraints

$$\{(c_{ij})^-_\Delta \leq (e_{ijrs})^-_{2\Delta}, (c_{ij})^-_{2\Delta} \leq (e_{ijrs})^-_{3\Delta}, \ldots, (c_{ij})^-_{1-\Delta} \leq (e_{ijrs})^-_1\} \tag{84}$$

$$(e_{ijrs})^-_1 < (f_{ijrs})^-_\Delta \tag{85}$$

$$\{(f_{ijrs})^-_\Delta \leq (a_{rs})^-_k, (f_{ijrs})^-_{2\Delta} \leq (a_{rs})^-_{k+\Delta}, \ldots, (f_{ijrs})^-_{1-k+\Delta} \leq (a_{rs})^-_1\} \tag{86}$$

$$\{(c_{rs})^-_\Delta \leq (e_{rsij})^-_{2\Delta}, (c_{rs})^-_{2\Delta} \leq (e_{rsij})^-_{3\Delta}, \ldots, (c_{rs})^-_{1-\Delta} \leq (e_{rsij})^-_1\} \tag{87}$$

$$(e_{rsij})^-_1 < (f_{rsij})^-_\Delta \tag{88}$$

$$\{(f_{rsij})^-_\Delta \leq (a_{ij})^-_k, (f_{rsij})^-_{2\Delta} \leq (a_{ij})^-_{k+\Delta}, \ldots, (f_{rsij})^-_{1-k+\Delta} \leq (a_{ij})^-_1\} \tag{89}$$

Figure 10 displays these linear constraints. In particular, (84)–(89) imply that $(c_{ij})^-_{1-\Delta} < (a_{rs})^-_k$ and $(c_{rs})^-_{1-\Delta} < (a_{ij})^-_k$. Assume that there exists an FI-model of Θ such that the corresponding P–model $\mathcal{I}'$ neither satisfies $(a_{ij})^-_1 < (c_{ij})^-_1$ nor $(a_{rs})^-_1 < (c_{rs})^-_1$. Then there exist a k_1 and a k_2 in M such that $k \leq k_1 \leq 1 - \Delta$ and $k \leq k_2 \leq 1 - \Delta$, and such that $\mathcal{I}'$ satisfies $(a_{ij})^-_{k_1} < (c_{ij})^-_{k_1}$ and $(a_{rs})^-_{k_2} < (c_{rs})^-_{k_2}$. We obtain $(c_{ij})^-_{1-\Delta} < (a_{rs})^-_k \leq (a_{rs})^-_{k_2} < (c_{rs})^-_{k_2} \leq (c_{rs})^-_{1-\Delta} < (a_{ij})^-_k \leq (a_{ij})^-_{k_1} < (c_{ij})^-_{k_1} \leq (c_{ij})^-_{1-\Delta}$, and thus that $(c_{ij})^-_{1-\Delta} < (c_{ij})^-_{1-\Delta}$ would hold. Hence, any FI-model of Θ corresponds to a P–model satisfying $(a_{ij})^-_1 < (c_{ij})^-_1$ or $(a_{rs})^-_1 < (c_{rs})^-_1$. If both $(a_{ij})^-_1 < (c_{ij})^-_1$ and $(a_{rs})^-_1 < (c_{rs})^-_1$ would be satisfied in an FI-model of Θ, we can arbitrarily choose to make either l_{ij} or l_{rs} true without making any of the clauses in $\mathcal{D}$ false. Therefore, we have established that whenever Θ is FI-satisfiable, $\mathcal{D}$ must be satisfiable.

To complete the proof, we also show the converse, i.e., whenever $\mathcal{D}$ is satisfiable, there exists an FI–model of Θ, or equivalently, a P–model of the linear constraints corresponding to Θ. If the literal l_{ij} is interpreted as true, we choose the disjunct $(a_{ij})^-_{1-\Delta} < (b_{ij})^-_{1-k}$ in (73), while if l_{ij} is interpreted as false, we choose the disjunct $(a_{ij})^-_1 < (b_{ij})^-_{1-k+\Delta}$. Thus we obtain a set Ψ of linear constraints without disjunctions whose P–satisfiability implies the FI–satisfiability of Θ. It holds that Ψ is P–satisfiable iff the graph representation of Ψ does not contain any cycles involving at least one edge labeled with $<$.

We begin by considering the edges corresponding to linear constraints of the form (73), (74) and (75)–(77), as depicted in Figure 9. Note that in the construction of Ψ, as mentioned above, we chose one specific disjunct in (75). Since at least one of the literals l_{i1}, l_{i2}, l_{i3} is interpreted as true, for at least one j in $\{1, 2, 3\}$, we chose the disjunct $(a_{ij})^-_{1-\Delta} < (b_{ij})^-_{1-k}$, resulting in $(a_{ij})^-_{1-\Delta} < (c_{ij})^-_{1-\Delta}$ instead of $(a_{ij})^-_1 < (c_{ij})^-_1$. For a cycle, however, we would need $(a_{i1})^-_1 < (c_{i1})^-_1$, $(a_{i2})^-_1 < (c_{i2})^-_1$ and $(a_{i3})^-_1 < (c_{i3})^-_1$. From this we conclude that the constraints of the form (73), (74) and (75)–(77) alone do not lead to cycles in the graph representation of Ψ.

Any cycle would therefore have to include at least one edge corresponding to a linear constraint of the form (84)–(89). Such a cycle can only occur if for some
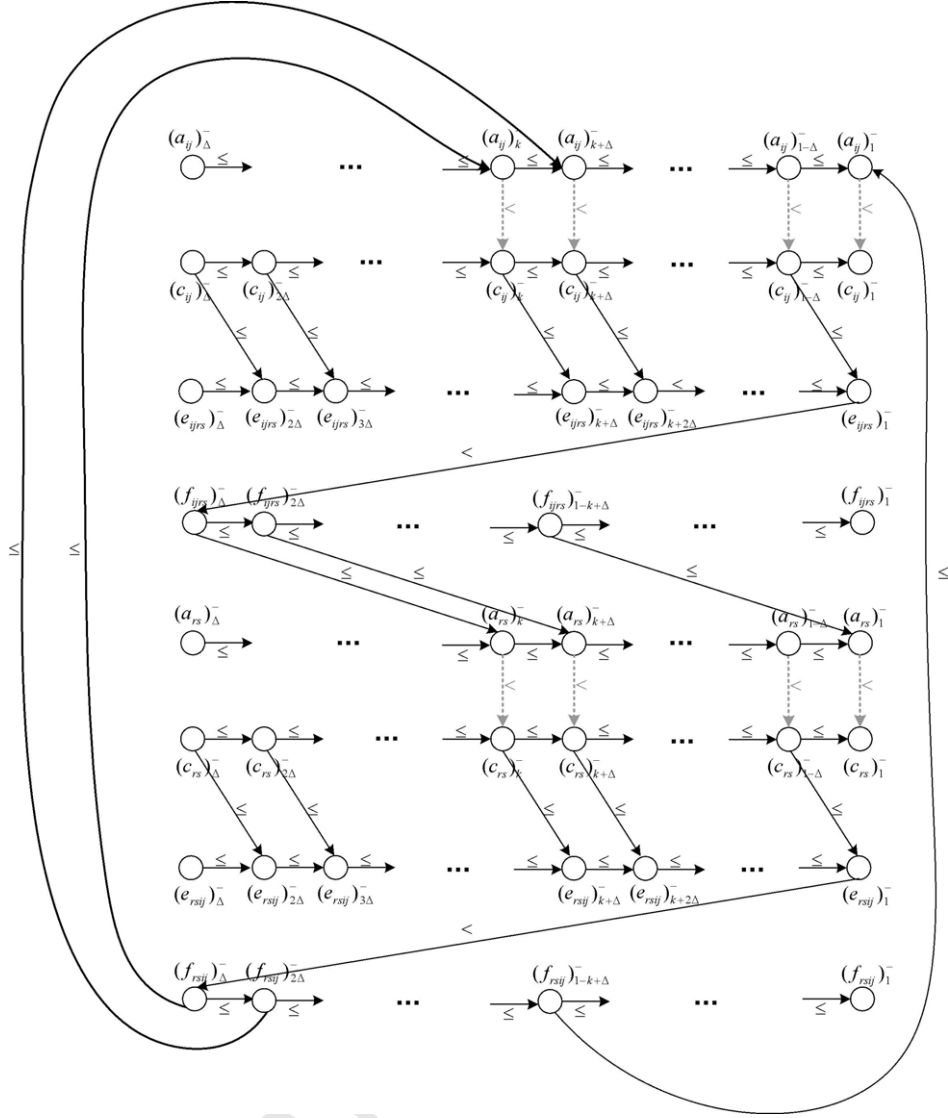
30

Fig. 10. Linear constraints (84)–(89).

i, j, r, s in $\{1, 2, \ldots, n\}$, we have that $(c_{ij})^-_{1-\Delta} < (a_{rs})^-_k$, $(a_{rs})^-_{1-\Delta} < (c_{rs})^-_{1-\Delta}$, $(c_{rs})^-_{1-\Delta} < (a_{ij})^-_k$ and $(a_{ij})^-_{1-\Delta} < (c_{ij})^-_{1-\Delta}$. By construction, $(c_{ij})^-_{1-\Delta} < (a_{rs})^-_k$ and $(c_{rs})^-_{1-\Delta} < (a_{ij})^-_k$ are only implied by Ψ iff $l_{ij} \equiv \neg l_{rs}$. However, if this is the case, either l_{ij} or l_{rs} is false, and $(a_{rs})^-_{1-\Delta} < (c_{rs})^-_{1-\Delta}$ and $(a_{ij})^-_{1-\Delta} < (c_{ij})^-_{1-\Delta}$ cannot both be contained in Ψ. Hence Ψ cannot contain any cycle, which completes the proof. $\qquad\square$

Proposition 4 shows that, when no restrictions on the variables are imposed, $\mathcal{F}^t_X$ cannot be extended with atomic FI–formulas without losing tractability. From the next proposition, it follows that this also holds for disjunctive FI-formulas.
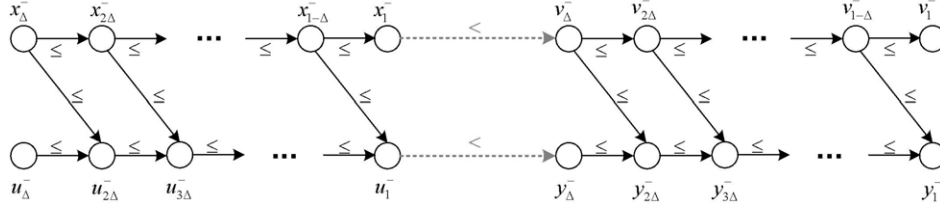
31

Fig. 11. Linear constraints (93)–(95).

Proposition 5. *Let r_d and s_d be $bb_d^\ll$, $ee_d^\ll$, $be_d^\ll$ or $eb_d^\ll$ ($d \in \mathbb{R}$). FISAT($\mathcal{A}$) is NP-complete if $\mathcal{A}$ contains any of the following sets of FI-formulas:*

$$\mathcal{F}_X^t \cup \bigcup_{(x,y,u,v) \in X^4} \{r_{d_1}(x,y) \geq l_1 \vee s_{d_2}(u,v) \geq l_2\} \tag{90}$$

$$\mathcal{F}_X^t \cup \bigcup_{(x,y,u,v) \in X^4} \{r_{d_1}(x,y) \geq l_1 \vee s_{d_2}(u,v) \leq k_2\} \tag{91}$$

$$\mathcal{F}_X^t \cup \bigcup_{(x,y,u,v) \in X^4} \{r_{d_1}(x,y) \leq k_1 \vee s_{d_2}(u,v) \leq k_2\} \tag{92}$$

for any $d_1, d_2 \in \mathbb{R}$, $l_1, l_2 \in M_0$ and $k_1, k_2 \in M_1$.

