Imprecise Temporal Interval Relations

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Abstract. When the time span of an event is imprecise, it can be represented by a fuzzy set, called a fuzzy time interval. In this paper we propose a representation for 13 relations that can hold between intervals. Since our model is based on fuzzy orderings of time points, it is not only suitable to express precise relationships between imprecise events ("the mid 1930's came *before* the late 1930's) but also imprecise relationships ("the late 1930's came *long before* the early 1990's). Furthermore we show that our model preserves many of the properties of the 13 relations Allen introduced for crisp time intervals.

Keywords: Temporal Reasoning, Fuzzy Relation, Fuzzy Ordering

1 Introduction

A significant part of the work on temporal representation and reasoning is concerned with time intervals. Allen [1] defined 13 qualitative relations that may hold between two intervals $A = [a^-, a^+]$ and $B = [b^-, b^+]$. Table 1 shows how these relations are expressed by means of constraints on the boundaries of the intervals. The relations are mutually exclusive and exhaustive, i.e. for any two intervals, exactly one of the relations holds. Temporal information is however often ill–defined, e.g. because the definition of some historical events is inherently subjective (e.g. the Renaissance), or because historical documents are usually written in a vague style (e.g. "in the late 1930s"). Ill–defined time intervals can either be intervals with uncertain boundaries [2], or imprecise intervals [4, 5]. In this paper we will focus on the latter, i.e. we will assume that we have complete knowledge about the time span of an event, but that it has a gradual beginning and/or ending. This kind of time intervals can be represented as fuzzy sets.

To our knowledge, Nagypál and Motik [4] were the first to extend Allen's work to fuzzy time intervals, generalizing the relations of Table 1 to fuzzy relations. However, their approach suffers from a number of important disadvantages, in particular concerning the relations e, m, s and f. For example, e is not reflexive in general; if A is a continuous fuzzy set in $\mathbb{R}$, it holds that $e(A, A) = s(A, A) = f(A, A) = 0.5$ while one would expect $e(A, A) = 1$ and $s(A, A) = f(A, A) = 0$.

Name		Definition
1. before	$b(A,B)$	$a^+ < b^-$
2. overlaps	$o(A,B)$	$a^- < b^-$ and $b^- < a^+$ and $a^+ < b^+$
3. during	$d(A,B)$	$b^- < a^-$ and $a^+ < b^+$
4. meets	$m(A,B)$	$a^+ = b^-$
5. starts	$s(A,B)$	$a^- = b^-$ and $a^+ < b^+$
6. finishes	$f(A,B)$	$a^+ = b^+$ and $b^- < a^-$
7. equals	$e(A,B)$	$a^- = b^-$ and $a^+ = b^+$

Inverse relations	
8. $bi(A,B) = b(B,A)$	11. $mi(A,B) = m(B,A)$
9. $oi(A,B) = o(B,A)$	12. $si(A,B) = s(B,A)$
10. $di(A,B) = d(B,A)$	13. $fi(A,B) = f(B,A)$

Table 1. Allen's temporal interval relations.

Furthermore they only consider precise relationships. The approach proposed by Ohlbach [5] allows to express some imprecise temporal relations (e.g. A *more or less* finishes B), but it does not deal with imprecise constraints such as "A was *long* before B". Moreover, many desirable properties that hold for Allen's relationships are not preserved in this fuzzification.

In Section 2 of this paper we introduce a generalization of Allen's 13 interval relations that can not only be used when the time intervals are fuzzy ("the mid 1930's came *before* the late 1930's), but is even powerful enough to express imprecise relationships ("the late 1930's came *long before* the early 1990's"). The magical ingredients are fuzzy orderings of time points; they are lifted into relationships between fuzzy time intervals through the use of relatedness measures for fuzzy sets. In Section 3, we show that our model preserves important properties regarding (ir)reflexivity, (a)symmetry and transitivity. To our knowledge, we are the first to introduce a generalization of Allen's relations that can be used for precise as well as imprecise temporal relationships between fuzzy time intervals, and at the same time preserves so many desirable properties.

