Available at WWW.MATHEMATICSWEB.ORG
POWERED BY SCIENCE @ DIRECT•

ELSEVIER

Fuzzy Sets and Systems 133 (2003) 137–153

FUZZY sets and systems

www.elsevier.com/locate/fss

On (un)suitable fuzzy relations to model approximate equality

Martine De Cock *, Etienne Kerre

Department of Applied Mathematics and Computer Science, Ghent University, Krijgslaan 291 (S9), B-9000 Gent, Belgium

Received 24 September 1999; received in revised form 12 July 2001; accepted 18 February 2002

Abstract

In this paper we state that fuzzy equivalence relations in general are not suitable to model approximate equality, since then the notion of transitivity is counter-intuitive. To substantiate this we investigate some of the undesirable results caused by transitivity, among other things in the case of approximate reasoning. We then introduce a new framework to model approximate equality, i.e. the concept of a pseudometric based resemblance relation. We go into the properties of this new kind of fuzzy relation and illustrate it by means of some examples. © 2002 Elsevier Science B.V. All rights reserved.

Keywords: Fuzzy relations; Fuzzy equivalence relations; Resemblance relations; Approximate reasoning

1. Introduction

An *equivalence relation*, i.e. a reflexive, symmetric and transitive relation, can be considered as a basic concept of mathematics. The most popular example of an equivalence relation is the *crisp equality* (see Fig. 1, $\stackrel{1}{\longleftarrow}$). The fuzzy counterpart of crisp equality is *approximate equality* (Fig. 1, $\downarrow_3$), while the concept of a crisp equivalence relation can be fuzzified to that of a *fuzzy equivalence relation* (Fig. 1, ${}_2\!\downarrow$). Unlike in the crisp case, however, where crisp equality is intuitively a crisp equivalence relation, we will show that in the fuzzy case approximate equality is intuitively not always a fuzzy equivalence relation. Taking into account the strong intuitive connection between *approximate equality* and *distance*, in this paper we replace the "problematic" condition of transitivity by a pseudometric based condition. Thus we come to the notion of *resemblance relation*, which is a suitable framework to model approximate equality.

After recalling some basic concepts from fuzzy set theory (Section 2), we go into paradoxical results that appear when representing approximate equality by fuzzy equivalence relations, among other things in the case of approximate reasoning (Section 3). Next, a study of the links between

* Corresponding author. Tel.: +32-9-264-4772; fax: +32-9-264-4995.
E-mail addresses: martine.decock@rug.ac.be (M. De Cock), etienne.kerre@rug.ac.be (E. Kerre).

PII: S0165-0114(02)00239-7

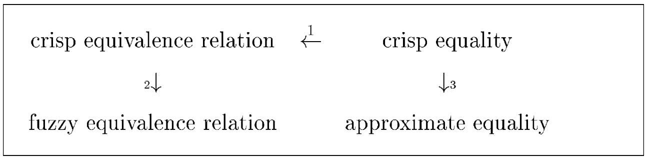

Fig. 1. Crisp relations and their fuzzy/approximate counterpart

distance functions and fuzzy equivalence relations will also indicate their counter-intuitive shortcomings for modelling approximate equality (Section 4). To overcome them, we define the concept of pseudometric based resemblance relation. We state and prove some of its properties and we conclude with some examples of resemblance relations in different universa (Section 5).

2. Basic concepts

Throughout this paper we will use the following triangular norms: the minimum operator M, the algebraic product P, the Lukasiewicz t-norm W and the drastic product D defined by (for x and y in $[0,1]$): $M(x,y) = \min(x,y)$, $P(x,y) = x.y$, $W(x,y) = \max(0, x+y-1)$ and $D(x,y) = \min(x,y)$ if $\max(x,y)=1$, $D(x,y)=0$ otherwise. These t-norms are ranked as: $D \subseteq W \subseteq P \subseteq M$. Furthermore we will use the following triangular conorms: the maximum operator M^*, the probabilistic sum P^*, the bounded sum W^* and the drastic sum D^* defined by (for x and y in $[0,1]$): $M^*(x,y) = \max(x,y)$, $P^*(x,y) = x + y - x.y$, $W^*(x,y) = \min(1, x+y)$ and $D^*(x,y) = \max(x,y)$ if $\min(x,y)=0$, $D^*(x,y)=1$ otherwise. These t-conorms are ranked as: $M^* \subseteq P^* \subseteq W^* \subseteq D^*$. We recall that a negation $\mathcal{N}$ is called involutive iff $\mathcal{N}(\mathcal{N}(x)) = x$, for all x in $[0,1]$. We will use the involutive standard negation $\mathcal{N}_s$ defined by $\mathcal{N}_s(x) = 1 - x$, for all x in $[0,1]$. The dual w.r.t. a negation $\mathcal{N}$ of a $([0,1]^2 - [0,1])$-mapping f is the $([0,1]^2 - [0,1])$-mapping $f^{\leftrightarrow \mathcal{N}}$ defined by $f^{\leftrightarrow \mathcal{N}}(x,y) = \mathcal{N}^{-1}(f(\mathcal{N}(x), \mathcal{N}(y)))$, for all x and y in $[0,1]$. If $\mathcal{N}$ is an involutive negation and $\mathcal{T}$ is a t-norm then $\mathcal{T}^{\leftrightarrow \mathcal{N}}$ is a t-conorm. Examples are $M^{\leftrightarrow \mathcal{N}_s} = M^*$, $P^{\leftrightarrow \mathcal{N}_s} = P^*$, $W^{\leftrightarrow \mathcal{N}_s} = W^*$, $D^{\leftrightarrow \mathcal{N}_s} = D^*$.