Proof. As an example, we show (90) for $r_{d_1} = s_{d_2} = bb_0^\ll$. First note that if $\cup_{(x,y) \in X^2} \{r_{d_1}(x,y) \geq l_1\} \not\subseteq \mathcal{F}_X^t$ or $\cup_{(u,v) \in X^2} \{s_{d_2}(u,v) \geq l_2\} \not\subseteq \mathcal{F}_X^t$, (90) follows straightforwardly from Proposition 4. Therefore, we only need to consider the case where $l_1 = l_2 = 1$. We will establish that FISAT($\mathcal{F}_X^t \cup \bigcup_{(x,y) \in X^2} \{bb^\ll(x,y) \geq 1 - \Delta\}$), which is NP-complete by Proposition 4, can be polynomially reduced to FISAT($\mathcal{F}_X^t \cup \bigcup_{(x,y,u,v) \in X^4} \{bb^\ll(x,y) \geq 1 \vee bb^\ll(u,v) \geq 1\}$).

Let Θ_1 be a set of FI–formulas from $\mathcal{F}_X^t \cup \bigcup_{(x,y) \in X^2} \{bb^\ll(x,y) \geq 1 - \Delta\}$. We construct a set Θ_2 of FI-formulas from FISAT($\mathcal{F}_X^t \cup \bigcup_{(x,y,u,v) \in X^4} \{bb^\ll(x,y) \geq 1 \vee bb^\ll(u,v) \geq 1\}$) by replacing every FI-formula in Θ_1 of the form $bb^\ll(x,y) \geq 1 - \Delta$ by the following FI-formulas

$$bb^\ll(x,v) \geq 1 \vee bb^\ll(u,y) \geq 1$$
$$bb^\ll(u,x) \leq \Delta$$
$$bb^\ll(y,v) \leq \Delta$$

giving rise to the following linear constraints:

$$x_1^- < v_\Delta^- \vee u_1^- < y_\Delta^- \tag{93}$$
$$\{x_\Delta^- \leq u_{2\Delta}^-, x_{2\Delta}^- \leq u_{3\Delta}^-, \ldots, x_{1-\Delta}^- \leq u_1^-\} \tag{94}$$
$$\{v_\Delta^- \leq y_{2\Delta}^-, v_{2\Delta}^- \leq y_{3\Delta}^-, \ldots, v_{1-\Delta}^- \leq y_1^-\} \tag{95}$$

These linear constraints are depicted in Figure 11. On the other hand, the corresponding FI–formula $bb^\ll(x,y) \geq 1 - \Delta$ from Θ_1 gives rise to

$$x_{1-\Delta}^- < y_\Delta^- \vee x_1^- < y_{2\Delta}^-$$

32

Let Ψ_1 and Ψ_2 be the sets of linear constraints corresponding to Θ_1 and Θ_2 respectively. By Proposition 3, it suffices to show that Ψ_1 is P–satisfiable iff Ψ_2 is P–satisfiable. Clearly, if $\mathcal{I}$ is a P–model of Ψ_2, $\mathcal{I}$ is also a P–model of Ψ_1. Conversely, we show that if $\mathcal{I}$ is a P–model of Ψ_1, there exists a P–model $\mathcal{I}'$ of Ψ_2. For all variables a occurring in Ψ_1, we define $\mathcal{I}'(a) = \mathcal{I}(a)$. Moreover, for additional variables occurring in (93)–(95), $\mathcal{I}'$ is defined as follows. For each k in $\{2\Delta, 3\Delta, \ldots, 1\}$, we define

$$\mathcal{I}'(u_k^-) = \mathcal{I}(x_{k-\Delta}^-)$$

while for each k in $\{\Delta, \ldots, 1 - 2\Delta, 1 - \Delta\}$, we define

$$\mathcal{I}'(v_k^-) = \mathcal{I}(y_{k+\Delta}^-)$$

Finally, we define

$$\mathcal{I}'(u_\Delta^-) = \mathcal{I}'(u_{2\Delta}^-)$$
$$\mathcal{I}'(v_1^-) = \mathcal{I}'(v_{1-\Delta}^-)$$

Note that $\mathcal{I}'(x_{1-\Delta}^-) < \mathcal{I}'(y_\Delta^-) \vee \mathcal{I}'(x_1^-) < \mathcal{I}'(y_{2\Delta}^-)$ implies that $\mathcal{I}'$ satisfies (93), as $\mathcal{I}'(x_{1-\Delta}^-) = \mathcal{I}'(u_1^-)$ and $\mathcal{I}'(v_\Delta^-) = \mathcal{I}'(y_{2\Delta}^-)$. Clearly, $\mathcal{I}'$ also satisfies (94) and (95), hence $\mathcal{I}'$ is a P–model of Ψ_2. $\qquad\square$

To find tractable sets of FI–formulas that are larger than $\mathcal{F}_X^t$, we can impose restrictions on the variables in the FI–formulas. For example, it can be shown that $bb_d^{\lll}(x, x) = ee_d^{\lll}(x, x) = eb_d^{\lll}(x, x) = 0$ for any $d \geq 0$ [41]. Hence, for example, $bb_d^{\lll}(x, x) \leq k$ is satisfied by any FI–interpretation for every $k \in M_1$, while no FI–interpretation can satisfy $bb_d^{\lll}(x, x) \geq l$ for $l \in M_0$. Therefore, if ϕ is an FI–formula from $\mathcal{F}_X^t$, FISAT($\mathcal{F}_X^t \cup \{\phi \vee bb_d^{\lll}(x, x) \leq k_1 \vee ee_d^{\lll}(x, x) \leq k_2 \vee bb_d^{\lll}(z, z) \geq l_1\}$) is still tractable. In the same way, if $k_1 \leq k_2$, a formula like $bb_d^{\lll}(x, y) \geq k_1 \vee bb_d^{\lll}(x, y) \leq k_2$ will be satisfied by any FI-interpretation.

These extensions of $\mathcal{F}_X^t$ are of limited practical value because of their rather trivial character. More useful tractable extensions can be derived by considering disjunctive FI–formulas that give rise to disjunctive linear constraints which are Horn. For example, let ϕ be an FI–formula from $\mathcal{F}_X^t$ and let the corresponding set of linear constraints be given by $\{\rho_1, \rho_2, \ldots, \rho_s\}$. An FI–formula like $\phi \vee bb_d^{\lll}(x, y) \geq \Delta \vee bb_{-d}^{\lll}(y, x) \geq \Delta$ gives rise to the set of linear constraints $\{\alpha_1, \alpha_2, \ldots, \alpha_s\}$, where

$$\begin{aligned}
\alpha_i &= \rho_i \vee x_\Delta^- < y_\Delta^- - d \vee x_{2\Delta}^- < y_{2\Delta}^- - d \vee \cdots \vee x_1^- < y_1^- - d \\
&\quad \vee y_\Delta^- - d < x_\Delta^- \vee y_{2\Delta}^- - d < x_{2\Delta}^- \vee \cdots \vee y_1^- - d < x_1^- \\
&= \rho_i \vee x_\Delta^- \neq y_\Delta^- - d \vee x_{2\Delta}^- \neq y_{2\Delta}^- - d \vee \cdots \vee x_1^- \neq y_1^- - d
\end{aligned}$$

In other words, each α_i is a Horn linear constraint ($i \in \{1, 2, \ldots, s\}$), hence FISAT($\mathcal{F}_X^t \cup \{\phi \vee bb_d^{\lll}(x, y) \geq \Delta \vee bb_d^{\lll}(y, x) \geq \Delta\}$) is tractable.

33

More generally, let the set $\mathcal{G}_X$ of FI–formulas be defined as follows:

$$\mathcal{G}_X = \bigcup_{(x,y)\in X^2} \bigcup_{d\in\mathbb{R}} \{bb_d^{\lll}(x,y) \geq \Delta \vee bb_{-d}^{\lll}(y,x) \geq \Delta,$$

$$ee_d^{\lll}(x,y) \geq \Delta \vee ee_{-d}^{\lll}(y,x) \geq \Delta,$$

$$be_d^{\lll}(x,y) \geq 1 \vee be_{-d}^{\lll}(y,x) \geq 1\}$$

Furthermore, let $\mathcal{H}_X$ be recursively defined as follows

(1) If $\phi \in \mathcal{F}_X^t$, then $\phi \in \mathcal{H}_X$
(2) If $\phi_1 \in \mathcal{H}_X$ and $\phi_2 \in \mathcal{G}_X$, then $(\phi_1 \vee \phi_2) \in \mathcal{H}_X$
(3) $\mathcal{H}_X$ contains no other elements

As any FI–formula in $\mathcal{H}_X$ corresponds to a Horn linear constraint, or a set of Horn linear constraints, we have that FISAT($\mathcal{H}_X$) is tractable.

When $\Delta = 1$ (i.e., $M = \{0, 1\}$), we know by Proposition 1 that a set of FI–formulas is FI–satisfiable iff there exists an interpretation that assigns a crisp interval to every variable. The set of FI–formulas $\mathcal{H}_X$ is then exactly equal to the set of all Horn linear constraints involving the endpoints of these crisp intervals. Hence, for $\Delta = 1$, our (tractable) fuzzy temporal reasoning framework degenerates to reasoning about (Horn) linear constraints. By decreasing the value of Δ to $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$, an increasingly higher expressiveness is achieved.

7 Entailment

Let Θ be a set of FI–formulas over X, and γ an FI–formula over X. We say that Θ entails γ, written $\Theta \models \gamma$, iff every FI–model of Θ is also an FI–model of $\{\gamma\}$. The notion of entailment is important for applications, because it allows to draw conclusions that are not explicitly contained in an initial set of assertions. Obviously, $\Theta \models \gamma$ if Θ and the negation of γ can never be satisfied at the same time. For example, $\Theta \models bb_d^{\lll}(x,y) \leq k$ iff $\Theta \cup \{bb_d^{\lll}(x,y) > k\}$ is not FI–satisfiable. Unfortunately, our procedure for checking FI–satifiability cannot be applied for strict inequalities like $bb_d^{\lll}(x,y) > k$, as Proposition 1 does not hold in this case. However, for every FI_M–interpretation $\mathcal{I}$, we have that $bb_d^{\lll}(x^{\mathcal{I}}, y^{\mathcal{I}}) > k$ iff $bb_d^{\lll}(x^{\mathcal{I}}, y^{\mathcal{I}}) \geq k + \Delta$. Inspired by this observation, we say that Θ weakly entails γ (w.r.t. M), written $\Theta \models_M \gamma$ iff every FI_M–model of Θ is also an FI_M–model of $\{\gamma\}$. Checking weak entailment can straightforwardly be reduced to checking FI–satisfiability.

