Intuitionistic Fuzzy Relational Images

Martine De Cock, Chris Cornelis, and Etienne E. Kerre

Ghent University
Dept. of Mathematics and Computer Science
Fuzziness and Uncertainty Modelling Research Unit
Krijgslaan 281 (S9), B-9000 Gent, Belgium
{Martine.DeCock, Chris.Cornelis, Etienne.Kerre}@UGent.be

Abstract: By tracing intuitionistic fuzzy sets back to the underlying algebraic structure that they are defined on (a complete lattice), they can be embedded in the well–known class of L–fuzzy sets, whose formal treatment allows the definition and study of order–theoretic concepts such as triangular norms and conorms, negators and implicators, as well as the development of intuitionistic fuzzy relational calculus. In this chapter we study the intuitionistic fuzzy relational direct and superdirect image. An important aspect of our work, differentiating it from the study of L–fuzzy relational images in general, concerns the construction of an intuitionistic fuzzy relational image from the separate fuzzy relational images of its membership and non–membership function. We illustrate our results with two applications: the representation of linguistic hedges, and the development of a meaningful concept of an intuitionistic fuzzy rough set.

Keywords: intuitionistic fuzzy relational calculus, direct and superdirect image, linguistic hedge, intuitionistic fuzzy rough set

1 INTRODUCTION

Intuitionistic fuzzy sets (IFSs for short), an extension of fuzzy sets, were introduced by Atanassov [1] and are currently generating a great deal of interest. IFS theory basically defies the claim that from the fact that an element x "belongs" to a given degree (say $\mu_A(x)$) to a fuzzy set A, naturally follows that x should "not belong" to A to the extent $1-\mu_A(x)$, an assertion implicit in the concept of a fuzzy set. On the contrary, IFSs assign to each element x of the universe both a degree of membership $\mu_A(x)$ and one of non–membership $\nu_A(x)$ such that

$$\mu_A(x) + \nu_A(x) \leq 1$$

thus relaxing the enforced duality $\nu_A(x) = 1 - \mu_A(x)$ from fuzzy set theory. Obviously, when $\mu_A(x) + \nu_A(x) = 1$ for all elements of the universe, the traditional fuzzy set concept is recovered.

Wang and He [32], and later also Deschrijver and Kerre [17], noticed that IFSs can be considered as special instances of Goguen's L–fuzzy sets [19], so every concept definable for L–fuzzy sets is also available to IFS theory. In this spirit, in [16, 18] suitable definitions and representation theorems for the most important intuitionistic fuzzy connectives have been derived; negators, triangular norms and conorms, and implicators can be used to model the elementary set–theoretical operations of complementation, intersection, union, and inclusion as well as the logical operations of negation, conjunction, disjunction, and implication. In this way, slowly but surely, IFSs start giving away their secrets.

Using these building blocks, we can arrive at the study of more complex frameworks such as intuitionistic fuzzy relational calculus (IF relational calculus for short). The importance of L–fuzzy relational calculus in computer science can hardly be overestimated. It has already proven its usefuleness in fields such as approximate reasoning (modelling linguistic IF–THEN rules as fuzzy relations, see e.g. [34]), fuzzy morphology for image processing (see e.g. [25]), fuzzy preference modelling (see e.g. [9]), and for obvious reasons also fuzzy relational databases. Furthermore benefits of fuzzy relations for the search in unstructured environments is becoming more and more clear (see [14] for a recent overview on the construction of fuzzy term–term relationships). In this chapter we will illustrate our results in yet two other applications, namely intuitionistic fuzzy rough sets for knowledge discovery, and the mathematical representation of linguistic terms for computing with words.

At the heart of L–fuzzy relational calculus and very closely related to the composition of L–fuzzy relations, are the notions of direct and superdirect image of an L–fuzzy set under an L–fuzzy relation. Most of the research on L–fuzzy relational images so far has been carried out for $L = [0, 1]$. We refer to [11, 26] for an overview of both established theoretical properties and applications. The mapping of elements of the universe to the interval $[0, 1]$ however implies a crisp, linear ordering of these elements, making $[0, 1]$–valued fuzzy set theory inadequate to deal with incomparable information. Attention to other complete lattices L of membership degrees is growing. In [8] a thorough study of L–fuzzy relational images was carried out, and in [13] their use for the representation of linguistic hedges — such as very and more or less — was proposed and investigated. The approach boasts in general a lot of nice properties as well as many practical and intuitive advantages over "traditional" modifiers such as powering [33] and shifting hedges [24].

IF relational calculus is situated between the extremes of the traditional $[0, 1]$–fuzzy relational calculus (or simply called fuzzy relational calculus) on one hand, and the very general notion of L–fuzzy relational calculus on the other. As a consequence it provides a more expressive formalism than traditional fuzzy relational calculus, but at the same time treasures special properties and a specific behavior that is lost when moving onto the more general L–fuzzy relational calculus. This makes IF relational calculus an attractive topic of study. An important issue, differentiating it from the study of L–fuzzy relational images in general, is what we call the "divide and conquer" rationale: taking images of membership and non–membership function separately, ensuring that the resulting construct is still an IFS, thus effectively breaking up our original problem into simpler, better–understood tasks.

However the reader should not get the impression that IFS theory comes down to merely applying ideas from fuzzy set theory twice, once for the membership and once for the non–membership function. Indeed throughout this chapter it will become clear that the "divide and conquer" approach is rather a challenge than a triviality in IFS theory, and sometimes even impossible. However some conditions implied on the logical operators involved can allow for some results in this direction.

This chapter is an extended version of [4] in which the use of IF relational images for the representation of linguistic hedges was introduced. We will recall this in Section 4. In Section 5 we additionally discuss IF rough sets as a second application, drawing upon results from [6]. First however we give the necessary preliminaries on IFSs (Section 2), and we study IF relational images in general (Section 3).

2 Preliminaries

2.1 L–fuzzy sets

In 1967 Goguen formally introduced the notion of an L–fuzzy set with a membership function taking values in a complete lattice L [19]. In this paper we assume that $(L, \leq_L)$ is a complete lattice with smallest element 0_L and greatest element 1_L. An L–fuzzy set A in a universe X is a mapping from X to L, again called the membership function. The L–fuzzy set A in X is said to be included in the L–fuzzy set B in X, usually denoted by $A \subseteq B$, if

$$A(x) \leq_L B(x)$$

for all x in X. An L–fuzzy set R in $X \times X$ is called an L–fuzzy relation on X. For all x and y in X, $R(x, y)$ expresses the degree to which x and y are related through R. For every y in X, the R-foreset of y is a L–fuzzy set in X, denoted as Ry and defined by

$$Ry(x) = R(x, y)$$

for all x in X.