Furthermore throughout this paper, let X denote a universe and $\mathcal{F}(X)$ the class of fuzzy sets on X. For R a fuzzy relation on X (i.e. $R \in \mathcal{F}(X \times X)$) and for x in X, the R-afterset of x is a fuzzy set on X denoted by xR and defined by

$$(xR)(y) = R(x,y)$$

for all y in X [1]. For $\mathcal{T}$ a triangular norm and R and G fuzzy relations on X, the $\mathcal{T}$-composition of R and G is a fuzzy relation on X , denoted by $R \circ_{\mathcal{T}} G$ and defined as usual by

$$(R \circ_{\mathcal{T}} G)(x,z) = \sup_{y \in X} \mathcal{T}(R(x,y), G(y,z)).$$

Definition 1 (Fuzzy $\mathcal{T}$-equivalence relation, fuzzy $\mathcal{T}$-equality). Let $\mathcal{T}$ be a triangular norm. A fuzzy relation E on X is called a fuzzy $\mathcal{T}$-equivalence relation on X iff for all x, y and z in X:

(FE.1) $E(x,x) = 1$ (reflexivity)
(FE.2) $E(x,y) = E(y,x)$ (symmetry)
(FE.3) $\mathcal{T}(E(x,y), E(y,z)) \leqslant E(x,z)$ ($\mathcal{T}$-transitivity).

If also $E(x,y) = 1$ iff $x = y$ (separation), then E is called a fuzzy $\mathcal{T}$-equality.

The definition of fuzzy $\mathcal{T}$-equivalence relation is very popular and can be found e.g. in [2,3,7] (under the name "equality relation") and [11] (under the name "indistinguishability operator"). Note that for $\mathcal{T}=M$ this definition reduces to Zadeh's similarity relation and for $\mathcal{T}=W$ to Zadeh's likeness relation (see [13,6]).

3. Problems and paradoxes

Similarities, approximate equalities play a very important part in human reasoning. Even natural languages are based on them: we use similarities to construct concepts such as "game" [12]:

> Consider for example the proceedings that we call 'games'. [..] For if you look at them you will not see something that is common to *all*, but similarities, relationships, and a whole series of them at that. [..] Look for example at board-games with their multifarious relationships. Now pass to card-games; here you find many correspondences with the first group, but many common features drop out, and others appear. When we pass next to ball-games, much that is common is retained, but much is lost. Are they all 'amusing'? Compare chess with noughts and crosses. Or is there always winning and losing, or competition between players? Think of patience. [..] Look at the parts played by skill and luck; and at the difference between skill in chess and skill in tennis. [..]
>
> And the result of this examination is: we see a complicated network of similarities overlapping and criss-crossing: sometimes overall similarities, sometimes similarities of detail. I can think of no better expression to characterize these similarities than 'family resemblances'; for the various resemblances between members of a family: build, features, colour of eyes, gait, temperament, etc. overlap and criss-cross in the same way.

Approximate equality is a truly vague concept: some objects are definitely approximately equal, others are not, but in between is a group of objects for which it is hard to tell whether they are or are not approximately equal. In fact: they can be considered to be approximately equal *to some extent*.

Furthermore approximate equality is not transitive. Referring to the example of Wittgenstein above, a girl may be similar to her mother, the mother may be similar to the grandmother, but it is still possible that the girl and the grandmother have nothing in common at all! This is an example of the so-called Poincaré paradox [9,10], which is usually symbolized as

$$A = B, \quad B = C, \quad A \neq C, \tag{1}$$

in which the equality sign should be interpreted as indistinguishability. Although it is very clear from the intuitive perspective, in fuzzy set theoretical contexts approximate equality is usual modelled by fuzzy equivalence relations that respect some kind of transitivity. In this section we discuss counter-intuitive results arising from this representation.

3.1. The consequence of a lower boundary

Zadeh originally generalized the notion of a crisp equivalence relation to that of a fuzzy M-equivalence relation, also called similarity relation.

Example 1. Consider the ages 50, 52 and 56.

x_0	.	z_0	.	.	.	y_0
$=$		$=$				$=$
50		52				56

If a similarity relation E is used to model "approximately equal" then we would expect intuitively

$$E(50,56) < E(50,52) \tag{2}$$

because 50 and 52 are more alike than 50 and 56. Analogously

$$E(56,52) < E(50,52). \tag{3}$$

Eqs. (2) and (3) do not conflict with M-transitivity

$$M(E(50,56),E(56,52)) \leqslant E(50,52).$$

Example 2. Now consider the ages 20, 22 and 24.

x_1	.	y_1	.	z_1
$=$		$=$		$=$
20		22		24

We expect intuitively

$$E(20,22) > E(20,24) \tag{4}$$

because 20 and 22 are more alike than 20 and 24. Analogously

$$E(22,24) > E(20,24). \tag{5}$$

Both (4) and (5) however conflict with M-transitivity

$$M(E(20,22),E(22,24)) \leqslant E(20,24).$$

In general, it is not acceptable to put as a lower boundary on the degree to which two objects x and z are equal, the minimum of the degrees to which they are equal to a third object y. In particular when the object y is intermediate between x and z this lower boundary leads to counter-intuitive results.