Proposition 6. *Let Θ be a set of FI–formulas and let $r(x,y)$ be one of $bb_d^{\lll}(x,y)$, $ee_d^{\lll}(x,y)$, $be_d^{\lll}(x,y)$ and $eb_d^{\lll}(x,y)$ ($d \in \mathbb{R}$, $(x,y) \in X^2$). For k in M_1 and l in M_0 it holds that*

34

(1) $\Theta \models_M r(x,y) \geq l$ *iff* $\Theta \cup \{r(x,y) \leq l - \Delta\}$ *is not FI–satisfiable.*
(2) $\Theta \models_M r(x,y) \leq k$ *iff* $\Theta \cup \{r(x,y) \geq k + \Delta\}$ *is not FI–satisfiable.*

Proof. The proof follows trivially from the fact that for any FI_M–interpretation $\mathcal{I}$, $r(x^{\mathcal{I}}, y^{\mathcal{I}}) < l$ implies $r(x^{\mathcal{I}}, y^{\mathcal{I}}) \leq l - \Delta$ and $r(x^{\mathcal{I}}, y^{\mathcal{I}}) > k$ implies $r(x^{\mathcal{I}}, y^{\mathcal{I}}) \geq k + \Delta$. $\qquad\square$

As the name already suggests, weak entailment is a weaker notion than entailment, i.e., $(\Theta \models \gamma) \Rightarrow (\Theta \models_M \gamma)$. Nonetheless, weak entailment can still be used in applications to derive sound conclusions, by virtue of the following proposition.

Proposition 7. *Let Θ be a set of FI–formulas and let $r(x,y)$ be one of $bb_d^{\ll}(x,y)$, $ee_d^{\ll}(x,y)$, $be_d^{\ll}(x,y)$ and $eb_d^{\ll}(x,y)$ $(d \in \mathbb{R}$, $(x,y) \in X^2)$. For k in $M_1 \setminus \{1 - \Delta\}$ and l in $M_0 \setminus \{\Delta\}$ it holds that*

(1) If $\Theta \models_M r(x,y) \geq l$ then $\Theta \models r(x,y) \geq l - \Delta$
(2) If $\Theta \models_M r(x,y) \leq k$ then $\Theta \models r(x,y) \leq k + \Delta$

Proof. If $\Theta \models_M r(x,y) \geq l$, then by Proposition 6, $\Theta \cup \{r(x,y) \leq l - \Delta\}$ is not FI–satisfiable. Hence in every FI–interpretation of Θ, it holds that $r(x,y) > l - \Delta$, and in particular, $r(x,y) \geq l - \Delta$. The second implication is shown in the same way. $\qquad\square$

In the remainder of this section, we will investigate when weak entailment coincides with entailment, i.e., in which situations Proposition 6 also holds for (regular) entailment. Clearly $\Theta \cup \{\phi_1 \vee \phi_2 \vee \cdots \vee \phi_n\} \models \gamma$ iff $\Theta \cup \{\phi_1\} \models \gamma$ and $\Theta \cup \{\phi_2\} \models \gamma$ and ... and $\Theta \cup \{\phi_n\} \models \gamma$. Therefore, we can restrict ourselves to the case where Θ only contains atomic FI–formulas.

As we discussed in Section 5, for each set of FI–formulas Θ, we can find a set of linear constraints Ψ which is P–satisfiable iff Θ is FI–satisfiable. If Ψ does not contain any disjunctive linear constraints, we can represent Ψ as a graph G whose nodes correspond to variables like x_l^- or x_l^+ $(l \in M_0)$. If Ψ contains a linear constraint $x + d \leq y$, we add an edge from the node corresponding with x to the node corresponding with y which is labeled with $(\leq, d)$. Similarly, if Ψ contains a linear constraint $x + d < y$, we add an edge labeled with $(<, d)$. The sum of two labels $(\leq, d_1)$ and $(\leq, d_2)$ is defined as $(\leq, d_1 + d_2)$, while the sum of $(<, d_1)$ and $(\leq, d_2)$, $(\leq, d_1)$ and $(<, d_2)$, or $(<, d_1)$ and $(<, d_2)$, is defined as $(<, d_1 + d_2)$. A cycle for which the edge labels sum up to $(\leq, d)$, with $d > 0$, or to $(<, d')$, with $d' \geq 0$, is called a forbidden cycle. It holds that Ψ is P–satisfiable iff there are no forbidden cycles in G [25]. If Ψ does contain disjunctive linear constraints, every choice of the disjuncts leads to a different graph representation, and Ψ is P–satisfiable as soon as one of these graphs is free of forbidden cycles.

35

In the following, nodes corresponding to variables like x_l^- will be called beginning nodes, while nodes corresponding to variables like x_l^+ will be called ending nodes. Furthermore, we will sometimes assume that $\Delta = \frac{1}{2p}$ for some $p \in \mathbb{N} \setminus \{0\}$. Nodes like x_l^- or x_l^+ will then be called white nodes if $l \in \{2\Delta, 4\Delta, \ldots, 1\}$ and black nodes otherwise. Finally, for $l \in M \setminus \{0, \Delta\}$ and $k \in M_1$, $x_{l-\Delta}^-$ (resp. $x_{k+\Delta}^+$) will be called the left neighbour of x_l^- (resp. x_k^+), while $x_{k+\Delta}^-$ (resp. $x_{l-\Delta}^+$) will be called the right neighbour of x_k^- (resp. x_l^+).

Graphs representing linear constraints derived from a set of FI–formulas exhibit some interesting properties. In particular, the following two lemmas will be useful in reducing entailment checking to FI–satisfiability checking, or, equivalently, weak entailment checking.

Lemma 3. *Let $\Delta = \frac{1}{2p}$ for some $p \in \mathbb{N} \setminus \{0\}$, and let Θ be a (finite) set of FI–formulas. Let Ψ be the corresponding set of linear constraints and let G be the graph representation corresponding to a particular choice of disjuncts for the disjunctive constraints in Ψ. Furthermore, assume that there is a path in G from v to u in which each edge either corresponds to a linear constraint of the form (46)–(48), or is the result of an FI–formula in Θ of the form $bb_d^{\ll}(x,y) \leq k$, $ee_d^{\ll}(x,y) \leq k$, $be_d^{\ll}(x,y) \leq k$, or $eb_d^{\ll}(x,y) \geq l$ for some k in $\{0, 2\Delta, 4\Delta, \ldots, 1 - 2\Delta\}$ and l in $\{2\Delta, 4\Delta, \ldots, 1\}$). Assume, moreover, that:*

(1) v is a black beginning node and u is a white beginning node, or
(2) v is a white ending node and u is a black ending node, or
(3) v is a white ending node and u is a white beginning node, or
(4) v is a black beginning node and u is a black ending node.

It holds that there is a path in G from v to the left neighbour of u, as well as a path from the right neighbour of v to u. Moreover, for both paths, the edge labels sum up to the same value as for the original path.

Proof. The proof is presented in Appendix A.3. □

We define the right(resp. left) neighbour of an edge from v to u as the edge from the right (resp. left) neighbour of v to the right (resp. left) neighbour of u.

Lemma 4. *Let $\Delta = \frac{1}{2p}$ for some p in $\mathbb{N} \setminus \{0\}$, and let Θ and Ψ be defined as before. Moreover, assume that all upper and lower bounds in Θ are taken from $\{0, 2\Delta, 4\Delta, \ldots, 1\}$. Let $\mathcal{I}$ be a P–model of Ψ, and let G_1 be the corresponding graph representation of Ψ without forbidden cycles. Let the graph G_2 be constructed from G_1 by replacing*

(1) edges resulting from an FI–formula of the form $bb_d^{\ll}(x,y) \geq l$ by their right neighbour if they start from a black beginning node;
(2) edges resulting from an FI–formula of the form $ee_d^{\ll}(x,y) \geq l$ by their right neighbour if they start from a white ending node;

(3) edges resulting from an FI–formula of the form $be_d^{\lll}(x,y) \geq l$ or $eb_d^{\lll}(x,y) \leq k$ by their right neighbour if they start from a black beginning node.

It holds that G_2 does not contain any forbidden cycles.

Proof. The proof is presented in Appendix A.4 □

Using Lemma 3 and Lemma 4, we can show the following lemma about FI–satisfiability when all upper and lower bounds are of the form $2i\Delta$.

Lemma 5. *Let $\Delta = \frac{1}{2p}$ for some p in $\mathbb{N} \setminus \{0\}$, let Θ be a set of atomic FI–formulas in which all upper and lower bounds are taken from $\{0, 2\Delta, 4\Delta, \ldots, 1\}$. For l in $\{0, 2\Delta, 4\Delta, \ldots, 1 - 2\Delta\}$ and k in $\{2\Delta, 4\Delta, \ldots, 1\}$, it holds that:*

(1) $\Theta \cup \{bb_d^{\lll}(x,y) \geq l + \Delta\}$ is FI–satisfiable iff $\Theta \cup \{bb_d^{\lll}(x,y) \geq l + 2\Delta\}$ is FI–satisfiable;

(2) $\Theta \cup \{ee_d^{\lll}(x,y) \geq l + \Delta\}$ is FI–satisfiable iff $\Theta \cup \{ee_d^{\lll}(x,y) \geq l + 2\Delta\}$ is FI–satisfiable;

(3) $\Theta \cup \{be_d^{\lll}(x,y) \geq l + \Delta\}$ is FI–satisfiable iff $\Theta \cup \{be_d^{\lll}(x,y) \geq l + 2\Delta\}$ is FI–satisfiable;

(4) $\Theta \cup \{eb_d^{\lll}(x,y) \leq k - \Delta\}$ is FI–satisfiable iff $\Theta \cup \{eb_d^{\lll}(x,y) \leq k - 2\Delta\}$ is FI–satisfiable.

Proof. The proof is presented in Appendix A.5. □

Finally, we can show the following characterization of entailment in terms of FI–satisfiability for FI–formulas of the form $bb_d^{\lll}(x,y) \leq k$, $ee_d^{\lll}(x,y) \leq k$, $be_d^{\lll}(x,y) \leq k$, and $eb_d^{\lll}(x,y) \geq l$.

Proposition 8. *Let Θ be a set of atomic FI–formulas. It holds for k in M_1 and l in M_0 that*

(1) $\Theta \models bb_d^{\lll}(x,y) \leq k$ iff $\Theta \cup \{bb_d^{\lll}(x,y) \geq k + \Delta\}$ is not FI–satisfiable;

(2) $\Theta \models ee_d^{\lll}(x,y) \leq k$ iff $\Theta \cup \{ee_d^{\lll}(x,y) \geq k + \Delta\}$ is not FI–satisfiable;

(3) $\Theta \models be_d^{\lll}(x,y) \leq k$ iff $\Theta \cup \{be_d^{\lll}(x,y) \geq k + \Delta\}$ is not FI–satisfiable;

(4) $\Theta \models eb_d^{\lll}(x,y) \geq l$ iff $\Theta \cup \{eb_d^{\lll}(x,y) \leq l - \Delta\}$ is not FI–satisfiable.

Proof. As an example, we show that $\Theta \models bb_d^{\lll}(x,y) \leq k$ iff $\Theta \cup \{bb_d^{\lll}(x,y) \geq k + \Delta\}$ is not FI–satisfiable. Clearly, if $\Theta \cup \{bb_d^{\lll}(x,y) \geq k + \Delta\}$ is FI–satisfiable, then $\Theta \not\models bb_d^{\lll}(x,y) \leq k$. Therefore, we only need to show that if there is an FI–model of Θ which does not satisfy $bb_d^{\lll}(x,y) \leq k$, it holds that $\Theta \cup \{bb_d^{\lll}(x,y) \geq k + \Delta\}$ is FI–satisfiable.