L–fuzzy–set–theoretical operations such as complementation, intersection, and union, can be defined by means of suitable generalizations of the well-known connectives from boolean logic. Negation, conjunction, disjunction and implication can be generalized respectively to negator, triangular norm, triangular conorm and implicator, all mappings taking values in L. More specifically, a negator in L is any decreasing $L \to L$ mapping $\mathcal{N}$ satisfying $\mathcal{N}(0_L) = 1_L$. It is called involutive if $\mathcal{N}(\mathcal{N}(x)) = x$ for all x in L. A triangular norm (t-norm for short) $\mathcal{T}$ in L is any increasing, commutative and associative $L^2 \to L$ mapping satisfying $\mathcal{T}(1_L, x) = x$, for all x in L. A triangular conorm (t-conorm for short) $\mathcal{S}$ in L is any increasing, commutative and associative $L^2 \to L$ mapping satisfying $\mathcal{S}(0_L, x) = x$, for all x in L. The $\mathcal{N}$-complement of an L–fuzzy set A in X as well as the $\mathcal{T}$-intersection and the $\mathcal{S}$-union of L–fuzzy sets A and B in X are the L–fuzzy sets $co_{\mathcal{N}}(A)$, $A \cap_{\mathcal{T}} B$ and $A \cup_{\mathcal{S}} B$ defined by

$$co_{\mathcal{N}}(A)(x) = \mathcal{N}(A(x))$$
$$A \cap_{\mathcal{T}} B(x) = \mathcal{T}(A(x), B(x))$$
$$A \cup_{\mathcal{S}} B(x) = \mathcal{S}(A(x), B(x))$$

for all x in X. The dual of a t-conorm $\mathcal{S}$ in L w.r.t. a negator $\mathcal{N}$ in L is a t-norm $\mathcal{T}$ in L defined as

$$\mathcal{T}(x, y) = \mathcal{N}(\mathcal{S}(\mathcal{N}(x), \mathcal{N}(y)))$$

An implicator in L is any $L^2 \to L$–mapping $\mathcal{I}$ satisfying $\mathcal{I}(0_L, 0_L) = 1_L$, $\mathcal{I}(1_L, x) = x$, for all x in L. Moreover we require $\mathcal{I}$ to be decreasing in its first, and increasing in its second component. If $\mathcal{S}$ and $\mathcal{N}$ are respectively a t-conorm and a negator in L, then it is well-known that the mapping $\mathcal{I}_{\mathcal{S},\mathcal{N}}$ defined by

$$\mathcal{I}_{\mathcal{S},\mathcal{N}}(x,y) = \mathcal{S}(\mathcal{N}(x), y)$$

is an implicator in L, usually called S-implicator (induced by $\mathcal{S}$ and $\mathcal{N}$). Note that if $\mathcal{N}$ is involutive then

$$\mathcal{S}(x,y) = \mathcal{I}_{\mathcal{S},\mathcal{N}}(\mathcal{N}(x), y)$$

for all x and y in L. For ease of notation we also use the concept of S–implicator induced by a t–norm $\mathcal{T}$ and an involutive negator $\mathcal{N}$

$$\mathcal{I}_{\mathcal{T},\mathcal{N}}(x,y) = \mathcal{N}(\mathcal{T}(x, \mathcal{N}(y)))$$

If $\mathcal{S}$ is the dual of $\mathcal{T}$ then $\mathcal{I}_{\mathcal{S},\mathcal{N}} = \mathcal{I}_{\mathcal{T},\mathcal{N}}$. Furthermore, if $\mathcal{T}$ is a t-norm in L, the mapping $\mathcal{I}_{\mathcal{T}}$ defined by

$$\mathcal{I}_{\mathcal{T}}(x,y) = \sup\{\lambda | \lambda \in L \text{ and } \mathcal{T}(x,\lambda) \leq_L y\}$$

is an implicator in L, usually called the residual implicator (of $\mathcal{T}$). The partial mappings of a t-norm $\mathcal{T}$ in L are sup-morphisms if

$$\mathcal{T}\left(\sup_{i \in I} x_i, y\right) = \sup_{i \in I} \mathcal{T}(x_i, y)$$

for every family I of indexes. Every implicator induces a negator in the following way

$$\mathcal{N}(x) = \mathcal{I}(x, 0)$$

for all x in L. The negator induced by an S–implicator $\mathcal{I}_{\mathcal{S},\mathcal{N}}$ coincides with $\mathcal{N}$. The negator induced by the residual implicator $\mathcal{I}_{\mathcal{T}}$ is denoted by $\mathcal{N}_{\mathcal{T}}$.

It is easy to verify that the meet and the join operation on L are respectively a t-norm and a t-conorm in L. We denote them by $\mathcal{T}_{\mathrm{M}}$ and $\mathcal{S}_{\mathrm{M}}$ respectively. Also $A \cap B$ is a shorter notation for $A \cap_{\mathcal{T}_M} B$, while $A \cup B$ corresponds to $A \cup_{\mathcal{S}_M} B$. The $[0,1] \to [0,1]$ mapping N_s defined as

$$N_s(x) = 1 - x$$

for all x in $[0,1]$ is a negator on $[0,1]$, often called the standard negator. For a $[0,1]$-fuzzy set A, $co_{N_s}(A)$ is commonly denoted by $co(A)$. Table 1 depicts the values of well-known t-norms and t-conorms on $[0,1]$, for all x and y in $[0,1]$. The first column

Table 1. Triangular norms and conorms on $[0,1]$

t-norm	t-conorm
$T_{\mathrm{M}}(x,y) = \min(x,y)$	$S_{\mathrm{M}}(x,y) = \max(x,y)$
$T_{\mathrm{P}}(x,y) = x \cdot y$	$S_{\mathrm{P}}(x,y) = x + y - x \cdot y$
$T_{\mathrm{W}}(x,y) = \max(x+y-1, 0)$	$S_{\mathrm{W}}(x,y) = \min(x+y, 1)$

of Table 2 shows the values of the S-implicators in $[0,1]$ induced by the t-conorms of Table 1 and the standard negator N_s, while the second column lists the values of the corresponding residual implicators.