2 Fuzzy Temporal Interval Relations

Throughout this paper, we represent time points as real numbers. A real number can, for example, be interpreted as the number of milliseconds since January 1, 1970. Because we want to model imprecise temporal relations, we need a way to express that a certain time point a is long before a time point b, and a way to express that a is before or approximately at the same time as b. Fuzzy relations are particularly well suited for this purpose, due to the vague nature of these concepts.

Definition 1 (Fuzzy Ordering of Time Points). *For $\beta \in]0, +\infty[$, the fuzzy relation $L_\beta^{\ll}$ in $\mathbb{R}$ is defined as*

$$L_\beta^{\ll}(a,b) = \begin{cases} 1 & \text{if } b - a > \beta \\ 0 & \text{if } b - a \leq 0 \\ \frac{b-a}{\beta} & \text{otherwise} \end{cases} \tag{1}$$

for all a and b in $\mathbb{R}$. $L_0^{\ll}$ is defined by $L_0^{\ll}(a,b) = 1$ if $a < b$ and $L_0^{\ll}(a,b) = 0$ otherwise. The fuzzy relation $L_\beta^{\preccurlyeq}$ in $\mathbb{R}$ is defined as

$$L_\beta^{\preccurlyeq}(a,b) = 1 - L_\beta^{\ll}(b,a) \tag{2}$$

$L_\beta^{\ll}(a,b)$ represents the extent to which a is much smaller than b. Note that the parameter β defines how the concept "much smaller than" should be interpreted. Likewise, $L_\beta^{\preccurlyeq}(a,b)$ represents the extent to which b is not "much smaller than a", in other words, the extent to which a is smaller than or approximately equal to b. Moreover, $L_0^{\preccurlyeq}(a,b) = 1$ if $a \leq b$ and $L_0^{\preccurlyeq}(a,b) = 0$ otherwise, i.e $L_\beta^{\preccurlyeq}$ is a generalization of the crisp ordering $\leq$. We use these ordering relations between time points as a stepping stone for the representation of imprecise relations that may hold between fuzzy time intervals.

Proposition 1 *Let $\beta \geq 0$; it holds that for every a, b and c in $\mathbb{R}$*

$$T_W(L_\beta^{\ll}(a,b), L_\beta^{\ll}(b,c)) \leq L_\beta^{\ll}(a,c) \tag{3}$$

$$T_W(L_\beta^{\preccurlyeq}(a,b), L_\beta^{\preccurlyeq}(b,c)) \leq L_\beta^{\preccurlyeq}(a,c) \tag{4}$$

$$T_W(L_\beta^{\preccurlyeq}(a,b), L_\beta^{\ll}(b,c)) \leq L_\beta^{\ll}(a,c) \tag{5}$$

$$T_W(L_\beta^{\ll}(a,b), L_\beta^{\preccurlyeq}(b,c)) \leq L_\beta^{\ll}(a,c) \tag{6}$$

$$T_W(L_\beta^{\ll}(a,b), L_\beta^{\preccurlyeq}(b,a)) = 0 \tag{7}$$

where T_W denotes the Łukasiewicz t–norm $T_W(x,y) = \max(0, x+y-1)$.

Recall that a fuzzy set A in $\mathbb{R}$ is convex and upper semicontinuous iff for each α in $]0,1]$ the set $\{x|A(x) \geq \alpha\}$ is a closed interval.

Definition 2 (Fuzzy Time Period). *A fuzzy time period is a normalised fuzzy set in $\mathbb{R}$ which is interpreted as the time span of some event. A fuzzy (time) interval is a convex and upper semicontinuous normalised fuzzy set in $\mathbb{R}$. A fuzzy time period A is called nondegenerate w.r.t. β iff $A \circ L_\beta^{\ll} \circ A = 1$, i.e. if the beginning of A is long before the end of A.*

As recalled in Section 1, Allen's definitions are based on constraints on the boundaries of the intervals. If A and B are fuzzy time intervals, the boundaries of A and B can be gradual. Hence, we cannot refer to these boundaries in the same way we refer to the boundaries of crisp intervals. Therefore, as shown in Table 2, we propose using relatedness measures to express the relations between