Let $\mathcal{I}$ be an FI–model of Θ, and assume that $bb_d^{\lll}(x^{\mathcal{I}}, y^{\mathcal{I}}) > k$. There exists an n in $\mathbb{N}$ such that $bb_d^{\lll}(x^{\mathcal{I}}, y^{\mathcal{I}}) \geq k + \frac{\Delta}{2^n}$. Obviously, we have that $\Theta \cup \{bb_d^{\lll}(x,y) \geq k + \frac{\Delta}{2^n}\}$ is FI–satisfiable. By letting $\frac{\Delta}{2^n}$ play the role of Δ, we obtain using

Lemma 5 that $\Theta \cup \{bb_d^{\lll}(x,y) \geq k + \frac{\Delta}{2^{n-1}}\}$ is FI–satisfiable. Again applying Lemma 5 reveals that also $\Theta \cup \{bb_d^{\lll}(x,y) \geq k + \frac{\Delta}{2^{n-2}}\}$ is FI–satisfiable. By repeating this argument n times, we find that $\Theta \cup \{bb_d^{\lll}(x,y) \geq k + \Delta\}$ is FI–satisfiable. $\qquad\square$

To find a characterization of entailment for FI–formulas of the form $bb_d^{\lll}(x,y) \geq l$, $ee_d^{\lll}(x,y) \geq l$, $be_d^{\lll}(x,y) \geq l$, and $eb_d^{\lll}(x,y) \leq k$, we restrict ourselves to the case where Θ only contains FI–formulas from $\mathcal{F}_X^t$.

Lemma 6. *Let $\Delta = \frac{1}{2p}$ for some $p \in \mathbb{N} \setminus \{0\}$, let Θ be a set of FI–formulas from $\mathcal{F}_X^t$ in which all upper and lower bounds are taken from $\{0, 2\Delta, 4\Delta, \ldots, 1\}$. For l in $\{0, 2\Delta, 4\Delta, \ldots, 1 - 2\Delta\}$ and k in $\{2\Delta, 4\Delta, \ldots, 1\}$, it holds that:*

(1) $\Theta \cup \{be_d^{\lll}(x,y) \leq k - \Delta\}$ *is FI–satisfiable iff* $\Theta \cup \{be_d^{\lll}(x,y) \leq k - 2\Delta\}$ *is FI–satisfiable;*

(2) $\Theta \cup \{eb_d^{\lll}(x,y) \geq l + \Delta\}$ *is FI–satisfiable iff* $\Theta \cup \{eb_d^{\lll}(x,y) \geq l + 2\Delta\}$ *is FI–satisfiable.*

Moreover, for k in $\{4\Delta, 6\Delta, \ldots, 1\}$, it holds that

(1) $\Theta \cup \{bb_d^{\lll}(x,y) \leq k - \Delta\}$ *is FI–satisfiable iff* $\Theta \cup \{bb_d^{\lll}(x,y) \leq k - 2\Delta\}$ *is FI–satisfiable;*

(2) $\Theta \cup \{ee_d^{\lll}(x,y) \leq k - \Delta\}$ *is FI–satisfiable iff* $\Theta \cup \{ee_d^{\lll}(x,y) \leq k - 2\Delta\}$ *is FI–satisfiable.*

Proof. The proof is presented in Appendix A.6. $\qquad\square$

Note that the FI–satisfiability of $\Theta \cup \{bb_d^{\lll}(x,y) \leq \Delta\}$ does not necessarily imply that $\Theta \cup \{bb_d^{\lll}(x,y) \leq 0\}$ is FI–satisfiable. For example, for $d' + d > 0$, it holds that $\{bb_{d'}^{\lll}(y,x) \leq 0, bb_d^{\lll}(x,y) \leq \Delta\}$ is FI–satisfiable, while $\{bb_{d'}^{\lll}(y,x) \leq 0, bb_d^{\lll}(x,y) \leq 0\}$ is not. Similarly, we have that $\{ee_{d'}^{\lll}(y,x) \leq 0, ee_d^{\lll}(x,y) \leq \Delta\}$ is FI–satisfiable and $\{ee_{d'}^{\lll}(y,x) \leq 0, e_d^{\lll}(x,y) \leq 0\}$ is not.

Proposition 9. *Let Θ be a set of atomic FI–formulas from $\mathcal{F}_X^t$. For k in M_1 and l in M_0, it holds that*

(1) $\Theta \models be_d^{\lll}(x,y) \geq l$ *iff* $\Theta \cup \{be_d^{\lll}(x,y) \leq l - \Delta\}$ *is not FI–satisfiable;*

(2) $\Theta \models eb_d^{\lll}(x,y) \leq k$ *iff* $\Theta \cup \{eb_d^{\lll}(x,y) \geq k + \Delta\}$ *is not FI–satisfiable.*

Moreover, for l in $M_0 \setminus \{\Delta\}$, it holds that

(1) $\Theta \models bb_d^{\lll}(x,y) \geq l$ *iff* $\Theta \cup \{bb_d^{\lll}(x,y) \leq l - \Delta\}$ *is not FI–satisfiable;*

(2) $\Theta \models ee_d^{\lll}(x,y) \geq l$ *iff* $\Theta \cup \{ee_d^{\lll}(x,y) \leq l - \Delta\}$ *is not FI–satisfiable.*

Proof. The proof is entirely analogous to the proof of Proposition 8, using Lemma 6 instead of Lemma 5. $\qquad\square$

38

Proposition 9 does not hold in general when Θ contains atomic FI–formulas from $\mathcal{F}_X \setminus \mathcal{F}_X^t$. As a counterexample, let $\Delta = 0.25$ and $\Theta = \{bb^{\ll}(a,e) \geq 0.75, bb^{\ll}(d,g) \geq 0.75, bb^{\ll}(e,f) \geq 1, bb^{\ll}(b,c) \geq 1, bb^{\ll}(b,a) \leq 0.5, bb^{\ll}(d,c) \leq 0.5, bb^{\ll}(g,f) \leq 0.5, bb^{\ll}(d,e) \leq 0.75\}$. It holds that $\Theta \cup \{bb^{\ll}(a,g) \leq 0.375\}$ is FI–satisfiable, implying that $\Theta \not\models bb^{\ll}(a,g) \geq 0.5$, whereas $\Theta \cup \{bb^{\ll}(a,g) \leq 0.25\}$ is not FI–satisfiable.

Note that Proposition 9 provides no characterization of entailment for the case where $bb_d^{\ll}(x,y) \geq \Delta$ or $ee_d^{\ll}(x,y) \geq \Delta$. However, to check entailment for $bb_d^{\ll}(x,y) \geq \Delta$ or $ee_d^{\ll}(x,y) \geq \Delta$, we can always redefine the set M as $\{0, \frac{\Delta}{2}, \Delta, \ldots, 1 - \frac{\Delta}{2}, 1\}$, i.e., we let $\frac{\Delta}{2}$ play the role of Δ. Also note that from Proposition 8 and 9, it follows that the tractability of $\mathcal{F}_X^t$ w.r.t. FI–satisfiability carries over to entailment checking. Indeed, if Θ only contains FI–formulas from $\mathcal{F}_X^t$, $\Theta \models \gamma$ can be checked by checking the FI–satisfiability of a set of FI–formulas which contains at most one FI–formula which is not in $\mathcal{F}_X^t$. Although this one FI–formula may correspond to a disjunctive linear constraint, the number of disjuncts is bounded by $|M|$. Therefore, FI–satisfiability can be checked in polynomial time, using $O(|M|)$ P–satisfiability checks of sets of linear constraints without disjuncts.

In addition to entailment checking, it may also be of interest to know what the strongest upper bound or lower bound is for the value of $bb_d^{\ll}(x,y)$, $ee_d^{\ll}(x,y)$, $be_d^{\ll}(x,y)$ or $eb_d^{\ll}(x,y)$, given that a set of FI–formulas Θ is satisfied. As a corollary of Proposition 8, we find that the strongest upper bound of $bb_d^{\ll}(x,y)$, $ee_d^{\ll}(x,y)$ and $be_d^{\ll}(x,y)$, as well as the strongest lower bound of $eb_d^{\ll}(x,y)$, is always a value from M:

Corollary 1. *Let Θ be a set of atomic FI–formulas. It holds that ($d \in \mathbb{R}$, $(x,y) \in X^2$)*

$$\inf\{k | k \in [0,1] \wedge \Theta \models bb_d^{\ll}(x,y) \leq k\} = \min\{k | k \in M \wedge \Theta \models bb_d^{\ll}(x,y) \leq k\}$$
$$\inf\{k | k \in [0,1] \wedge \Theta \models ee_d^{\ll}(x,y) \leq k\} = \min\{k | k \in M \wedge \Theta \models ee_d^{\ll}(x,y) \leq k\}$$
$$\inf\{k | k \in [0,1] \wedge \Theta \models be_d^{\ll}(x,y) \leq k\} = \min\{k | k \in M \wedge \Theta \models be_d^{\ll}(x,y) \leq k\}$$
$$\sup\{k | k \in [0,1] \wedge \Theta \models eb_d^{\ll}(x,y) \geq k\} = \max\{k | k \in M \wedge \Theta \models eb_d^{\ll}(x,y) \geq k\}$$

In the same way, as a corollary of Proposition 9, we can establish the strongest lower bound of $bb_d^{\ll}(x,y)$, $ee_d^{\ll}(x,y)$ and $be_d^{\ll}(x,y)$, as well as the strongest upper bound of $eb_d^{\ll}(x,y)$, given that a set of atomic FI–formulas from $\mathcal{F}_X^t$ is satisfied.

Corollary 2. *Let Θ be a set of atomic FI–formulas from $\mathcal{F}_X^t$. It holds that ($d \in \mathbb{R}$, $(x,y) \in X^2$)*

$$\sup\{k | k \in [0,1] \wedge \Theta \models be_d^{\ll}(x,y) \geq k\} = \max\{k | k \in M \wedge \Theta \models be_d^{\ll}(x,y) \geq k\}$$
$$\inf\{k | k \in [0,1] \wedge \Theta \models eb_d^{\ll}(x,y) \leq k\} = \min\{k | k \in M \wedge \Theta \models eb_d^{\ll}(x,y) \leq k\}$$

If $\Theta \models bb_d^{\ll}(x,y) \geq \Delta$ or $\Theta \cup \{bb_d^{\ll}(x,y) \leq 0\}$ is FI–satisfiable, resp. $\Theta \models$

39

$ee_d^{\lll}(x, y) \geq \Delta$ *or* $\Theta \cup \{ee_d^{\lll}(x, y) \leq 0\}$ *is FI–satisfiable, it holds that*

$$\sup\{k|k \in [0, 1] \wedge \Theta \models bb_d^{\lll}(x, y) \geq k\} = \max\{k|k \in M \wedge \Theta \models bb_d^{\lll}(x, y) \geq k\}$$
$$\sup\{k|k \in [0, 1] \wedge \Theta \models ee_d^{\lll}(x, y) \geq k\} = \max\{k|k \in M \wedge \Theta \models ee_d^{\lll}(x, y) \geq k\}$$

Finally, if $\Theta \models bb_d^{\lll}(x, y) \geq \Delta$ *while* $\Theta \cup \{bb_d^{\lll}(x, y) \leq 0\}$ *is not FI–satisfiable, resp.* $\Theta \models ee_d^{\lll}(x, y) \geq \Delta$ *while* $\Theta \cup \{ee_d^{\lll}(x, y) \leq 0\}$ *is not FI–satisfiable, it holds that in any FI–model* $\mathcal{I}$ *of* Θ

$$bb_d^{\lll}(x^{\mathcal{I}}, y^{\mathcal{I}}) > 0$$
$$ee_d^{\lll}(x^{\mathcal{I}}, y^{\mathcal{I}}) > 0$$

while for any $k > 0$*, there exists an FI–model* $\mathcal{I}$ *of* Θ *in which*

$$bb_d^{\lll}(x^{\mathcal{I}}, y^{\mathcal{I}}) < k$$
$$ee_d^{\lll}(x^{\mathcal{I}}, y^{\mathcal{I}}) < k$$

In other words, in this last case, the strongest lower bound implied by Θ *is a strict lower bound.*

As becomes clear from Corollary 1 and 2, finding the strongest upper and lower bounds on $bb_d^{\lll}(x, y)$, $ee_d^{\lll}(x, y)$, $be_d^{\lll}(x, y)$, or $eb_d^{\lll}(x, y)$ implied by Θ can be done by $O(log(|M|))$ FI–satisfiability checks, using binary search.