An L–fuzzy relation R on X is called an L–fuzzy $\mathcal{T}$-equivalence relation if for all x, y, and z in X

Table 2. S-implicators and residual implicators on $[0,1]$

S-implicator	residual implicator
$I_{S_{\mathrm{M}},N_s}(x,y)=\max(1-x,y)$	$I_{T_{\mathrm{M}}}(x,y)=\begin{cases}1, & \text{if } x\leq y\\ y, & \text{else}\end{cases}$
$I_{S_{\mathrm{P}},N_s}(x,y)=1-x+x.y$	$I_{T_{\mathrm{P}}}(x,y)=\begin{cases}1, & \text{if } x\leq y\\ \frac{y}{x}, & \text{else}\end{cases}$
$I_{S_{\mathrm{W}},N_s}(x,y)=\min(1-x+y,1)$	$I_{T_{\mathrm{W}}}(x,y)=\min(1-x+y,1)$

(E1) $R(x,x)=1_L$ (reflexivity)
(E2) $R(x,y)=R(y,x)$ (symmetry)
(E3) $\mathcal{T}(R(x,y),R(y,z))\leq R(x,z)$ ($\mathcal{T}$-transitivity)

When $L=\{0,1\}$, L–fuzzy set theory coincides with traditional set theory, in this context also called crisp set theory. $\{0,1\}$-fuzzy sets and $\{0,1\}$-fuzzy relations are usually also called crisp sets and crisp relations. When $L=[0,1]$, fuzzy set theory in the sense of Zadeh is recovered. $[0,1]$-fuzzy sets and $[0,1]$-fuzzy relations are commonly called fuzzy sets and fuzzy relations. Furthermore it is customary to omit the indication "in $[0,1]$" when describing the logical operators, and hence to talk about negators, triangular norms, etc.

2.2 Intuitionistic Fuzzy Sets

IFSs can also be considered as special instances of L–fuzzy sets [16]. Let $(L^*,\leq_{L^*})$ be the complete, bounded lattice defined by:

$$L^*=\{(x_1,x_2)\in[0,1]^2\mid x_1+x_2\leq 1\}$$
$$(x_1,x_2)\leq_{L^*}(y_1,y_2)\Leftrightarrow x_1\leq y_1 \text{ and } x_2\geq y_2$$

The units of this lattice are denoted $0_{L^*}=(0,1)$ and $1_{L^*}=(1,0)$. For each element $x\in L^*$, by x_1 and x_2 we denote its first and second component, respectively. An IFS A in a universe X is a mapping from X to L^*. For every $x\in X$, the value $\mu_A(x)=(A(x))_1$ is called the membership degree of x to A; the value $\nu_A(x)=(A(x))_2$ is called the non–membership degree of x to A; and the value $\pi_A(x)$ is called the hesitation degree of x to A. Just like L^*-fuzzy sets are called IFSs, L^*-fuzzy relations are called IF relations.

By complementing the membership degree with a non-membership degree that expresses to what extent the element does not belong to the IFS, such that the sum of the degrees does not exceed 1, a whole spectrum of knowledge not accessible to fuzzy sets can be accessed. The applications of this simple idea are manyfold indeed: it may be used to express positive as well as negative preferences; in a logical context, with a proposition a degree of truth and one of falsity may be associated; within databases, it can serve to evaluate the satisfaction as well as the violation of relational constraints. More generally, IFSs address the fundamental two-sidedness of knowledge, of positive versus negative information, and by not treating the two sides as exactly complementary (like fuzzy sets do), a margin of hesitation is created. This hesitation is quantified for each x in X by the number

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$$

The terms IF negator, IF t-norm, IF t-conorm and IF implicator are used to denote respectively a negator in L^*, a t-norm in L^*, a t-conorm in L^* and an implicator in L^*. A t-norm $\mathcal{T}$ in L^* (resp. t-conorm $\mathcal{S}$) is called t-representable [16] if there exists a t-norm T and a t-conorm S in $[0,1]$ (resp. a t-conorm S' and a t-norm T' in $[0,1]$) such that, for $x = (x_1, x_2), y = (y_1, y_2) \in L^*$,

$$\begin{aligned} \mathcal{T}(x,y) &= (T(x_1,y_1), S(x_2,y_2)) \\ \mathcal{S}(x,y) &= (S'(x_1,y_1), T'(x_2,y_2)) \end{aligned}$$

T and S (resp. S' and T') are called the representants of $\mathcal{T}$ (resp. $\mathcal{S}$).

Finally, denoting the first projection mapping on L^* by pr_1, we recall from [16] that the $[0,1] - [0,1]$ mapping N defined by

$$N(a) = pr_1\mathcal{N}(a, 1-a)$$

for all a in $[0,1]$ is an involutive negator in $[0,1]$, as soon as $\mathcal{N}$ is an involutive negator in L^*. N is called the negator induced by $\mathcal{N}$. Furthermore

$$\mathcal{N}(x_1, x_2) = (N(1-x_2), 1 - N(x_1))$$

for all x in L^*.

The standard IF negator is defined by

$$\mathcal{N}_s(x) = (x_2, x_1)$$

for all x in L^*. The meet and the join operators on L^* are respectively the IF t-norm $\mathcal{T}_{\mathrm{M}}$ and the IF t-conorm $\mathcal{S}_{\mathrm{M}}$ defined by

$$\mathcal{T}_{\mathrm{M}}(x,y) = (\min(x_1,y_1), \max(x_2,y_2))$$

$$\mathcal{S}_{\mathrm{M}}(x,y) = (\max(x_1,y_1), \min(x_2,y_2))$$

Combining T_{W} and S_{W} of Table 1 gives rise to the t-representable IF t-norm $\mathcal{T}_{\mathrm{W}}$ and IF t-conorm $\mathcal{S}_{\mathrm{W}}$ defined by

$$\mathcal{T}_{\mathrm{W}}(x,y) = (\max(0, x_1 + y_1 - 1), \min(1, x_2 + y_2))$$

$$\mathcal{S}_{\mathrm{W}}(x,y) = (\min(1, x_1 + y_1), \max(0, x_2 + y_2 - 1))$$

However also $\mathcal{T}_{\mathrm{L}}$ and $\mathcal{S}_{\mathrm{L}}$ are possible extensions of T_{W} and S_{W} to IFS theory

$$\mathcal{T}_{\mathrm{L}}(x,y) = (\max(0, x_1 + y_1 - 1), \min(1, x_2 + 1 - y_1, y_2 + 1 - x_1))$$

$$\mathcal{S}_{\mathrm{L}}(x,y) = (\min(1, x_1 + 1 - y_2, y_1 + 1 - x_2), \max(0, x_2 + y_2 - 1))$$