It is neither acceptable to put a similar upper boundary on the degree to which x and z are alike. This is done in [8] under the name of M^*-transitivity

$$M^*(E(x,y),E(y,z)) \geqslant E(x,z).$$

It is clear that in this case the intuitive expectations (1) and (2) from Example 1 will conflict with

$$M^*(E(50,56),E(56,52))\geqslant E(50,52).$$

3.2. All objects are approximately equal to each other?

In the previous section we have illustrated the intuitive shortcomings of M-transitive relations to model approximate equality. In this section we will show that for *every* t-norm $\mathscr{T}$ the $\mathscr{T}$-transitivity leads to undesirable results.

Example 3. Let X be the universe of possible heights of men, $\mathscr{T}$ a t-norm and E a fuzzy $\mathscr{T}$-equivalence relation on X. If E is used to represent "approximately equal", we could expect intuitively

$$E(1.50m,1.51m) = 1, \tag{6}$$

$$E(1.51m,1.52m) = 1, \tag{7}$$

$$E(1.52m,1.53m) = 1, \tag{8}$$

$$E(1.53m,1.54m) = 1, \tag{9}$$

$$\vdots$$

Now we derive from the $\mathscr{T}$-transitivity of E

$$\mathscr{T}(E(1.50m,1.51m),E(1.51m,1.52m)) \leqslant E(1.50m,1.52m)$$

and hence

$$E(1.50m,1.52m) = 1. \tag{10}$$

Combining (10) with (8) and using $\mathscr{T}$-transitivity, we can derive in a similar way that

$$E(1.50m,1.53m) = 1. \tag{11}$$

Combining (11) with (9) leads to

$$E(1.50m,1.54m) = 1, \tag{12}$$

etc. Finally, we can prove that all heights are approximately equal to degree 1.

In general we have to be very careful with every fuzzy $\mathscr{T}$-equivalence (for an arbitrary(!) t-norm $\mathscr{T}$) relation E on a universe X, that is not a fuzzy $\mathscr{T}$-equality. For then the intuitive very acceptable

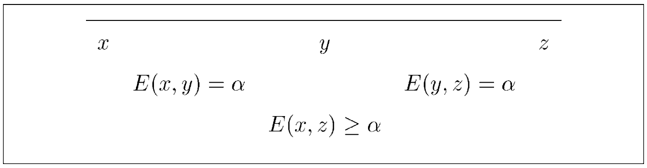

Fig. 2. M-transitivity

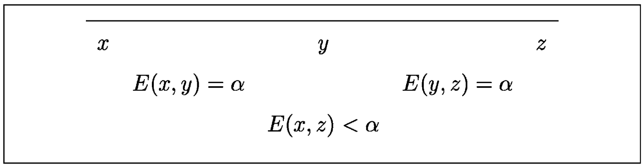

Fig. 3. Intuitive expectation

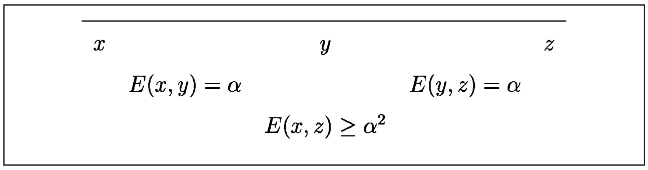

Fig. 4. P-transitivity

situation in which we have different objects x, y and z in X

$$\overline{x\ .\ .\ y\ .\ .\ z}$$

such that $E(x, y) = 1$ and $E(y, z) = 1$ whereas $E(x, z) < 1$, conflicts with $\mathscr{T}$-transitivity. Clearly in this case the representation of approximate equality by E is not compatible with the Poincaré Paradox (Formula 1).

In the case of a M-transitive relation, a similar problem occurs on each α-level ($\alpha \in [0, 1]$). If for x, y and z in X: $E(x, y) = \alpha$ and $E(y, z) = \alpha$ then M-transitivity implies that $E(x, z) \geqslant M(E(x, y), E(y, z)) = \alpha$ (see Fig. 2). However, if y is an intermediate object between x and z, we would intuitively expect $E(x, z) < \alpha$ (see Fig. 3). The lower boundary on $E(x, z)$ is obviously too high. M is the largest t-norm. The smaller the t-norm $\mathscr{T}$ we use to model $\mathscr{T}$-transitivity, the lower the boundary on $E(x, z)$ will be, and the more the derived result will agree with our intuition (see Figs. 4 and 5).

This could explain the success of W-transitive relations in relational databases [6, p. 123]. Nevertheless, even by using the smallest t-norm D we cannot avoid the paradoxical results in Example 3.