8 Concluding remarks

In this paper, we have shown how temporal reasoning about fuzzy time intervals can be reduced to reasoning about linear constraints. An important advantage of this approach is that we can draw upon well–established results for solving disjunctive temporal reasoning problems, as well as reuse existing, optimized tools for crisp temporal reasoning. The problem of satisfiability checking was shown to be NP–complete. Hence, introducing vagueness in temporal reasoning does not increase the computational complexity. Moreover, an important tractable subfragment $\mathcal{H}_X$ was identified in this paper, which for $\Delta = 1$ degenerates to the well–known framework of Horn linear constraints. In general, for $\Delta = 1$ our framework degenerates to reasoning about crisp intervals, i.e., if only 0 and 1 are used as upper and lower bounds, a set of FI–formulas can be satisfied by fuzzy time intervals iff it can be satisfied by crisp intervals. For $\Delta = 0.5$, the framework degenerates to reasoning about three valued intervals. Such intervals can be represented as a pair of crisp intervals $(\underline{a}, \overline{a})$, where $\overline{a}$ contains the dates which fully belong to the vague time period and $\underline{a}$ contains the dates which at least belong to the vague time period to some extent ($\overline{a} \subseteq \underline{a}$). This essentially corresponds to a temporal counterpart

of the Egg-Yolk calculus [11] for spatial reasoning about vague regions. Further decreasing the value of Δ leads to an increasingly higher expressiveness, requiring, however, an increasing amount of computation time.

In contrast to crisp temporal reasoning frameworks, entailment checking in our framework cannot straightforwardly be reduced to satisfiability checking. To cope with this, we have introduced the notion of weak entailment, which can be used to derive sound conclusions. Next, we have investigated in Proposition 8 and Proposition 9 how entailment relates to weak entailment. Finally, we have discussed how the strongest upper and lower bound on the possible values of a fuzzy temporal relation, applied to a particular pair of variables, can be obtained.

Our work is complementary to existing approaches for fuzzy temporal reasoning, which have focused on modelling possibilistic uncertainty and preferences (e.g., [4]). An interesting direction for future work might be to combine our framework with, for example, the IA^{fuz} framework from [4] to allow temporal reasoning with uncertain information about vague time periods, or with fuzzy temporal constraint networks to allow imprecise metric constraints like "A happened *about* three weeks before B". Further investigation is also needed to optimize the reasoning procedures. Properties about the specific structure of the linear constraints that arise from a set of FI–formulas may be very useful to prune the search space.

A Proofs

A.1 Proof of Lemma 1

First, we consider (12):

$$
\begin{aligned}
&bb_d^{\lll}(A, B) \geq l \\
&\Leftrightarrow \sup_{p \in \mathbb{R}} T_W(A(p), \inf_{q \in \mathbb{R}} I_W(B(q), L_d^{\lll}(p, q))) \geq l \\
&\Leftrightarrow (\forall \varepsilon > 0)(\sup_{p \in \mathbb{R}} T_W(A(p), \inf_{q \in \mathbb{R}} I_W(B(q), L_d^{\lll}(p, q))) > l - \varepsilon) \\
&\Leftrightarrow (\forall \varepsilon > 0)(\exists p \in \mathbb{R})(T_W(A(p), \inf_{q \in \mathbb{R}} I_W(B(q), L_d^{\lll}(p, q))) > l - \varepsilon)
\end{aligned}
$$

If $A(p) = 0$, it holds that $T(A(p), \inf_{q \in \mathbb{R}} I_T(B(q), L_d^{\lll}(p, q))) = 0$. Therefore, we can assume that $A(p) > 0$. Hence, there must exist a λ in $]0, 1]$ such that $\lambda = A(p)$ and thus $p \in A_\lambda$:

$$
\Leftrightarrow (\forall \varepsilon > 0)(\exists \lambda \in]0, 1])(\exists p \in A_\lambda)(T_W(\lambda, \inf_{q \in \mathbb{R}} I_W(B(q), L_d^{\lll}(p, q))) > l - \varepsilon)
$$

41

$$\Leftrightarrow (\forall \varepsilon > 0)(\exists \lambda \in]0,1])(\exists p \in A_\lambda)(\lambda + \inf_{q \in \mathbb{R}} I_W(B(q), L_d^{\ll}(p,q)) - 1 > l - \varepsilon)$$

$$\Leftrightarrow (\forall \varepsilon > 0)(\exists \lambda \in]0,1])(\exists p \in A_\lambda)(\inf_{q \in \mathbb{R}} I_W(B(q), L_d^{\ll}(p,q)) > l + 1 - \lambda - \varepsilon)$$

As any fuzzy time interval is upper semi–continuous, the mapping defined by $1 - B(q)$ for each q in $\mathbb{R}$ is lower semi–continous. Moreover, as $L_d^{\ll}(p,q)$ is lower semi–continuous, the mapping defined by $I_W(B(q), L_d^{\ll}(p,q))$ for each q in $\mathbb{R}$ is lower semi–continuous as well. Hence, the infimum $\inf_{q \in \mathbb{R}} I_W(B(q), L_d^{\ll}(p,q))$ is attained for some q in $\mathbb{R}$. We therefore find:

$$\Leftrightarrow (\forall \varepsilon > 0)(\exists \lambda \in]0,1])(\exists p \in A_\lambda)(\forall q \in \mathbb{R})$$
$$(I_W(B(q), L_d^{\ll}(p,q)) > l + 1 - \lambda - \varepsilon)$$

For $\lambda \leq l - \varepsilon$, $I_W(B(q), L_d^{\ll}(p,q)) > l + 1 - \lambda - \varepsilon$ can never be satisfied, hence:

$$\Leftrightarrow (\forall \varepsilon > 0)(\exists \lambda \in]l - \varepsilon, 1])(\exists p \in A_\lambda)(\forall q \in \mathbb{R})$$
$$(I_W(B(q), L_d^{\ll}(p,q)) > l + 1 - \lambda - \varepsilon)$$

If $L_d^{\ll}(p,q) = 1$, then $I_W(B(q), L_d^{\ll}(p,q)) = 1$, while $I_W(B(q), L_d^{\ll}(p,q)) = 1 - B(q)$ if $L_d^{\ll}(p,q) = 0$. We thereby obtain:

$$\Leftrightarrow (\forall \varepsilon > 0)(\exists \lambda \in]l - \varepsilon, 1])(\exists p \in A_\lambda)(\forall q \in \mathbb{R})$$
$$(L_d^{\ll}(p,q) = 1 \vee 1 - B(q) > l + 1 - \lambda - \varepsilon)$$
$$\Leftrightarrow (\forall \varepsilon > 0)(\exists \lambda \in]l - \varepsilon, 1])(\exists p \in A_\lambda)(\forall q \in \mathbb{R})$$
$$(L_d^{\ll}(p,q) = 1 \vee \neg(B(q) \geq \lambda + \varepsilon - l))$$

If $\varepsilon > l$, then $\neg(B(q) \geq \lambda + \varepsilon - l))$ can always be satisfied by choosing $\lambda = 1$. This yields:

$$\Leftrightarrow (\forall \varepsilon \in]0,l])(\exists \lambda \in]l - \varepsilon, 1])(\exists p \in A_\lambda)(\forall q \in \mathbb{R})$$
$$(L_d^{\ll}(p,q) = 1 \vee \neg(B(q) \geq \lambda + \varepsilon - l))$$
$$\Leftrightarrow (\forall \varepsilon \in]0,l])(\exists \lambda \in]l - \varepsilon, 1])(\exists p \in A_\lambda)(\forall q \in \mathbb{R})$$
$$(L_d^{\ll}(p,q) = 1 \vee \neg(q \in B_{\lambda + \varepsilon - l}))$$
$$\Leftrightarrow (\forall \varepsilon \in]0,l])(\exists \lambda \in]l - \varepsilon, 1])(\exists p \in A_\lambda)(\forall q \in \mathbb{R})(q \in B_{\lambda + \varepsilon - l} \Rightarrow L_d^{\ll}(p,q) = 1)$$
$$\Leftrightarrow (\forall \varepsilon \in]0,l])(\exists \lambda \in]l - \varepsilon, 1])(bb_d^{\ll}(A_\lambda, B_{\lambda + \varepsilon - l}))$$

proving (12).

Turning now to (13), we find:

$$bb_d^{\ll}(A,B) \leq k$$
$$\Leftrightarrow \sup_{p \in \mathbb{R}} T_W(A(p), \inf_{q \in \mathbb{R}} I_W(B(q), L_d^{\ll}(p,q))) \leq k$$
$$\Leftrightarrow (\forall p \in \mathbb{R})(T_W(A(p), \inf_{q \in \mathbb{R}} I_W(B(q), L_d^{\ll}(p,q))) \leq k) \tag{A.1}$$

42

If $A(p) = 0$, then $T_W(A(p), \inf_{q \in \mathbb{R}} I_W(B(q), L_d^{\ll}(p,q))) \leq k$ is trivially satisfied. Consequently, it is sufficient to show that for every p satisfying $A(p) > 0$, it holds that $T_W(A(p), \inf_{q \in \mathbb{R}} I_W(B(q), L_d^{\ll}(p,q))) \leq k$, or equivalently, to show that for every $\lambda \in]0, 1]$ and every p in A_λ:

$$\Leftrightarrow (\forall \lambda \in]0,1])(\forall p \in A_\lambda)(T_W(A(p), \inf_{q \in \mathbb{R}} I_W(B(q), L_d^{\ll}(p,q))) \leq k) \qquad \text{(A.2)}$$

which implies

$$(\forall \lambda \in]0,1])(\forall p \in A_\lambda)(T_W(\lambda, \inf_{q \in \mathbb{R}} I_W(B(q), L_d^{\ll}(p,q))) \leq k) \qquad \text{(A.3)}$$

since $p \in A_\lambda$ means that $A(p) \geq \lambda$. Conversely, we also have that (A.3) implies (A.2). Indeed, if (A.2) is violated, i.e., $T_W(A(p_0), \inf_{q \in \mathbb{R}} I_W(B(q), L_d^{\ll}(p_0,q))) > k$ for some $\lambda_0 \in]0, 1]$ and some $p_0 \in A_{\lambda_0}$, then (A.3) is violated for $\lambda = A(p_0)$ and $p = p_0$. We conclude that (A.1) is equivalent to (A.3). We furthermore find

$$(\forall p \in \mathbb{R})(T_W(A(p), \inf_{q \in \mathbb{R}} I_W(B(q), L_d^{\ll}(p,q))) \leq k)$$
$$\Leftrightarrow (\forall \lambda \in]0,1])(\forall p \in A_\lambda)(T_W(\lambda, \inf_{q \in \mathbb{R}} I_W(B(q), L_d^{\ll}(p,q))) \leq k)$$
$$\Leftrightarrow (\forall \lambda \in]0,1])(\forall p \in A_\lambda)(\lambda + \inf_{q \in \mathbb{R}} I_W(B(q), L_d^{\ll}(p,q)) - 1 \leq k)$$
$$\Leftrightarrow (\forall \lambda \in]0,1])(\forall p \in A_\lambda)(\inf_{q \in \mathbb{R}} I_W(B(q), L_d^{\ll}(p,q)) \leq 1 - \lambda + k)$$