They are however not t-representable [16]. All of these IF t-conorms induce IF S-implicators

$$\mathcal{I}_{\mathcal{S}_{\mathrm{M}},\mathcal{N}_s}(x,y) = (\max(x_2,y_1), \min(x_1,y_2))$$

$$\mathcal{I}_{\mathcal{S}_{\mathrm{W}},\mathcal{N}_s}(x,y) = (\min(1, x_2 + y_1), \max(0, x_1 + y_2 - 1))$$

$$\mathcal{I}_{\mathcal{S}_{\mathrm{L}},\mathcal{N}_s}(x,y) = (\min(1, y_1 + 1 - x_1, x_2 + 1 - y_2), \max(0, y_2 + x_1 - 1))$$

while the IF t-norms have residual IF implicators

$$\mathcal{I}_{\mathcal{T}_M}(x,y) = \begin{cases} 1_{L^*} & \text{if } x_1 \le y_1 \text{ and } x_2 \ge y_2 \\ (1-y_2, y_2) & \text{if } x_1 \le y_1 \text{ and } x_2 < y_2 \\ (y_1, 0) & \text{if } x_1 > y_1 \text{ and } x_2 \ge y_2 \\ (y_1, y_2) & \text{if } x_1 > y_1 \text{ and } x_2 < y_2 \end{cases}$$

$$\mathcal{I}_{\mathcal{T}_W}(x,y) = (\min(1, 1+y_1-x_1, 1+x_2-y_2), \max(0, y_2-x_2))$$

Finally we note that $\mathcal{I}_{\mathcal{T}_L}$ equals $\mathcal{I}_{\mathcal{S}_L,\mathcal{N}_s}$.

3 IF Relational Images

Next to the composition of relations, the direct image of a set under a relation is a basic operation in traditional relational calculus. Let R be a relation from X to X and A a subset of X, then the direct image of A under R is defined by

$$R{\uparrow}A = \{y | y \in X \text{ and } (\exists x \in X)(x \in A \text{ and } (x,y) \in R)\} \tag{1}$$

The direct image of A contains all elements of X that are related to at least one element of A. Furthermore the superdirect image of A under R

$$R{\downarrow}A = \{y | y \in X \text{ and } (\forall x \in X)((x,y) \in R \Rightarrow x \in A)\} \tag{2}$$

contains all elements of X that are related only to elements of A. These images can be generalized to the L–fuzzy relational case ([8, 21]). Since in our quest for a "divide and conquer" approach we attempt to express IF relational images as constructs of fuzzy relational images for membership and non–membership functions, we recall the most general definition on the level of L–fuzzy relational calculus.

Definition 1 (L–fuzzy relational images). *Let $\mathcal{T}$ and $\mathcal{I}$ be a t–norm and an implicator in L. Let A be an L–fuzzy set in X and R an L–fuzzy relation on X. The direct and superdirect image of A under R are the L–fuzzy sets in X respectively defined by*

$$R{\uparrow}_{\mathcal{T}}A(y) = \sup_{x \in X} \mathcal{T}(A(x), R(x,y))$$
$$R{\downarrow}_{\mathcal{I}}A(y) = \inf_{x \in X} \mathcal{I}(R(x,y), A(x))$$

for all y in X.

$R{\uparrow}_{\mathcal{T}}A(y)$ is the height of the $\mathcal{T}$–intersection of A and Ry, i.e. the degree to which A and Ry overlap. $R{\downarrow}_{\mathcal{I}}A(y)$ corresponds to a well–known measure of inclusion of Ry in A. The main aim in this section is to provide properties of these L–fuzzy relational images in general and of IF relational images in particular. In the following sections we will go into the semantics of the concepts of IF relational direct and superdirect images and their properties in the context of modelling linguistic hedges and IF rough sets.

Proposition 1. *[12] Let $\mathcal{T}$ and $\mathcal{I}$ be a t–norm and an implicator in L. Let R be an L–fuzzy relation on X. If R is reflexive then for every L–fuzzy set A in X*

$$R{\downarrow}_{\mathcal{I}}A \subseteq A \subseteq R{\uparrow}_{\mathcal{T}}A$$

Proposition 2. *[12] Let $\mathcal{T}$ and $\mathcal{I}$ be a t–norm and an implicator in L. Let R be an L–fuzzy relation on X. Let A and B be L–fuzzy sets in X. If $A \subseteq B$ then*

$$R{\downarrow_{\mathcal{I}}}A \subseteq R{\downarrow_{\mathcal{I}}}B$$

$$R{\uparrow_{\mathcal{T}}}A \subseteq R{\uparrow_{\mathcal{T}}}B$$

Proposition 3. *[12] Let $\mathcal{T}$ and $\mathcal{I}$ be a t–norm and an implicator in L. Let A be an L–fuzzy set in X. Let R_1 and R_2 be L–fuzzy relations on X. If $R_1 \subseteq R_2$ then*

$$R_1{\downarrow_{\mathcal{I}}}A \supseteq R_2{\downarrow_{\mathcal{I}}}A$$

$$R_1{\uparrow_{\mathcal{T}}}A \subseteq R_2{\uparrow_{\mathcal{T}}}A$$

Proposition 4. *[12] Let $\mathcal{T}$ and $\mathcal{I}_{\mathcal{T}}$ be a t–norm and its residual implicator in L. Let A be an L–fuzzy set in X and R a $\mathcal{T}$–transitive L–fuzzy relation on X. If the partial mappings of $\mathcal{T}$ are sup–morphisms then the following are equivalent*

(1) $A = R{\downarrow_{\mathcal{I}_{\mathcal{T}}}}A$
(2) $A = R{\uparrow_{\mathcal{T}}}A$

Proposition 5. *[12] Let $\mathcal{T}$ and $\mathcal{I}_{\mathcal{T}}$ be a t–norm and its residual implicator in L. Let A be an L–fuzzy set in X and R a $\mathcal{T}$–transitive L–fuzzy relation in X. If the partial mappings of $\mathcal{T}$ are sup–morphisms then*

$$R{\uparrow_{\mathcal{T}}}(R{\uparrow_{\mathcal{T}}}A) = R{\uparrow_{\mathcal{T}}}A$$

$$R{\downarrow_{\mathcal{I}_{\mathcal{T}}}}(R{\downarrow_{\mathcal{I}_{\mathcal{T}}}}A) = R{\downarrow_{\mathcal{I}_{\mathcal{T}}}}A$$