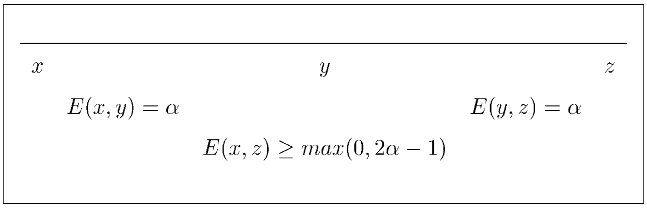

Fig. 5. W-transitivity

The M^*-transitivity mentioned by Kundu in [8] can be generalized to negative $\mathscr{S}$-transitivity for an arbitrary t-conorm $\mathscr{S}$ [4]

$$\mathscr{S}(E(x,y),E(y,z)) \geqslant E(x,z).$$

This definition of transitivity unfortunately gives rise to similar (or "dual") counter-intuitive results. E.g. for a negative-$\mathscr{S}$-transitive relation, from $E(20,70)=0$ and $E(70,20)=0$ we can now derive that $E(20,20)=0$. This makes E certainly unsuitable to model approximate equality.

In [5] it is proven that Lukasiewicz logic is compatible with the Poincaré Paradox. It should be noted however that in the proof the author uses a very specific relation to model equality, namely $E(x,y)=1-min((1/\varepsilon)|x-y|,1)$, for all x and y in $\mathbb{R}$ and ε denoting a fixed positive real number. Note that $E(x,y)=1$ iff $x=y$. In the same paper an alternative kind of fuzzy relation to model equality is defined:

Definition 2 ([0,1]-Valued equality). A fuzzy relation E on X is called a [0,1]-valued equality on X iff for all x, y and z in X:

(E.1) $E(x,y)\leqslant min(E(x,x),E(y,y))$
(E.2) $E(x,y)=E(y,x)$
(E.3) $E(x,y)-E(y,y)+E(y,z)\leqslant E(x,z)$.

The first condition is a weakened reflexivity. We feel that in the case of approximate equality it is justified to demand full reflexivity, i.e. $E(x,x)=1$, for all x in X, or "every object is definitely approximately equal to itself". The last condition is a variant of W-transitivity. It can easily be proven that the Poincaré Paradox is not satisfied for reflexive [0,1]-valued equalities that are not separated in a trivial way. I.e. if there exists x, y and z such that $E(x,y)=1$ and $E(y,z)=1$, due to condition (E.3) also necessarily $E(x,z)=1$.

3.3. Paradox in approximate reasoning

Since the intuitive shortcomings of fuzzy $\mathscr{T}$-equivalence relations in modelling approximate equality already become apparent when simply studying the defining characteristic of $\mathscr{T}$-transitivity, it is not very surprising that we will also meet them on a higher level, in case when we use them for approximate reasoning purposes.

Let us take a closer look at a particular form of the compositional rule of inference [15]. Let X_1, X_2 and X_3 be 3 universes. $\mathbf{v}_1$ is a variable in X_1, $\mathbf{v}_2$ is a variable in X_2, $\mathbf{v}_3$ is a variable in X_3,

$R_1 \in \mathscr{F}(X_1 \times X_2)$, $R_2 \in \mathscr{F}(X_2 \times X_3)$, $\mathscr{T}$ is a t-norm. From (1) and (2) we can derive (3):

$$\begin{array}{ll} (\mathbf{v}_1, \mathbf{v}_2) \text{ is } R_1 & (1) \\ (\mathbf{v}_2, \mathbf{v}_3) \text{ is } R_2 & (2) \\ \hline (\mathbf{v}_1, \mathbf{v}_3) \text{ is } R_1 \circ_{\mathscr{T}} R_2 & (3) \end{array}$$

Zadeh originally stated this rule of inference for $\mathscr{T} = M$.

Example 4. We study the inference:

The age of Alberik (26) and the age of Bart (23) is approximately equal.
The age of Bart (23) and the age of Chris (20) is approximately equal.

The age of Alberik and the age of Chris is ???

This inference is clearly a special case of the compositional rule of inference mentioned above with $X_1 = X_2 = X_3 = [0, 150]$, $\mathbf{v}_1 =$ age of Alberik, $\mathbf{v}_2 =$ age of Bart and $\mathbf{v}_3 =$ age of Chris. Furthermore, R_1 and R_2 both represent "*is approximately equal to*". Let us choose $R_1 = R_2 = E$ with E a fuzzy $\mathscr{T}_1$-equivalence relation for some t-norm $\mathscr{T}_1$. If we use a t-norm $\mathscr{T}_2$ to model the composition in the compositional rule of inference, we obtain:

$$\begin{array}{l} (\mathbf{v}_1, \mathbf{v}_2) \text{ is } E \\ (\mathbf{v}_2, \mathbf{v}_3) \text{ is } E \\ \hline (\mathbf{v}_1, \mathbf{v}_3) \text{ is } E \circ_{\mathscr{T}_2} E. \end{array}$$

If $\mathscr{T}_1 = \mathscr{T}_2$ (a popular choice is $\mathscr{T}_1 = \mathscr{T}_2 = M$), then $E \circ_{\mathscr{T}_2} E = E$ (this equality is a result of the reflexivity and the $\mathscr{T}_2$-transitivity of E). In other words the conclusion is: $\mathbf{v}_1$ and $\mathbf{v}_3$ are "*approximately equal*" in the same manner as $\mathbf{v}_1$ and $\mathbf{v}_2$ are approximately equal, and in the same manner as $\mathbf{v}_2$ and $\mathbf{v}_3$ are approximately equal. Now if we keep making similar inferences (e.g. by adding the proposition "The age of Wouter (29) and the age of Alberik (26) are approximately equal"), we obtain a similar paradox as in the previous section. Note that this is the case for *every* t-norm!