Crisp	Fuzzy	Crisp	Fuzzy
$a^- < b^-$	$A \circ (L_\beta^{\ll} \triangleright B)$	$a^- \leq b^-$	$(A \circ L_\beta^{\preccurlyeq}) \triangleright B$
$a^+ < b^+$	$(A \triangleleft L_\beta^{\ll}) \circ B$	$a^+ \leq b^+$	$A \triangleleft (L_\beta^{\preccurlyeq} \circ B)$
$a^+ < b^-$	$A \triangleleft L_\beta^{\ll} \triangleright B$	$a^+ \leq b^-$	$A \triangleleft L_\beta^{\preccurlyeq} \triangleright B$
$a^- < b^+$	$A \circ L_\beta^{\ll} \circ B$	$a^- \leq b^+$	$A \circ L_\beta^{\preccurlyeq} \circ B$

Table 2. Relation between the boundaries of the crisp intervals $[a^-, a^+]$ and $[b^-, b^+]$, and the fuzzy intervals A and B.

the boundaries of fuzzy intervals without actually referring to these boundaries. For an arbitrary fuzzy relation $\mathbb{R}$, these relatedness measures are defined as [3]:

$$A \circ_T R \circ_T B = \sup_{v\in\mathbb{R}} T(B(v), \sup_{u\in\mathbb{R}} T(A(u), R(u,v))) \tag{8}$$

$$A \triangleleft_I R \triangleright_I B = \inf_{v\in\mathbb{R}} I(B(v), \inf_{u\in\mathbb{R}} I(A(u), R(u,v))) \tag{9}$$

$$(A \triangleleft_I R) \circ_T B = \sup_{v\in\mathbb{R}} T(B(v), \inf_{u\in\mathbb{R}} I(A(u), R(u,v))) \tag{10}$$

$$A \triangleleft_I (R \circ_T B) = \inf_{u\in\mathbb{R}} I(A(u), \sup_{v\in\mathbb{R}} T(B(v), R(u,v))) \tag{11}$$

$$(A \circ_T R) \triangleright_I B = \inf_{v\in\mathbb{R}} I(B(v), \sup_{u\in\mathbb{R}} T(A(u), R(u,v))) \tag{12}$$

$$A \circ_T (R \triangleright_I B) = \sup_{u\in\mathbb{R}} T(A(u), \inf_{v\in\mathbb{R}} I(B(v), R(u,v))) \tag{13}$$

where T is a left–continuous t–norm and I its residual implicator. For example $A \circ (L_\beta^{\ll} \triangleright B)$ expresses the degree to which there is an element in A that is much smaller than all elements in B. In the remainder of this paper we assume that T is the Łukasiewicz t–norm and I its residual implicator $I_W(x,y) = \min(1, 1-x+y)$.

Note how the appearance of $<$ (resp. $\leq$) in Table 2 corresponds to the use of $L_\beta^{\ll}$ (resp. $L_\beta^{\preccurlyeq}$). If $\beta > 0$, the relations from Table 2 become imprecise relations (e.g. the beginning of A is *long* before the beginning of B). Using the expressions from Table 2, we define the temporal relations for fuzzy intervals as shown in Table 3. For convenience, we use the same notation for the temporal relations when fuzzy intervals are used instead of crisp intervals.

3 Properties

When A and B are crisp intervals and $\beta = 0$, our definitions are equivalent to Allen's original definitions. Note that in Table 3 we have used the minimum to generalize the conjunctions that appear in the crisp definitions. The use of the minimum as t–norm makes it possible to prove the following proposition.

Name	Definition
$b(A,B)$	$A \lhd L_\beta^{\ll} \rhd B$
$o(A,B)$	$\min(A \circ (L_\beta^{\ll} \rhd B), B \circ L_\beta^{\ll} \circ A, (A \lhd L_\beta^{\ll}) \circ B)$
$d(A,B)$	$\min(B \circ (L_\beta^{\ll} \rhd A), (A \lhd L_\beta^{\ll}) \circ B)$
$m(A,B)$	$\min(A \lhd L_\beta^{\preccurlyeq} \rhd B, B \circ L_\beta^{\preccurlyeq} \circ A)$
$s(A,B)$	$\min((A \circ L_\beta^{\succcurlyeq}) \rhd B, (B \circ L_\beta^{\succcurlyeq}) \rhd A, (A \lhd L_\beta^{\ll}) \circ B)$
$f(A,B)$	$\min(A \lhd (L_\beta^{\preccurlyeq} \circ B), B \lhd (L_\beta^{\preccurlyeq} \circ A), B \circ (L_\beta^{\ll} \rhd A))$
$e(A,B)$	$\min((A \circ L_\beta^{\succcurlyeq}) \rhd B, (B \circ L_\beta^{\succcurlyeq}) \rhd A, A \lhd (L_\beta^{\preccurlyeq} \circ B), B \lhd (L_\beta^{\preccurlyeq} \circ A))$