Corollary 1. *Let $\mathcal{T}$ and $\mathcal{I}_{\mathcal{T}}$ be a t–norm and its residual implicator on L. Let A be an L–fuzzy set in X and R a $\mathcal{T}$–transitive L–fuzzy relation on X. If the partial mappings of $\mathcal{T}$ are sup–morphisms then*

$$R{\uparrow_{\mathcal{T}}}(R{\downarrow_{\mathcal{I}_{\mathcal{T}}}}A) = R{\downarrow_{\mathcal{I}_{\mathcal{T}}}}A$$

$$R{\downarrow_{\mathcal{I}_{\mathcal{T}}}}(R{\uparrow_{\mathcal{T}}}A) = R{\uparrow_{\mathcal{T}}}A$$

We recall some results concerning the interaction of L–fuzzy relational images with union, intersection, and complementation of L–fuzzy sets.

Proposition 6. *[12] Let $\mathcal{T}$ and $\mathcal{I}_{\mathcal{T}}$ be a t–norm and its residual implicator in L. Let A and B be L–fuzzy sets in X and R an L–fuzzy relation on X. If the partial mappings of $\mathcal{T}$ are sup–morphisms then*

$$R{\uparrow_{\mathcal{T}}}(A \cup B) = R{\uparrow_{\mathcal{T}}}A \cup R{\uparrow_{\mathcal{T}}}B$$

$$R{\downarrow_{\mathcal{I}_{\mathcal{T}}}}(A \cap B) = R{\downarrow_{\mathcal{I}_{\mathcal{T}}}}A \cap R{\downarrow_{\mathcal{I}_{\mathcal{T}}}}B$$

Proposition 7. *[12] Let $\mathcal{T}$ and $\mathcal{I}_{\mathcal{T}}$ be a t–norm and its residual implicator in L. Let R be an L–fuzzy relation on X. For every L–fuzzy set A in X*

$$co_{\mathcal{N}_{\mathcal{T}}}(R \uparrow_{\mathcal{T}} A) \subseteq R{\downarrow_{\mathcal{I}_{\mathcal{T}}}}(co_{\mathcal{N}_{\mathcal{T}}}A) \tag{3}$$

If the partial mappings of $\mathcal{T}$ are sup–morphisms then

$$R{\uparrow_{\mathcal{T}}}(co_{\mathcal{N}_{\mathcal{T}}}A) \subseteq co_{\mathcal{N}_{\mathcal{T}}}(R{\downarrow_{\mathcal{I}_{\mathcal{T}}}}A) \tag{4}$$

If $\mathcal{N}_{\mathcal{T}}$ is involutive then the left and right hand sides in (3) and (4) are equal.

Proposition 8. *[12] Let $\mathcal{T}$ and $\mathcal{N}$ be a t–norm and an involutive negator in L, and let $\mathcal{I}_{\mathcal{T},\mathcal{N}}$ be the corresponding S–implicator. Let R be an L–fuzzy relation on X. For every L–fuzzy set A in X*

$$co_{\mathcal{N}_\mathcal{T}}(R\uparrow_\mathcal{T} A) \subseteq R{\downarrow}_{\mathcal{I}_{\mathcal{T},\mathcal{N}}}(co_\mathcal{N} A)$$

$$R{\uparrow}_\mathcal{T}(co_\mathcal{N} A) \subseteq co_\mathcal{N}(R{\downarrow}_{\mathcal{I}_{\mathcal{T},\mathcal{N}}} A)$$

An IFS A is characterized by means of a membership function μ_A and a non-membership function ν_A. A natural question which arises is whether the direct image and the superdirect image of A could be defined in terms of the direct and the superdirect image of μ_A and ν_A, (all under the proper L–fuzzy relations of course). Generally such a "divide and conquer" approach is everything but trivial in IFS theory, and sometimes even impossible. However some conditions implied on the logical operators involved can allow for some results in this direction. Particularly attractive are the t-representable t-norms and t-conorms, and the S-implicators that can be associated with them.

Proposition 9. *Let $\mathcal{T}$ be a t-representable IF t–norm such that $\mathcal{T} = (T, S)$, let N be an involutive negator in $[0,1]$, and let $I_{S,N}$ be the S-implicator in $[0,1]$ induced by S and N. Let R be an IF relation on X. For every IFS A in X*

$$R{\uparrow}_\mathcal{T} A = (\mu_R{\uparrow}_T\mu_A, (co_N(\nu_R)){\downarrow}_{I_{S,N}}\nu_A) \tag{5}$$

Proof. For all y in X we obtain successively

$$\begin{aligned}
R{\uparrow}_\mathcal{T} A(y) &= \sup_{x\in X} \mathcal{T}(R(x,y), A(x)) \\
&= \sup_{x\in X} (T(\mu_R(x,y), \mu_A(x)), S(\nu_R(x,y), \nu_A(x))) \\
&= \left(\sup_{x\in X} T(\mu_R(x,y), \mu_A(x)), \inf_{x\in X} I_{S,N}(N(\nu_R(x,y)), \nu_A(x))\right) \\
&= \Big((\mu_R{\uparrow}_T\mu_A)(y), (co_N(\nu_R){\downarrow}_{I_{S,N}}\nu_A)(y)\Big)
\end{aligned}$$

□

Proposition 10. *Let $\mathcal{S}$ be a t-representable IF t-conorm such that $\mathcal{S} = (S, T)$, let $\mathcal{N}$ be an involutive IF negator, let $\mathcal{I}_{\mathcal{S},\mathcal{N}}$ be the IF S–implicator induced by $\mathcal{S}$ and $\mathcal{N}$, let N be the negator in $[0,1]$ induced by $\mathcal{N}$ and let $I_{S,N}$ be the S-implicator induced by S and N. Let R be an IF relation on X. For every IFS A in X*

$$R{\downarrow}_{\mathcal{I}_{\mathcal{S},\mathcal{N}}} A = ((co\ \nu_R){\downarrow}_{I_{S,N}}\mu_A, co(co_N\mu_R){\uparrow}_\mathcal{T}\nu_A) \tag{6}$$