4. On the links between distance functions and fuzzy equivalence relations

Fuzzy equivalence relations are obviously not very suitable to represent approximate equality. To model approximate equality in a proper way, we could start by taking the very clear intuitive relationship between distance and equality into account:

Assumption 1. The closer two objects are to each other (i.e. the smaller the distance between them), the more they are (approximately) equal.

The concept of resemblance relation that will be introduced in the next section is indeed directly pseudometric based. However to enlighten the reader who is still not convinced about the unsuitability of fuzzy equivalence relations to model approximate equality we will first study the relationship between distance functions and fuzzy equivalence relations. Once more we will meet problems on the intuitive level.

Throughout this section we will assume that X is a universe, and that d is a $X^2 - [0,1]$ mapping.

Definition 3 (Pseudometric). d is a pseudometric on X iff for all x, y and z in X:

(PM.1) $d(x,x)=0$,
(PM.2) $d(x,y)=d(y,x)$,
(PM.3) $d(x,y)+d(y,z)\geqslant d(x,z)$.

The couple (X,d) is called a pseudometric space. If also $d(x,y)=0$ implies $x=y$, then d is called a metric and (X,d) is a metric space.

For simplicity, we restrict ourself to $[0,1]$-valued pseudometrics. Note that every pseudometric d' can be turned into a $[0,1]$-valued pseudometric d by defining $d(x,y)=min(1,d'(x,y))$, for all x and y in X.

The concept of a pseudometric reflects our intuitive understanding of the notion of distance. The third condition (PM.3) can be replaced by one that involves a t-conorm $\mathscr{S}$, which gives rise to the definition of a $\mathscr{S}$-pseudometric [11].

Definition 4 ($\mathscr{S}$-pseudometric). d is a $\mathscr{S}$-pseudometric on X, iff for all x, y and z in X:

(SPM.1) $d(x,x)=0$,
(SPM.2) $d(x,y)=d(y,x)$,
(SPM.3) $\mathscr{S}(d(x,y),d(y,z))\geqslant d(x,z)$.

If also $d(x,y)=0$ implies $x=y$, then d is called a $\mathscr{S}$-metric.

Especially the case where $\mathscr{S}=M^*$ is well known.

Definition 5 (Pseudo-ultrametric). A pseudo-ultrametric on X is a M^*-pseudometric on X.

Note that a pseudo-ultrametric is a very counter-intuitive concept to represent distance. Indeed, if y is some intermediate point between x and z, then $M^*(d(x,y), d(y,z))\geqslant d(x,z)$ does not correspond to our intuition. On the other hand for a D^*-pseudometric, the condition (SPM.3) gives almost no information at all: if neither $d(x,y)$ nor $d(y,z)$ is 0, then (SPM.3) corresponds to $1\geqslant d(x,z)$ which is trivial considering the definition of distance functions we use (namely $[0,1]$-valued).

Some $\mathscr{S}$-metrics are however intuitively useful to model distance. Namely the class of W^*-pseudometrics on X coincides with the class of pseudometrics on X. Taking into account that every $\mathscr{S}_1$-pseudometric on X is also a $\mathscr{S}_2$-pseudometric on X for $\mathscr{S}_1\subseteq\mathscr{S}_2$, the second diagram in Fig. 6 depicts the class of $\mathscr{S}$-pseudometrics with its subclasses. As we stated before, the condition imposed by M^*-pseudometrics is too severe from the intuitive point of view. Leaving the sub-

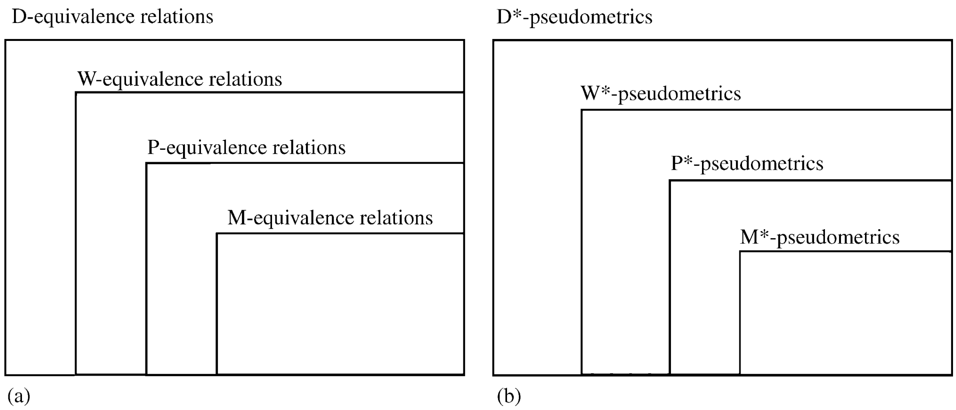

Fig. 6. a) $\mathscr{T}$-equivalence relations. b) $\mathscr{S}$-pseudometrics

set of M^*-pseudometrics and approaching to the borderline of the class of W^*-pseudometrics, this condition relaxes. Once we cross the border and leave the set of W^*-pseudometrics however, the $\mathscr{S}$-pseudometrics are no longer pseudometrics. Therefore, in our opinion the best $\mathscr{S}$-pseudometrics to model distance are the pseudometrics in the neighbourhood of the W^*-border.