Table 3. Fuzzy temporal interval relations.

Proposition 2 (Exhaustivity) *Let A and B be fuzzy time periods. It holds that*

$$\begin{aligned}S_W(&b(A,B), bi(A,B), o(A,B), oi(A,B), d(A,B), di(A,B), m(A,B), mi(A,B),\\ &s(A,B), si(A,B), f(A,B), fi(A,B), e(A,B)) = 1 \end{aligned} \tag{14}$$

where S_W is the Łukasiewicz t–conorm defined by $S_W(x,y) = \min(1, x+y)$ for all x and y in $[0,1]$.

Proposition 3 (Mutual Exclusiveness) *Let A and B be nondegenerate fuzzy time periods w.r.t. β. Moreover, let R and S both be one of the 13 fuzzy temporal relations. If $R \neq S$, then it holds that*

$$T_W(R(A,B), S(A,B)) = 0 \tag{15}$$

Proposition 4 (Reflexivity and Symmetry) *The relations b, bi, o, oi, d, di, s, si, f and fi are irreflexive and asymmetric w.r.t. T_W, i.e. let R be one of the aforementioned fuzzy relations and let A and B be fuzzy time periods. It holds that*

$$R(A,A) = 0 \tag{16}$$

$$T_W(R(A,B), R(B,A)) = 0 \tag{17}$$

Furthermore, it holds that

$$e(A,A) = 1 \tag{18}$$

$$e(A,B) = e(B,A) \tag{19}$$

$$m(A,A) = A \lhd L_\beta^{\preccurlyeq} \rhd A \tag{20}$$

$$T_W(m(A,B), m(B,A)) \leq \min(A \lhd L_\beta^{\preccurlyeq} \rhd A, B \lhd L_\beta^{\preccurlyeq} \rhd B) \tag{21}$$

The crisp meets relation m (between crisp intervals) is irreflexive, provided that the beginning of each interval is strictly before the end of the interval, i.e. provided singletons (time points) are not allowed as time intervals; (20)–(21) is a generalization of this observation in the sense that our meets relation (and therefore also mi) is irreflexive and asymmetric if the beginning of A (resp. B) is not approximately equal to the end of A (resp. B). From (14)–(21) it can be concluded that the fuzzy temporal interval relations are mutually exclusive and exhaustive w.r.t. the Łukasiewicz t–norm and t–conorm. Moreover, the reflexivity and symmetry properties of our definitions are in accordance with the corresponding properties of the temporal relations between crisp intervals.

Proposition 5 (Transitivity) *The relations b, bi, d, di, s, si, f, fi and e are T_W–transitive, i.e. let R be one of the aforementioned fuzzy relations and let A, B and C be fuzzy time periods. It holds that*

$$T_W(R(A,B), R(B,C)) \leq R(A,C)$$

No kind of transitivity holds for o, oi, m and mi in general. Thus the transitivity properties of our definitions are in accordance with the transitivity properties of the (crisp) temporal relations between crisp intervals.

The properties in this section are valid for arbitrary fuzzy time periods. In practice however, it seems often more natural to consider only fuzzy time intervals in this context.

4 Concluding Remarks

In this paper we have introduced a new approach to define possibly imprecise, temporal interval relations between fuzzy time intervals. It can be shown that, unlike in previous approaches, generalizations of all the important properties of the crisp interval relations are valid. Further work will focus on the use of our approach for temporal reasoning. The reader can verify that, for example

$$T_W(d(A,B), b(B,C)) \leq b(A,C)$$

which expresses that from "A takes place during B", and "B happens before C", we deduce that "A takes place before C".

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