Proof. For all y in X we obtain successively

$$
\begin{aligned}
R\downarrow_{\mathcal{I}_{S,\mathcal{N}}} A(y) &= \inf_{x\in X} \mathcal{I}_{S,\mathcal{N}}(R(x,y), A(x)) \\
&= \inf_{x\in X} \mathcal{S}(\mathcal{N}(R(x,y)), A(x)) \\
&= \Big(\inf_{x\in X} S(N(1-\nu_R(x,y)), \mu_A(x)), \\
&\quad \sup_{x\in X} T(1 - N(\mu_R(x,y)), \nu_A(x)) \Big) \\
&= \Big(\inf_{x\in X} I_{S,N}(co(\nu_R)(x,y), \mu_A(x)), \\
&\quad \sup_{x\in X} T(co(co_N(\mu_R))(x,y)), \nu_A(x)) \Big) \\
&= \Big((co(\nu_R)\downarrow_{I_{S,N}} \mu_A)(y), (co(co_N(\mu_R))\uparrow_T \nu_A)(y) \Big)
\end{aligned}
$$

□

Observe that in both (5) and (6) on the "fuzzy level" the images are taken under the membership function μ_R, or something semantically very much related, such as the N-complement of the non-membership function ν_R or once even the standard complement of the N-complement of μ_R. Presented in this way, the resulting formulas look quite complicated. For better understanding, let N be the standard negator, and let the IF relation R be a fuzzy relation, i.e. $\mu_R = co(\nu_R)$, then formulas (5) and (6) reduce to

$$R\uparrow_{\mathcal{T}} A = (\mu_R \uparrow_T \mu_A, \mu_R \downarrow_{I_{S,N}} \nu_A)$$

$$R\downarrow_{\mathcal{I}_{S,\mathcal{N}}} A = (\mu_R \downarrow_{I_{S,N}} \mu_A, \mu_R \uparrow_T \nu_A)$$

Apparently in this case the membership function of the direct image of A is the direct image of the membership function of A, while the non-membership function of the direct image of A is the superdirect image of the non-membership function of A. For the superdirect image of A the dual proposition holds.

Finally let us recall a proposition that helps to construct non trivial $\mathcal{T}$–transitive IF relations (i.e. μ_R not necessarily equal to $co(\nu_R)$).

Proposition 11. *[6] Let $\mathcal{T}$ be a t-representable IF t-norm such that $\mathcal{T} = (T, S)$ and such that $S(x,y) = 1 - T(1-x, 1-y)$, and let R_1, R_2 be two fuzzy T–equivalence relations such that $R_1(x,y) \leq R_2(x,y)$, for all x and y in X. Then R defined by, for x and y in X,*

$$R(x,y) = (R_1(x,y), 1 - R_2(x,y))$$

is an IF $\mathcal{T}$–equivalence relation.

Example 1. Let $X = [0, 100]$, and let the fuzzy T_W-equivalence relation E_c on X be defined by

$$E_c(x,y) = \max\left(1 - \frac{|\, x - y \,|}{c}, 0\right)$$

for all x and y in X, and with real parameter $c > 0$. Obviously, if $c_1 \leq c_2$ then $E_{c_1}(x,y) \leq E_{c_2}(x,y)$. By proposition 11, $(E_{c_1}, co(E_{c_2}))$ is an IF $\mathcal{T}_W$-equivalence relation.

4 Representation of Linguistic Hedges

Since its introduction in the 1960's, fuzzy set theory [34] has rapidly acquired an immense popularity as a formalism for the representation of vague linguistic information. Over the years many researchers have studied the automatic computation of membership functions for modified linguistic terms (such as very cool) from those of atomic ones (such as cool). Whether we are working with fuzzy sets, IFSs or L–fuzzy sets in general, establishing a concrete mathematical model for a given linguistic expression is typically one of the most difficult tasks when developing an application. Therefore it is very useful to have standard representations of linguistic modifiers such as very and more or less at hand, since they allow for the automatic construction of representations for the modified terms from representations of the original terms.

The first proposal in this direction was made by Zadeh [33] who suggested to transform the membership function of a fuzzy set A in X into membership functions for very A and more or less A in the following way

$$\text{very } A(y) = A(y)^2$$
$$\text{more or less } A(y) = A(y)^{\frac{1}{2}}$$

for all y in X. One can easily verify that the following natural condition, called semantical entailment, [24] is respected:

$$\text{very } A \subseteq A \subseteq \text{more or less } A$$

These representations have the significant shortcoming of keeping the kernel and the support, which are defined as

$$\ker A = \{y | y \in X \wedge A(y) = 1\}$$
$$\text{supp } A = \{y | y \in X \wedge A(y) > 0\}$$

As a consequence they do not make any distinction between e.g. being old to degree 1 and being very old to degree 1, while intuition might dictate to call a woman of 85 old to degree 1 but very old only to a lower degree. Many representations developed in the same period are afflicted with these and other disadvantages on the level of intuition as well as on the level of applicability (we refer to [22] for an overview), in our opinion due to the fact that these operators are only technical tools, lacking inherent meaning. In fact it wasn't until the second half of the 1990's that new models with a clear semantics started to surface, such as the horizon approach [27] and the context (or fuzzy relational) based approach [10, 15]. The latter can be elegantly generalized to L–fuzzy sets [13] which accounts for its strength.

A characteristic of the "traditional" approaches is that they do not really look at the context: when computing the degree to which y is very A, Zadeh's representation for instance only looks at the degree to which y is A. It completely ignores all the other objects of the universe and their degree of belonging to A. In the context based approach the objects in the context of y are taken into account as well. This context is defined as the set of objects that are related to y by some relation R that models approximate equality. Specifically it is the R-foreset of y.

One could say that somebody is more or less adult "if he resembles an adult". Likewise a park is more or less large "if it resembles a large park". In general: v is more or less A if y resembles an x that is A. Hence y is more or less A if the intersection of A and Ry is not empty. Or to state it more fuzzy–set–theoretically: y is more or less A to the degree to which Ry and A overlap, i.e.

$$\text{more or less } A(y) = R \uparrow_{\mathcal{T}} A(y)$$

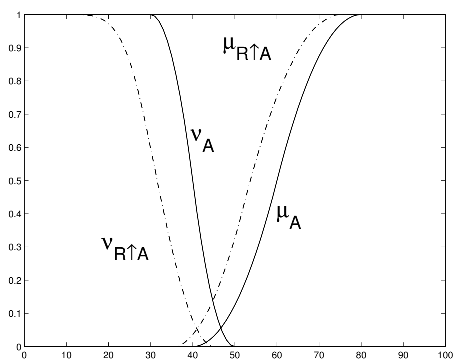
For the representation of very an analogous scheme can be used. Indeed: if all men resembling Alberik in height are tall, then Alberik must be very tall. Likewise Krista is very kind "if everyone resembling Krista is kind". In general: y is very A if all x resembling y are A. Hence y is very A if Ry is included in A. To state it more fuzzy–set–theoretically: y is very A to the degree to which Ry is included in A, i.e.

$$\text{very } A(y) = R \downarrow_{\mathcal{I}} A(y)$$

Under the natural assumption that R is reflexive (every object is approximately equal to itself to the highest degree), semantical entailment holds (Proposition 1). As mentioned in the introduction, since IFSs are also L–fuzzy sets, a representation for more or less and very is readily obtained.