If $\lambda \leq k$, then $I_W(B(q), L_d^{\ll}(p,q)) \leq 1 - \lambda + k$ is trivially satisfied. Therefore, we have

$$\Leftrightarrow (\forall \lambda \in]k,1])(\forall p \in A_\lambda)(\inf_{q \in \mathbb{R}} I_W(B(q), L_d^{\ll}(p,q)) \leq 1 - \lambda + k)$$
$$\Leftrightarrow (\forall \lambda \in]k,1])(\forall p \in A_\lambda)(\exists q \in \mathbb{R})(I_W(B(q), L_d^{\ll}(p,q)) \leq 1 - \lambda + k)$$
$$\Leftrightarrow (\forall \lambda \in]k,1])(\forall p \in A_\lambda)(\exists q \in \mathbb{R})(L_d^{\ll}(p,q) = 0 \wedge B(q) \geq \lambda - k)$$
$$\Leftrightarrow (\forall \lambda \in]k,1])(\forall p \in A_\lambda)(\exists q \in \mathbb{R})(L_d^{\ll}(p,q) = 0 \wedge q \in B_{\lambda-k})$$
$$\Leftrightarrow (\forall \lambda \in]k,1])(\forall p \in A_\lambda)(\exists q \in B_{\lambda-k})(L_d^{\ll}(p,q) = 0)$$
$$\Leftrightarrow (\forall \lambda \in]k,1])(\forall p \in A_\lambda)(\exists q \in B_{\lambda-k})(\neg(L_d^{\ll}(p,q) = 1))$$
$$\Leftrightarrow (\forall \lambda \in]k,1])(\neg(\exists p \in A_\lambda)(\forall q \in B_{\lambda-k})(L_d^{\ll}(p,q) = 1))$$
$$\Leftrightarrow (\forall \lambda \in]k,1])(\neg bb_d^{\ll}(A_\lambda, B_{\lambda-k}))$$
$$\Leftrightarrow (\forall \lambda \in]k,1])(bb_d^{\preceq}(B_{\lambda-k}, A_\lambda))$$

which proves (13). The characterizations (14)–(19) can be shown in the same way as (12) or (13).

43

A.2   Proof of Lemma 2

By definition of $bb_d^{\lll}$, we obtain

$$
\begin{aligned}
bb_d^{\lll}&(A,B)\\
&= \sup_{p\in\mathbb{R}} T_W(A(p), \inf_{q\in\mathbb{R}} I_W(B(q), L_d^{\lll}(p,q)))\\
&= \max(\sup_{p\geq m_b-d} T_W(A(p), \inf_{q\in\mathbb{R}} I_W(B(q), L_d^{\lll}(p,q))),\\
&\qquad\quad \sup_{p<m_b-d} T_W(A(p), \inf_{q\in\mathbb{R}} I_W(B(q), L_d^{\lll}(p,q))))
\end{aligned}
$$

From the convexity of B, we establish that B is increasing for values smaller than m_b and decreasing for values greater than m_b. Hence, we obtain

$$
\inf_{q\in\mathbb{R}} I_W(B(q), L_d^{\lll}(p,q)) =
\begin{cases}
I_W(B(p+d), L_d^{\lll}(p,p+d)) & \text{if } p < m_b - d\\
I_W(B(m_b), L_d^{\lll}(p,m_b)) & \text{if } p \geq m_b - d
\end{cases}
$$

We thus find

$$
\begin{aligned}
\max(&\sup_{p\geq m_b-d} T_W(A(p), \inf_{q\in\mathbb{R}} I_W(B(q), L_d^{\lll}(p,q))),\\
&\sup_{p<m_b-d} T_W(A(p), \inf_{q\in\mathbb{R}} I_W(B(q), L_d^{\lll}(p,q))))\\
=\max(&\sup_{p\geq m_b-d} T_W(A(p), I_W(B(m_b), L_d^{\lll}(p,m_b))),\\
&\sup_{p<m_b-d} T_W(A(p), I_W(B(p+d), L_d^{\lll}(p,p+d))))\\
=\max(&\sup_{p\geq m_b-d} T_W(A(p), I_W(1,0)), \sup_{p<m_b-d} T_W(A(p), I_W(B(p+d),0)))\\
=\max(&\sup_{p\geq m_b-d} T_W(A(p),0), \sup_{p<m_b-d} T_W(A(p), 1-B(p+d)))\\
=&\sup_{p<m_b-d} T_W(A(p), 1-B(p+d))
\end{aligned}
$$

Due to its convexity, A is increasing for values smaller than m_a and decreasing for values greater than m_a. Hence if $p < m_b - d$ and $p > m_a$, it holds that $T_W(A(p), 1-B(p+d))) \leq T_W(A(m_a), 1-B(m_a+d))$. Therefore, we have that

$$
\sup_{p<m_b-d} T_W(A(p), 1-B(p+d)) = \sup_{p<m_b-d, p\leq m_a} T_W(A(p), 1-B(p+d))
$$

proving (20). Eq. (21) is shown entirely analogously.

44

A.3 Proof of Lemma 3

As an example, we show that there is a path from v to the left neighbour of u when v is a black beginning node and u is a white beginning node. Let $v_0 = v, v_1, v_2, \ldots, v_n = u$ be a path in G from v to u. If an edge from v_j to v_{j+1} in G corresponds to a linear constraint that is the result of an FI–formula of the form $bb_d^{\lll}(x, y) \leq k$, with $k \in \{0, 2\Delta, 4\Delta, \ldots, 1 - 2\Delta\}$, then v_j and v_{j+1} are either both white beginning nodes, or both black beginning nodes. If this edge is the result of an FI–formula of the form $ee_d^{\lll}(x, y) \leq k$, v_j and v_{j+1} are both white ending nodes or both black ending nodes. Finally, if the edge from v_j to v_{j+1} is the result of an FI–formula of the form $be_d^{\lll}(x, y) \leq k$, or an FI–formula of the form $eb_d^{\lll}(x, y) \geq l$, with $l \in \{2\Delta, 4\Delta, \ldots, 1\}$, either v_j is a white ending node and v_{j+1} a black beginning node, or v_j is a black ending node and v_{j+1} a white beginning node. The only remaining possibility is that the edge from v_j to v_{j+1} corresponds to a linear constraint of the form (46)–(48).

First assume that none of the edges on the path from v to u corresponds to a linear constraint of the form (46)–(48). Then all of the nodes $v_1, \ldots, v_{n-1}$ need to be beginning nodes, as none of the remaining types of edges starts at a beginning node and ends at an ending node. This means that all edges $(v_0, v_1), (v_1, v_2), \ldots, (v_{n-1}, v_n)$ would correspond to a linear constraint that is the result of an FI–formula of the form $bb_d^{\lll}(x, y) \leq k$. Thus, from the fact that v is a black node, we establish that $v_1, v_2, \ldots, v_n$ are all black nodes. This, however, is not possible since $u = v_n$ is a white beginning node.

Hence, at least one of the edges corresponds to a linear constraint of the form (46)–(48). Let (v_s, v_{s+1}) be the last of these edges. If (v_s, v_{s+1}) corresponds to an edge of the form (46), v_{s+1} is a white ending node. Then all edges between v_{s+1} and v_n correspond to FI–formulas of the form $ee_d^{\lll}(x, y) \leq k$, $be_d^{\lll}(x, y) \leq k$, or $eb_d^{\lll}(x, y) \geq l$. This would imply that the nodes $v_{s+2}, v_{s+3}, \ldots, v_n$ are all white ending nodes or black beginning nodes. This, however, is not possible since $u = v_n$ is a white beginning node. Therefore, (v_s, v_{s+1}) has to correspond to either (47) or (48). In both cases, v_{s+1} is the right neighbour of v_s, and the path $v_0, v_1, \ldots, v_s, v'_{s+2}, v'_{s+3}, \ldots, v'_n$, where v'_i denotes the left neighbour of v_i, is a path from v to the left neighbour v'_n of u. Moreover, the edge labels of $v'_{s+2}, v'_{s+3}, \ldots, v'_n$ are the same as those of $v_{s+2}, v_{s+3}, \ldots, v_n$ (i.e., $(\leq, 0)$), and the edge label of (v_s, v_{s+1}) is $(\leq, 0)$, which adds nothing to the sum of the edge labels on the original path.
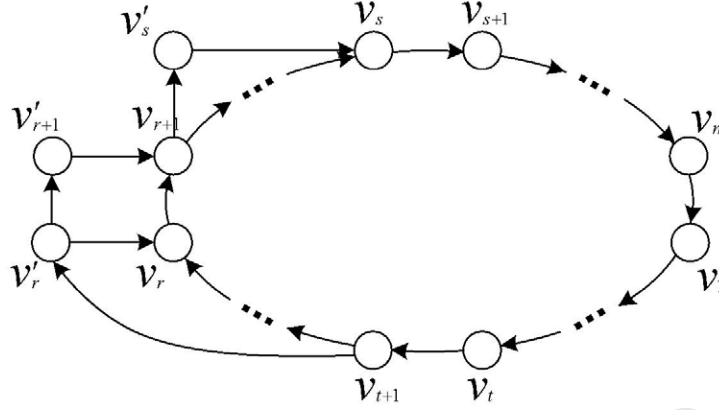
45

Fig. A.1. The forbidden cycle in G_2 is independent from the fact that the edge (v'_r, v'_{r+1}) in G_1 was replaced by (v_r, v_{r+1}).

A.4 Proof of Lemma 4

Assume that G_2 contains a forbidden cycle $v_1, v_2, \ldots, v_n, v_1$, and let (v_r, v_{r+1}) be an edge in G_2 that does not occur in G_1. Then (v_r, v_{r+1}) is the right neighbour of the edge (v'_r, v'_{r+1}) from G_1.