The links between fuzzy equivalence relations and the $\mathscr{S}$-pseudometrics are well known. We recall a very important one.

Proposition 1 (Valverde [11]). *For $\mathscr{T}$ a* t-*norm and $\mathscr{N}$ an involutive negation, the following statements are equivalent*:

(1) *E is a fuzzy $\mathscr{T}$-equivalence relation on X*,
(2) $\mathrm{co}_{\mathscr{N}}E$ *is a $\mathscr{T}^{\leftrightarrow\mathscr{N}}$–pseudometric on X.*

For the standard negation $\mathscr{N}_s$ this proposition is visualized in Fig. 6. At first sight it seems to reflect the intuitive reverse connection between distance and approximate equality (Assumption 1) very good. An immediate result of Proposition 1 however is that E is a fuzzy M-equivalence relation on X iff $\mathrm{co}_{\mathscr{N}_s}E$ is a pseudo-ultrametric on X. In other words Proposition 1 shows the link between fuzzy M-equivalence relations with a counter-intuitive concept to model distance! Likewise E is a fuzzy W-equivalence relation on X iff $\mathrm{co}_{\mathscr{N}_s}E$ is a pseudometric on X, which indicates that fuzzy W-equivalence relations are more natural. Or as Valverde puts it [11]

> ...in fact that property is used as one of the major arguments to introduce likeness relations because the triangle inequality refines the ultrametric inequality given by the similarity relations.

In Section 3.2 we already came to a similar conclusion (cfr. their success in database applications). Nevertheless as we have explained in Sections 3.2 and 3.3, problems on the intuitive level remain even for fuzzy W-equivalence relations. Therefore, we feel the need to introduce a new kind of relation to model approximate equality.

5. Resemblance relations

Instead of making a straightforward generalization of crisp equivalence relations to fuzzy ones and then trying to find the most suitable t-norm to have the least of problems on the intuitive level for representing approximate equality, we suggest using a kind of relation that immediately satisfies Assumption 1, e.g. resemblance relations.

Definition 6 ((g,d)-resemblance relation). For X a universe, $(\mathcal{M},d)$ a pseudometric space and g a $X-\mathcal{M}$ mapping, a fuzzy relation E on X is called a (g,d)-resemblance relation on X iff for all x,y,z and u in X:

(R.1) $E(x,x)=1$,
(R.2) $E(x,y)=E(y,x)$,
(R.3) $d(g(x),g(y))\leqslant d(g(z),g(u))$ implies $E(x,y)\geqslant E(z,u)$.

The definition above is directly inspired by general intuitive expectations about approximate equality. The first two defining criteria are obvious: every object x resembles to itself to degree 1 ((R.1) reflexivity). Furthermore an object x resembles to an object y to the same degree as y resembles to x ((R.2) symmetry). The third criterion (R.3) expresses Assumption 1.

If X is already equipped with a suitable pseudometric—as is the case for most numerical universes —then g can be the identical mapping on X, i.e. $\mathbb{I}_X(x)=x$, for all x in X. In this case (R.3) reduces to

$$d(x,y)\leqslant d(z,u) \text{ implies } E(x,y)\geqslant E(z,u).$$

Later on, however, we will also provide an example in which the use of a non-trivial mapping g is useful.

5.1. Related concepts

In [4] a kind of fuzzy relation on the unit interval is defined, that is related to that of a resemblance relation on $[0,1]$, namely "equivalence". In the following propositions we relate the concept of resemblance relation to those of crisp equality, fuzzy equivalence relation and point fuzzification. Let X be a universe, $(\mathcal{M},d)$ a pseudometric space and g a $X-\mathcal{M}$ mapping.

Proposition 2. *If $(\mathcal{M},d)$ is a metric space and g is injective then the crisp equality "=" is a (g,d)-resemblance relation on X.*

Proof. The crisp equality "=" is reflexive and symmetric, so (R.1) and (R.2) are trivial. To prove (R.3), we consider a quadruple (x,y,z,u) in X^4 with $d(g(x),g(y))\leqslant d(g(z),g(u))$. If $z\neq u$ ($E(z,u)=0$) then (R.3) is automatically satisfied. On the other hand if $z=u$ then $g(z)=g(u)$ and hence $d(g(z),g(u))=0$. Then $d(g(x),g(y))=0$ must hold and hence $g(x)=g(y)$. Since g is injective this implies $x=y$. This means that if $z=u$ ($E(z,u)=1$) we obtain $x=y$ ($E(x,y)=1$) and hence (R.3). □