Example 2. Figure 1 depicts the membership function μ_A and non-membership function ν_A of an IFS A in $\mathbb{R}$. A is modified by taking the direct image by means of

Fig. 1. Membership and non-membership functions of A and $R{\uparrow}A$



$\mathcal{T}_W$ under an IF relation R with a membership function based on the general shape $\mathbb{S}$-membership function

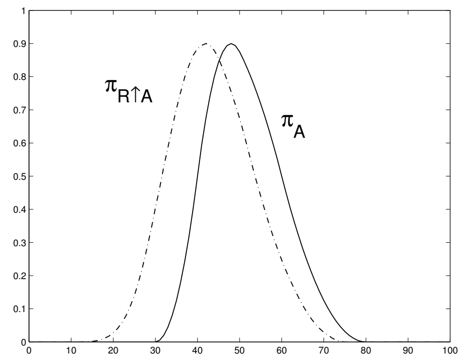
$$\mathbb{S}(x;\alpha,\gamma) = \begin{cases} 0, & x \leq \alpha \\ \frac{2(x-\alpha)^2}{(\gamma-\alpha)^2}, & \alpha \leq x \leq (\alpha+\gamma)/2 \\ 1 - \frac{2(x-\gamma)^2}{(\gamma-\alpha)^2}, & (\alpha+\gamma)/2 \leq x \leq \gamma \\ 1, & \gamma \leq x \end{cases}$$

for x, α and γ in $\mathbb{R}$ and $\alpha < \gamma$. Specifically R is defined as

$$\mu_R(x,y) = \begin{cases} \mathbb{S}(x; y-20, y-5) & \text{if } x \leq y-5 \\ 1 & \text{if } y-5 < x < y+5 \\ 1 - \mathbb{S}(x; y+5, y+20) & \text{if } y+5 \leq x \end{cases}$$

and $\nu_R(x,y) = 1 - \mu_R(x,y)$, for all x and y in $\mathbb{R}$. This results in the membership and the non-membership function for the modified IFS $R{\uparrow}A$ also depicted in Figure 1. As Figure 2 illustrates, the modification does not preserve the local hesitation: depending on its context, the hesitation degree of y in A increases, decreases or remains unaltered when passing to $R{\uparrow}A$. On the global level however, the overall

Fig. 2. Hesitation



hesitation seems to be invariant, but this is not in general the case. Let us assume that the conditions of Proposition 9 are fulfilled. Under the natural assumption that R is reflexive, we have

$$(co_{\mathcal{N}}(\nu_R)) \downarrow_{\mathcal{I}} \nu_A \subseteq \nu_A$$

and

$$\mu_A \subseteq \mu_R {\uparrow}_{\mathcal{T}} \mu_A$$

If ν_A is the constant $[0,1] \to \{0\}$ mapping, modification of the non–membership function will have no effect. Any change in the membership function will therefore give rise to a decrease of the overall hesitation. Note that this seems natural: the hesitation to call objects A might be greater than the hesitation to call them more or less A.

As far as the authors are aware, the only other existing approach to the modification of linguistic terms modeled by IFSs is due to De, Biswas and Roy [7]. They proposed an extension of Zadeh's representation; it is based on the so–called product $A \cap_{\mathcal{T}_\mathrm{P}} B$ of IFSs A and B. One can easily verify that

$$A^2(u) = (\mu_A(u)^2, 1 - (1 - \nu_A(u))^2) \tag{7}$$

in which A^2 is used as a shorthand notation for $A \cap_{\mathcal{T}_\mathrm{P}} A$. Furthermore for $A^{\frac{1}{2}}$ defined in a similar manner as

$$A^{\frac{1}{2}}(u) = (\mu_A(u)^{\frac{1}{2}}, 1 - (1 - \nu_A(u))^{\frac{1}{2}}) \tag{8}$$

one can verify that $A^{\frac{1}{2}} \cap_{\mathcal{T}_\mathrm{P}} A^{\frac{1}{2}} = A$ which justifies the notation. Entirely in the line of Zadeh's work, in [7] the authors propose to use $A^{\frac{1}{2}}$ and A^2 for the representation of more or less and very respectively. As a consequence, the drawbacks listed in Section 4 are also inherited, making the approach less interesting from the semantical point of view.

Nevertheless Equations (7) and (8) reveal some interesting semantical clues. Indeed, these formulas actually suggest to model very A by

$$(\text{very } \mu_A, \text{ not (very not } \nu_A))$$

and more or less A by

$$(\text{more or less } \mu_A, \text{ not (more or less not } \nu_A))$$

As such it is an example of what we have called the divide–and–conquer approach. The resulting expressions for the non–membership functions are clearly more complicated than those for the membership functions; they stem from the observation that the complement of the non–membership function can be interpreted loosely as a kind of second membership function.

As Proposition 9 indicates, taking the IF direct image ("more or less") involves both a fuzzy direct image ("more or less") and a fuzzy superdirect image ("very"). A dual observation can be made for Proposition 10. Interestingly enough, De, Biswas and Roy [7] do not use both hedges at the same time, but their approach involves negation of the non-membership function. Possible connections between intensifying hedges (like very) and weakening hedges (such as more or less) by means of negation have already intrigued several researchers. In [2] the meaning of "not overly bright" is described as "rather underly bright" which gives rise (albeit simplified) to the demand for equality of the mathematical representations for not very A and more or less not A. Propositions 7, 8, 9, and 10 show that under certain conditions on the connectives involved, the IF relational model indeed behaves in this way.