Moreover, first assume that v_r is a white beginning node and v_{r+1} is a black beginning node. Suppose that the edge from v_r to v_{r+1} is the only edge in the cycle that corresponds to an FI–formula of the form $bb_d^{\lll}(x,y) \geq l$, $ee_d^{\lll}(x,y) \geq l$, $be_d^{\lll}(x,y) \geq l$ or $eb_d^{\lll}(x,y) \leq k$. This means that the path $v_{r+1}, v_{r+2}, \ldots, v_n, v_1, \ldots, v_r$ in G_2 also exists in G_1. Indeed, none of the constraints on the edges of this path fulfills the conditions for replacement in the construction process of G_2 from G_1. Furthermore, all of the constraints on the edges of this path fulfill the conditions of Lemma 3. Hence, we establish that in G_1 there is a path from v_{r+1} to v'_r whose edge labels sum up to the same value as the edge labels of the path $v_{r+1}, v_{r+2}, \ldots, v_{r-1}, v_r$. This would mean that G_1 contains the forbidden cycle consisting of the path from v_{r+1} to v'_r, the edge from v'_r to v'_{r+1} and the edge from v'_{r+1} to v_{r+1}. Note that the latter edge exists since v'_{r+1} is the left neighbour of v_{r+1}.

Therefore, at least two edges in the forbidden cycle have to correspond to an FI–formula of the form $bb_d^{\lll}(x,y) \geq l$, $ee_d^{\lll}(x,y) \geq l$, $be_d^{\lll}(x,y) \geq l$ or $eb_d^{\lll}(x,y) \leq k$. Let the edge from v_s to v_{s+1} be the first such edge in the forbidden cyle after v_{r+1}, and let the edge from v_t to v_{t+1} be the last such edge in the forbidden cycle before v_r (where $r + 1 = s$ or $t + 1 = r$ are also allowed). Figure A.1 depicts the forbidden cycle. It holds that v_s is either a white beginning node or a black ending node, because of the way we transformed G_1 to G_2. In both cases, we can establish by Lemma 3 that there is a path from v_{r+1} to the left neighbour v'_s of v_s whose edge labels sum up to the same value

46

as those of the path from v_{r+1} to v_s. In the same way, we have by construction of G_2 that v_{t+1} is either a black beginning node or a white ending node. From Lemma 3, we obtain that there is a path from v_{t+1} to v'_r, whose edge labels sum up to the same value as those of the path from v_{t+1} to v_r.

Thus we have established that the forbidden cycle in G_2 is independent from the fact that the edge (v'_r, v'_{r+1}) in G_1 was replaced by (v_r, v_{r+1}). In a similar way, we can show this result when v_r is a black ending node and v_{r+1} is a white ending node, or when v_r is a white beginning node and v_{r+1} is a white ending node. We can repeat this argument for every edge that was changed in the transformation from G_1 to G_2. Hence, if G_2 contained a forbidden cycle, then G_1 would contain a forbidden cycle as well.

A.5   Proof of Lemma 5

As an example, we show that $\Theta \cup \{bb_d^{\lll}(x,y) \geq l + \Delta\}$ is FI–satisfiable iff $\Theta \cup \{bb_d^{\lll}(x,y) \geq l + 2\Delta\}$ is FI–satisfiable. If Θ is not FI–satisfiable, or x or y does not occur in the FI–formulas in Θ, the proof is trivial. Therefore, assume that Θ is FI–satisfiable and contains both FI–formulas involving x and FI–formulas involving y. If $\Theta \cup \{bb_d^{\lll}(x,y) \geq l + \Delta\}$ is not FI–satisfiable, then clearly $\Theta \cup \{bb_d^{\lll}(x,y) \geq l + 2\Delta\}$ is not FI–satisfiable either. Hence, we only need to show that if $\Theta \cup \{bb_d^{\lll}(x,y) \geq l + 2\Delta\}$ is not FI–satisfiable, $\Theta \cup \{bb_d^{\lll}(x,y) \geq l + \Delta\}$ cannot be FI–satisfiable.

Let Ψ be the set of linear constraints corresponding to the FI–formulas in Θ, and let $\mathcal{I}$ be a P–model of Ψ. The linear constraint corresponding to $bb_d^{\lll}(x,y) \geq l + \Delta$ is given by:

$$x_{l+\Delta}^- < y_\Delta^- - d \vee x_{l+2\Delta}^- < y_{2\Delta}^- - d \vee \cdots \vee x_1^- < y_{1-l}^- - d \qquad (A.4)$$

while the linear constraint corresponding to $bb_d^{\lll}(x,y) \geq l + 2\Delta$ is given by

$$x_{l+2\Delta}^- < y_\Delta^- - d \vee x_{l+3\Delta}^- < y_{2\Delta}^- - d \vee \cdots \vee x_1^- < y_{1-l-\Delta}^- - d \qquad (A.5)$$

If $\Theta \cup \{bb_d^{\lll}(x,y) \geq l+2\Delta\}$ is not FI–satisfiable, a forbidden cycle emerges when adding an edge corresponding to any of the disjuncts of (A.5) to the graph representation of Θ which corresponds with $\mathcal{I}$. This means that any P–model of Θ will correspond to a choice of disjuncts that leads to a graph representation G of Θ in which there is a path from y_Δ^- to $x_{l+2\Delta}^-$, a path from $y_{2\Delta}^-$ to $x_{l+3\Delta}^-$, etc. Moreover, the edge labels of the path from $y_{(i+1)\Delta}^-$ to $x_{l+(2+i)\Delta}^-$ sum up to a value $(d_i, \leq)$ or $(d_i, <)$ such that $d_i + d \geq 0$ ($i \in \{0, 1, 2, \ldots, \frac{1-l}{\Delta} - 2\}$).

We now transform the graph G to a graph G' by applying the transformation from Lemma 4. The changing of edges in this transformation corresponds to

choosing different disjuncts for the disjunctive linear constraints in Ψ. As this transformation cannot introduce forbidden cycles, the graph G' corresponds to a P–model of Ψ. Therefore G' contains a path from $y^-_{(i+1)\Delta}$ to $x^-_{l+(2+i)\Delta}$ for every i in $\{0, 1, 2, \ldots, \frac{1-l}{\Delta} - 2\}$. Let $y^-_\Delta = v_0, v_1, v_2, \ldots, v_n = x^-_{l+2\Delta}$ be a path from y^-_Δ to $x^-_{l+2\Delta}$.

If this path contains no edges that correspond to an FI–formula of the form $bb^{\lll}_{d'}(x', y') \geq l'$, $ee^{\lll}_{d'}(x', y') \geq l'$, $be^{\lll}_{d'}(x', y') \geq l'$ or $eb^{\lll}_{d'}(x', y') \leq k'$, we can apply Lemma 3 to establish that there is a path in G' from y^-_Δ to $x^-_{l+\Delta}$, the left neighbour of v_n, and a path from $y^-_{2\Delta}$, the right neighbour of y^-_Δ, to $x^-_{l+2\Delta}$. As none of the edges in these paths are changed in the transformation from G to G', these paths also occur in G.

Next, assume that the path from v_0 to v_n contains at least one edge which corresponds to an FI–formula of the form $bb^{\lll}_{d'}(x', y') \geq l'$, $ee^{\lll}_{d'}(x', y') \geq l'$, $be^{\lll}_{d'}(x', y') \geq l'$ or $eb^{\lll}_{d'}(x', y') \leq k'$. Let (v_s, v_{s+1}) and (v_r, v_{r+1}) be the first and last of these edges respectively. Then v_s is either a white beginning node or a black ending node, because of the nature of the transformation from G to G'. The path between v_0 and v_s therefore satisfies the conditions of Lemma 3. Thus we find that G' contains a path from $y^-_{2\Delta}$, the right neighbour of y^-_Δ, to $x^-_{l+2\Delta}$. Similarly, v_{r+1} is either a black beginning node, or a white ending node. By Lemma 3 we find that G' contains a path from y^-_Δ to $x^-_{l+\Delta}$, the left neighbour of v_n.

In the same way, we find from the fact that G' contains a path from $y^-_{3\Delta}$ to $x^-_{l+4\Delta}$ that G' also contains a path from $y^-_{3\Delta}$ to $x^-_{l+3\Delta}$ and from $y_{4\Delta}$ to $x^-_{l+4\Delta}$, etc. Adding an edge to G' corresponding to any of the disjuncts in (A.4) therefore leads to a forbidden cycle in G'. Using Lemma 4, we can conclude from this that adding an edge to G corresponding to any of the disjuncts in (A.4) would lead to a forbidden cycle as well. Hence, in any P–model of Θ, it holds that neither $x^-_{l+\Delta} < y^-_\Delta - d$, $x^-_{l+2\Delta} < y^-_{2\Delta} - d$, $\ldots$, or $x^-_1 < y^-_{1-l} - d$ can be satisfied, or, in other words, that $\Theta \cup \{bb^{\lll}_d(x, y) \geq l + \Delta\}$ is not FI–satisfiable.

A.6 Proof of Lemma 6

As an example, we show that for $k \in \{4\Delta, 6\Delta, \ldots, 1\}$, $\Theta \cup \{bb^{\lll}_d(x, y) \leq k - \Delta\}$ is FI–satisfiable iff $\Theta \cup \{bb^{\lll}_d(x, y) \leq k - 2\Delta\}$ is FI–satisfiable. Clearly, if $\Theta \cup \{bb^{\lll}_d(x, y) \leq k - 2\Delta\}$ is FI–satisfiable, then also $\Theta \cup \{bb^{\lll}_d(x, y) \leq k - \Delta\}$ is FI–satisfiable. Conversely, we show that if $\Theta \cup \{bb^{\lll}_d(x, y) \leq k - 2\Delta\}$ is not FI–satisfiable, then also $\Theta \cup \{bb^{\lll}_d(x, y) \geq k - \Delta\}$ is not FI–satisfiable.

Let Ψ be the set of linear constraints corresponding to Θ. The linear con-

straints corresponding to $bb_d^{\lll}(x,y) \leq k - 2\Delta$ are given by:

$$\{y_\Delta^- \leq x_{k-\Delta}^- + d, y_{2\Delta}^- \leq x_k^- + d, \ldots, y_{1-k+2\Delta}^- \leq x_1^- + d\} \qquad (A.6)$$

while the linear constraints corresponding to $bb_d^{\lll}(x,y) \leq k - \Delta$ are given by

$$\{y_\Delta^- \leq x_k^- + d, y_{2\Delta}^- \leq x_{k+\Delta}^- + d, \ldots, y_{1-k+\Delta}^- \leq x_1^- + d\} \qquad (A.7)$$

Assume that $\Theta \cup \{bb_d^{\lll}(x,y) \leq k - 2\Delta\}$ is not FI–satisfiable. This means that the graph G corresponding to the linear constraints in Ψ contains a path from $x_{k-\Delta}^-$ to y_Δ^-, or a path from x_k^- to $y_{2\Delta}^-$, or ..., or a path from x_1^- to $y_{1-k+2\Delta}^-$. Moreover, the edge labels in this path sum up to $(<,d')$ where $d' + d \geq 0$, or $(\leq, d'')$ where where $d'' + d > 0$, i.e., adding the edges corresponding to (A.6) would introduce a forbidden cycle in the graph. Note that there is only one graph G corresponding to Ψ, as Ψ contains no disjunctive linear constraints.