Proposition 3. *Let $\mathscr{T}$ be a* t*-norm such that $W \subseteq \mathscr{T}$. If E is a fuzzy $\mathscr{T}$-equivalence relation on X then E is a $(\mathbb{I}_X, co_{\mathscr{N}_s}(E))$-resemblance relation on X.*

Proof. Since E is a fuzzy $\mathscr{T}$-equivalence relation on X and $W \subseteq \mathscr{T}$, $\mathrm{co}_{\mathscr{N}_s}(E)$ is a pseudometric on X (cf. Proposition 1). A fuzzy $\mathscr{T}$-equivalence relation is reflexive and symmetric, so (R.1) and (R.2) are trivial. Let (x, y, z, u) in X^4. Furthermore from

$$\mathrm{co}_{\mathscr{N}_s}(E)(x, y) \leqslant \mathrm{co}_{\mathscr{N}_s}(E)(z, u)$$

we obtain (since $\mathrm{co}_{\mathscr{N}_s}$ is decreasing):

$$\mathrm{co}_{\mathscr{N}_s}(\mathrm{co}_{\mathscr{N}_s}(E))(x, y) \geqslant \mathrm{co}_{\mathscr{N}_s}(\mathrm{co}_{\mathscr{N}_s}E)(z, u)$$

and hence (since $\mathrm{co}_{\mathscr{N}_s}$ is involutive)

$$E(x, y) \geqslant E(z, u). \qquad \square$$

In [14], Zadeh describes a process of point fuzzification which transforms a singleton $\{x\}$ of X into a fuzzy set on X that is concentrated around x. Following Kerre [6], we will name this fuzzy set after the process used to generate it, and thus define the concept of "point fuzzification centered around $x \in X$ w.r.t. a distance function d".

Definition 7 (Point fuzzification). For (X, d) a pseudometric space and x in X, a fuzzy set K on X is called a d-point fuzzification of x iff for all a and b in X:

(PF.1) $K(x) = 1$,
(PF.2) $d(x, a) < d(x, b)$ implies $K(a) \geqslant K(b)$.

Proposition 4. *Let x in $\mathscr{M}$. If E is an $(\mathbb{I}_{\mathscr{M}}, d)$-resemblance relation on $\mathscr{M}$ then the E-afterset of x is a d-point fuzzification of x.*

Proof. $(xE)(x) = E(x, x) = 1$. Furthermore, since E is an $(\mathbb{I}_X, d)$-resemblance relation for all a and b in $\mathscr{M}$: if $d(x, a) < d(x, b)$ then $E(x, a) \geqslant E(x, b)$. Applying the definition of afterset we obtain $(xE)(a) \geqslant (xE)(b)$. $\square$

5.2. Properties

Let X be a universe, $(\mathscr{M}, d)$ a pseudometric space and g an $X - \mathscr{M}$ mapping. Let $\mathscr{T}$ be a t-norm, $\mathscr{S}$ a t-conorm and $\mathscr{N}$ a negation.

Proposition 5. *If E is a (g, d)-resemblance relation on X then $d(g(x), g(y)) = d(g(z), g(u))$ implies $E(x, y) = E(z, u)$, for all x, y, z and u* in *X.*

Proof. From $d(g(x),g(y))=d(g(z),g(u))$ we obtain

$$d(g(x),g(y)) \geqslant d(g(z),g(u))$$

and hence $E(x,y)\leqslant E(z,u)$. Similarly we obtain $E(x,y)\geqslant E(z,u)$ and hence the equality. □

The following three propositions are straightforward: [1]

Proposition 6. *$co_{\mathcal{N}}\, d\circ(g\times g)$ is a (g,d)-resemblance relation on X.*

Proposition 7. *For $a\geqslant 1$, the fuzzy relation R defined by $R(x,y)=max(0,min(1,a-d(x,y))$, for all x and y in $\mathcal{M}$, is an $(\mathbb{I}_{\mathcal{M}},d)$-resemblance relation on $\mathcal{M}$.*

Proposition 8. *If E is a (g,d)-resemblance relation on X and f is an increasing $[0,1]-[0,1]$ mapping satisfying $f(1)=1$ then: $f\circ E$ is a (g,d)-resemblance relation on X.*

Proposition 9. *If E_1 and E_2 are (g,d)-resemblance relations on X then*

(1) *$E_1\cap_{\mathcal{T}}E_2$ is a (g,d)-resemblance relation on X.*
(2) *$E_1\cup_{\mathcal{S}}E_2$ is a (g,d)-resemblance relation on X.*

Proof. $E_1\cap_{\mathcal{T}}E_2$ and $E_1\cup_{\mathcal{S}}E_2$ are indeed fuzzy relations on X. We check that (R.1), (R.2) and (R.3) hold for $E_1\cap_{\mathcal{T}}E_2$. (The proofs for $E_1\cup_{\mathcal{S}}E_2$ are analogous.) For all x, y, z and u in X:

$$(E_1\cap_{\mathcal{T}}E_2)(x,x)=\mathcal{T}(E_1(x,x),E_2(x,x))=\mathcal{T}(1,1)=1 \quad \text{(R.1)}.$$