5 IF Rough Sets

Pawlak [28] launched rough set theory as a framework for the construction of approximations of concepts when only incomplete information is available. Since its introduction it has become a popular tool for knowledge discovery (see e.g. [23] for a recent overview of the theory and its applications). As a new trend in the attempts to combine the best of several worlds, very recently all kinds of suggestions for approaches merging rough set theory and IFS theory start to pop up [3, 20, 30, 31]. In the literature there exist many views on the notion "rough set" which can be grouped into two main streams. Several suggested options for fuzzification have led to an even greater number of views on the notion "fuzzy rough set". Typically, under the same formal umbrella, they can be further generalized to the notion "L–fuzzy rough set". Needless to say that, when trying to compare and/or to combine rough set theory, fuzzy set theory and IFS theory, one finds oneself at a complicated crossroads with an abundance of possible ways to proceed. The proposals referred to above all suffer from various drawbacks making them less eligible for applications. In [6] we made the definition of fuzzy rough set by Radzikowska and Kerre [29] — which exists already in a more specific form for more than a decade — undergo the natural transformation process towards intuitionistic fuzzy rough set theory (IFRS theory for short), which lead to a mathematically elegant and semantically interpretable concept.

Definition 2 (IF rough set). *Let $\mathcal{T}$ and $\mathcal{I}$ be an IF t–norm and an IF implicator respectively. Let R be an IF $\mathcal{T}$–equivalence relation on X. We say that a couple of IFSs (A_1, A_2) is called an intuitionistic fuzzy rough set (IFRS) in the approximation space $(X, R, \mathcal{T}, \mathcal{I})$ if there exists an IFS A such that $R{\downarrow}_{\mathcal{I}}A = A_1$ and $R{\uparrow}_{\mathcal{T}}A = A_2$. $R{\downarrow}_{\mathcal{I}}A$ and $R{\uparrow}_{\mathcal{T}}A$ are called the lower and upper approximation of A respectively.*

Proposition 1 ensures that the lower approximation of A is included in A, while A is included in its upper approximation. Propositions 2 and 3 describe how the lower and upper approximations behave w.r.t. a refinement of the IFS to be approximated, or a refinement of the IF relation that defines the approximation space.

Definition 3. *[6] A is called definable in $(X, R, \mathcal{T}, \mathcal{I})$ iff $R{\downarrow}_{\mathcal{I}}A = R{\uparrow}_{\mathcal{T}}A$*

In classical rough set theory, a set is definable if and only if it is a union of equivalence classes. This property no longer holds in IFRS theory. However, if we imply sufficient conditions on the IF t-norm $\mathcal{T}$ and the IF implicator $\mathcal{I}$ defining the approximation space, we can still establish a weakened theorem, relying on Propositions 4 and 5.

Theorem 1. *[6] Let $\mathcal{T}$ and $\mathcal{I}_{\mathcal{T}}$ be an IF t-norm and its residual implicator. If the partial mappings of $\mathcal{T}$ are sup-morphisms then any union of R-foresets is definable, i.e.*

$$(\exists B)\left(B \subseteq X \text{ and } A = \bigcup_{z \in B} Rz\right) \text{ implies } R{\downarrow}_{\mathcal{I}_{\mathcal{T}}}A = R{\uparrow}_{\mathcal{T}}A$$

Under the same conditions implied on $\mathcal{T}$ and $\mathcal{I}$ as in Theorem 1, the $\mathcal{S}_{\mathrm{M}}$-union and $\mathcal{T}_{\mathrm{M}}$-intersection of two definable IFSs is definable. This is a corollary of Proposition 6. Finally, the following examples illustrate the concept of an IFRS computed in approximation spaces involving t–representable as well as non t–representable IF t–norms.

Example 3. Figure 3 shows the membership function μ_A and the non–membership function ν_A of the IFS A in the universe $X = [0, 100]$. Using the non-t-representable IF t-norm $\mathcal{T}_{\mathrm{L}}$, its residual IF implicator $\mathcal{I}_{\mathcal{T}_{\mathrm{L}}}$ and the IF relation R defined by

$$R(x, y) = (E_{40}(x, y), 1 - E_{40}(x, y))$$

for all x and y in $[0, 100]$ (see Example 1 for the definition of E_{40}) we computed the lower approximation of A ($= A_1$) as well as the upper approximation of A ($= A_2$). They are both depicted as well in Figure 3.

Example 4. Figure 4 shows the same IFS A we used in Example 3. However to compute its lower approximation A_1 and its upper approximation A_2, this time we used the t-representable IF t-norm $\mathcal{T}_{\mathrm{W}}$, its residual IF implicator and the IF relation R defined by

$$R(x, y) = (E_{30}(x, y), E_{50}(x, y))$$

for all x and y in $[0, 100]$.

Fig. 3. A (solid lines); upper approximation of A (dashed lines); lower approximation of A (dotted lines)

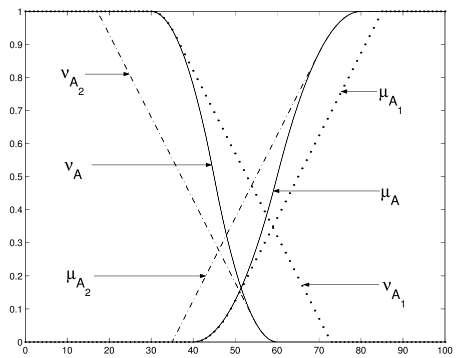
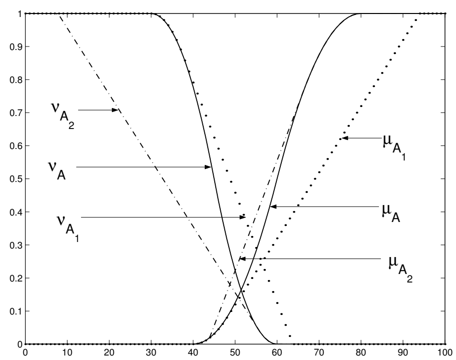


Fig. 4. dashed line: upper approximation of A; dotted line: lower approximation of A



6 Conclusion

Partly due to existing studies on connectives in the lattice L^*, the direct and superdirect relational images are readily obtained in an intuitionistic fuzzy setting. Under certain conditions on the connectives used in the formulas of direct and superdirect image, a meaningful representation for the image of the whole in terms of that of its constituting parts is established. This is not only interesting from the computational point of view, but, using the images to represent linguistic hedges, also helps us to gain more insight in the semantics of the linguistic modification process. Furthermore the IF relational images lead to a mathematically elegant concept of an IF rough set, where they are used to construct upper and lower approximations.

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