Let $v_1, v_2, \ldots, v_n$ be a path from $x_{k+(i-1)\Delta}^-$ to $y_{(1+i)\Delta}^-$, for some i in $\{0, 1, \ldots, 1 + \frac{1-k}{\Delta}\}$, where v_1 and v_n are both white beginning nodes or both black beginning nodes. First assume that this path contains no edges corresponding to an FI–formula of the form $bb_{d'}^{\lll}(x',y') \geq 1$, $ee_{d'}^{\lll}(x',y') \geq 1$, $be_{d'}^{\lll}(x',y') \geq 1$ or $eb_{d'}^{\lll}(x',y') \leq 0$. Note that edges corresponding to FI–formulas of the form $ee_d^{\lll}(x,y) \leq k$, $be_d^{\lll}(x,y) \leq k$, and $eb_d^{\lll}(x,y) \geq l$ always start at an ending node. Hence, either the path from v_1 to v_n contains no edges of the form $ee_d^{\lll}(x,y) \leq k$, $be_d^{\lll}(x,y) \leq k$, and $eb_d^{\lll}(x,y) \geq l$, or this path contains at least one edge corresponding to (46). In the former case, however, it is not possible to obtain a path from a node $a_{k_1}^-$ to a node $b_{k_2}^-$ if $k_1 > k_2$. Hence, since $k > 2\Delta$, the path from v_1 to v_n needs to contain at least one edge of the form (46). Assume that v_1 and v_n are white beginning nodes, and let the edge from v_i to v_{i+1} be the last edge of the form (46). Then v_{i+1} is a white ending node, and by Lemma 3 there exists a path from v_{i+1} to the left neighbour of v_n. Hence, there is a path from $x_{k+(i-1)\Delta}^-$ to $y_{i\Delta}^-$. In particular, we obtain that adding the edges corresponding to (A.7) would introduce a forbidden cycle, in other words, that $\Theta \cup \{bb_d^{\lll}(x,y) \geq k - \Delta\}$ cannot be FI–satisfiable. Next, assume that v_1 and v_n are black beginning nodes and let the edge from v_j to v_{j+1} be the first edge of the form (46). Using Lemma 3, we now find that there must exist a path from the right neighbour of v_1 to v_j, and again, that $\Theta \cup \{bb_d^{\lll}(x,y) \geq k - \Delta\}$ is not FI–satisfiable.

Finally, assume that the path from v_1 to v_n contains at least one edge corresponding to an FI–formula of the form $bb_{d'}^{\lll}(x',y') \geq 1$, $ee_{d'}^{\lll}(x',y') \geq 1$, $be_{d'}^{\lll}(x',y') \geq 1$ or $eb_{d'}^{\lll}(x',y') \leq 0$. Let the edge from v_i to v_{i+1} and the edge from v_j to v_{j+1} be the first and the last of these edges respectively. Then v_{j+1} is either a black beginning node or a white ending node and v_i is either a white beginning node or a black ending node. Using Lemma 3, we find that there must exist a path from v_1 to the left neighbour of v_n if v_1 and v_n are white

49

beginning nodes, and a path from the right neighbour of v_1 to v_n if v_1 and v_n are black beginning nodes. In either case, we find that adding the edges corresponding to (A.7) would introduce a forbidden cycle.

References

[1] J. Allen, Maintaining knowledge about temporal intervals, Communications of the ACM 26 (11) (1983) 832–843.

[2] J. Allen, Planning as temporal reasoning, in: Proceedings of the Second International Conference on Principles of Knowledge Representation and Reasoning, 1991.

[3] E. André, T. Rist, Coping with temporal constraints in multimedia presentation planning, in: Proceedings of the Thirteenth National Conference on Artificial Intelligence (AAAI-96), 1996.

[4] S. Badaloni, M. Giacomin, The algebra IA^{fuz}: a framework for qualitative fuzzy temporal reasoning, Artificial Intelligence 170 (10) (2006) 872–908.

[5] S. Barro, R. Marín, J. Mira, A. Patón, A model and a language for the fuzzy representation and handling of time, Fuzzy Sets and Systems 61 (2) (1994) 153–175.

[6] R. Barzilay, M. Elhadad, K. McKeown, Inferring strategies for sentence ordering in multidocument news summarization, Journal of Artificial Intelligence Research 17 (2002) 35–55.

[7] U. Bodenhofer, A new approach to fuzzy orderings, Tatra Mountains Mathematical Publications 16 (1999) 21–29.

[8] A. Bosch, M. Torres, R. Marin, Reasoning with disjunctive fuzzy temporal constraint networks, in: Proceedings of the 9th International Symposium on Temporal Representation and Reasoning, 2002.

[9] F. Bry, B. Lorenz, H. Ohlbach, S. Spranger, On reasoning on time and location on the web, in: Proceedings of the International Workshop on Principles and Practice of Semantic Web Reasoning (PPSWR 2003), LNCS 2901, 2003.

[10] M. Buchanan, P. Zellweger, Scheduling multimedia documents using temporal constraints, in: Proceedings of the Third International Workshop on Network and Operating System Support for Digital Audio and Video, 1992.

[11] A. Cohn, N. Gotts, The 'egg-yolk' representation of regions with indeterminate boundaries, in: Geographic Objects with Indeterminate Boundaries (P.A. Burrough and A.U. Frank, eds.), Taylor and Francis Ltd., 1996.

[12] R. Dechter, I. Meiri, J. Pearl, Temporal constraint networks, Artificial Intelligence 49 (1–3) (1991) 61–95.

50

[13] T. Drakengren, P. Jonsson, Eight maximal tractable subclasses of Allen's algebra with metric time, Journal of Artificial Intelligence Research 7 (1997) 25–45.

[14] T. Drakengren, P. Jonsson, Twenty-one large tractable subclasses of Allen's algebra, Artificial Intelligence 93 (1–2) (1997) 297–319.

[15] D. Dubois, H. Prade, Ranking fuzzy numbers in the setting of possibility theory, Information Sciences 30 (3) (1983) 183–224.

[16] D. Dubois, H. Prade, Processing fuzzy temporal knowledge, IEEE Transactions on Systems, Man, and Cybernetics 19 (4) (1989) 729–744.

[17] A. El-Kholy, B. Richards, Temporal and resource reasoning in planning: the parcPLAN approach, in: Proceedings of the 12th European Conference on Artificial Intelligence (ECAI-96), 1996.

[18] R. Erfle, Specification of temporal constraints in multimedia documents using HyTime, Electronic Publishing 6 (4) (1993) 397–411.

[19] M. Fox, D. Long, PDDL2.1: an extension to PDDL for expressing temporal planning domains, Journal of Artificial Intelligence Research 20 (2003) 61–124.

[20] A. Gerevini, M. Cristani, On finding a solution in temporal constraint satisfaction problems, in: Proceedings of the International Joint Conference on Articifical Intelligence, 1997.

[21] H. Guesgen, J. Hertzberg, A. Philpott, Towards implementing fuzzy Allen relations, in: Proceedings of the ECAI-94 Workshop on Spatial and Temporal Reasoning, 1994.

[22] S. Harabagiu, C. Bejan, Question answering based on temporal inference, in: AAAI-2005 Workshop on Inference for Textual Question Answering, 2005.

[23] P. Jonsson, C. Bäckström, A unifying approach to temporal constraint reasoning, Artificial Intelligence 102 (1) (1998) 143–155.

[24] P. Kalczynski, A. Chou, Temporal document retrieval model for business news archives, Information Processing and Management 41 (2005) 635–650.

[25] H. Kautz, P. Ladkin, Integrating metric and qualitative temporal reasoning, in: Proceedings of the Ninth National Conference on Artificial Intelligence (AAAI-91), 1991.

[26] L. Khatib, P. Morris, R. Morris, F. Rossi, Temporal constraint reasoning with preferences, in: Proceedings of the 17th International Joint Conference on Artificial Intelligence, 2001.

[27] M. Koubarakis, Tractable disjunctions of linear constraints: basic results and applications to temporal reasoning, Theoretical computer science 266 (1) (2001) 311–339.

[28] A. Krokhin, P. Jeavons, P. Jonsson, Reasoning about temporal relations: The tractable subalgebras of Allen's interval algebra, Journal of the ACM 50 (5) (2003) 591–640.

[29] M. Lapata, A. Lascarides, Proceedings of the north american chapter of the assocation of computational linguistics, in: Proceedings of the International Conference on Ontologies, Databases and Applications of Semantics (ODBASE), LNCS 2888, 2004.

[30] R. Marín, S. Barro, A. Bosch, J. Mira, Modelling the representation of time from a fuzzy perspective, Cybernetics and Systems 25 (2) (1994) 217–231.

[31] I. Meiri, Combining qualitative and quantitative constraints in temporal reasoning, Artificial Intelligence 87 (1) (1996) 343–385.

[32] D. Moldovan, C. Clark, S. Harabagiu, Temporal context representation and reasoning, in: Proceedings of the 19th International Joint Conference on Artificial Intelligence, 2005.

[33] G. Nagypál, B. Motik, A fuzzy model for representing uncertain, subjective and vague temporal knowledge in ontologies, in: Proceedings of the International Conference on Ontologies, Databases and Applications of Semantics (ODBASE), LNCS 2888, 2003.

[34] A. Nakhimovsky, Temporal reasoning in natural language understanding: the temporal structure of the narrative, in: Third Conference of the European Chapter of the Association for Computational Linguistics, 1987.

[35] I. Navarrete, A. Sattar, R. Wetprasit, R. Marin, On point-duration networks for temporal reasoning, Artificial Intelligence 140 (1–2) (2002) 39–70.

[36] B. Nebel, Solving hard qualitative temporal reasoning problems: Evaluating the efficiency of using the ORD-Horn class, Constraints 1 (3) (1997) 175–190.

[37] B. Nebel, H.-J. Bürckert, Reasoning about temporal relations: a maximal tractable subset of Allen's interval algebra, Journal of the ACM 42 (1) (1995) 43–66.

[38] H. Ohlbach, Relations between fuzzy time intervals, in: Proceedings of the 11th International Symposium on Temporal Representation and Reasoning, 2004.

[39] E. Saquete, P. Martínez-Barco, R. Muñoz, J. Vicedo, Splitting complex temporal questions for question answering systems, in: Proceedings of the 42nd Annual Meeting of the ACL, 2004.

[40] S. Schockaert, D. Ahn, M. De Cock, E. Kerre, Question answering with imperfect temporal information, in: Proceedings of the 7th International Conference on Flexible Query Answering Systems, LNAI 4027, 2006.

[41] S. Schockaert, M. De Cock, E. Kerre, Fuzzifying Allen's temporal interval relations, IEEE Transactions on Fuzzy Systems (to appear).

[42] S. Schockaert, M. De Cock, E. Kerre, Imprecise temporal interval relations, in: Proceedings of the 6th International Workshop on Fuzzy Logic and Applications, LNAI 3849, 2006.

[43] S. Schockaert, M. De Cock, E. Kerre, Qualitative temporal reasoning about vague events, in: Proceedings of the 20th International Joint Conference on Artificial Intelligence, 2007.

[44] E. Sontag, Real addition and the polynomial time hierarchy, Information Processing Letters 20 (3) (1985) 115–120.

[45] K. Stergiou, M. Koubarakis, Backtracking algorithms for disjunctions of temporal constraints, Artificial Intelligence 120 (1) (2000) 81–117.

[46] I. Tsamardinos, M. E. Pollack, Efficient solution techniques for disjunctive temporal reasoning problems, Artificial Intelligence 151 (1–2) (2003) 43–90.

[47] J. Zacks, B. Tversky, Event structure in perception and conception, Psychological Bulletin 127 (1) (2001) 3–21.

[48] L. Zadeh, Fuzzy sets, Information and Control 8 (3) (1965) 338–353.