Furthermore,

$$\begin{aligned}(E_1\cap_{\mathcal{T}}E_2)(x,y)&=\mathcal{T}(E_1(x,y),E_2(x,y))\\ &=\mathcal{T}(E_1(y,x),E_2(y,x))\\ &=(E_1\cap_{\mathcal{T}}E_2)(y,x) \quad \text{(R.2)}.\end{aligned}$$

Finally, $d(g(x),g(y))\leqslant d(g(z),g(u))$ implies $(E_1(x,y)\geqslant E_1(z,u))$ and $(E_2(x,y)\geqslant E_2(z,u))$ and hence since $\mathcal{T}$ is increasing:

$$\mathcal{T}(E_1(x,y),E_2(x,y)) \geqslant \mathcal{T}(E_1(z,u),E_2(z,u)).$$

Applying the definition of $\cap_{\mathcal{T}}$ we obtain:

$$(E_1\cap_{\mathcal{T}}E_2)(x,y) \geqslant (E_1\cap_{\mathcal{T}}E_2)(z,u) \quad \text{(R.3)}. \qquad \square$$

[1] $\circ$ without an index is used to denote the common composition of mappings: for X,Y and Z universa, f a mapping from X to Y and g a mapping from Y to Z, the composition $g\circ f$ is a mapping from X to Z with $(\forall x\in X)((g\circ f)(x)=g(f(x)))$. This should not be confused with the $\mathcal{T}$-composition of fuzzy relations.
The product $f_1\times f_2$ of a X_1-Y_1 mapping f_1 and a X_2-Y_2 mapping f_2 is a $(X_1\times X_2)-(Y_1\times Y_2)$ mapping defined by $f_1\times f_2(x_1,x_2)\ =(f_1(x_1),f_2(x_2))$, for all (x_1,x_2) in $(X_1\times X_2)$.

Corollary 10. $\mathrm{co}_{\mathcal{N}}\, d$ *is an* $(\mathbb{I}_{\mathcal{M}}, d)$*-resemblance relation on* $\mathcal{M}$.

5.3. Examples

Comparing ages: $X = [0, 150]$ is a universe of ages. The $X^2 - \mathbb{R}$-mapping d defined by

$$d(x, y) = \frac{|x - y|}{10}$$

for all x and y in X, is a pseudometric on X, so E defined by $E(x, y) = \max(0, \min(1, 1.1 - d(x, y)))$ for all x and y in X, is an $(\mathbb{I}_X, d)$-resemblance relation on X (see Proposition 7). We compute some degrees of membership

$$E(50, 56) = 0.5,$$

$$E(50, 52) = 0.9,$$

$$E(56, 52) = 0.7.$$

Clearly $E(50, 56) < E(50, 52)$ and $E(56, 52) < E(50, 52)$. This corresponds to the intuitive expectations we stated in Example 1 of Section 3.1.

$$E(20, 22) = 0.9,$$

$$E(22, 24) = 0.9,$$

$$E(20, 24) = 0.7.$$

Clearly $E(20, 22) > E(20, 24)$ and $E(22, 24) > E(20, 24)$. This also corresponds to our intuition. Notice that we could not obtain these results if E was a fuzzy M-equivalence relation (see Example 2 of Section 3.1). Furthermore,

$$E(20, 20) = 1,$$

$$E(20, 20.5) = 1.$$

Hence different ages can be approximately equal to degree 1. The paradoxical situation in which all objects are approximately equal to degree 1 (see Section 3.2) however does not arise.

Comparing heights: $X = [1.00, 2.50]$ is a universe of heights of men (expressed in meters). The $X^2 - \mathbb{R}$-mapping d defined by

$$d(x, y) = \frac{|x - y|}{5}$$

for all x and y in X, is a pseudometric on X, so E defined by

$$E(x, y) = max(0, min(1, 1.2 - d(x, y)))$$

for all x and y in X, is an $(\mathbb{I}_X, d)$-resemblance relation on X (see Proposition 7). We compute some degrees of membership

$$E(1.50, 1.51) = 1,$$

$$E(1.51, 1.52) = 1,$$

$$E(1.50, 1.52) = 0.8.$$

It is clear from this example that E is not a fuzzy equivalence relation since for any t-norm $\mathscr{T}$ we have $\neg(\mathscr{T}(E(1.50, 1.51), E(1.51, 1.52)) \leqslant E(1.50, 1.52))$.

Comparing beauty: X is a universe of fairytale characters

$$X = \{snowwhite,\ witch,\ wolf,\ dwarf,\ prince,\ little\text{-}red\text{-}riding\text{-}hood\}$$

The fuzzy sets *beautiful, average* and *ugly* in X are given as follows:

	Beautiful	*Average*	*Ugly*
Snowwhite	1.00	0.00	0.00
Witch	0.00	0.30	0.70
Wolf	0.00	0.00	1.00
Dwarf	0.10	0.70	0.20
Prince	0.80	0.20	0.00
Red-hood	0.50	0.50	0.00

g is an $X - [0,1]^3$ mapping defined by

$$g(x) = (beautiful(x), average(x), ugly(x))$$

for all x in X. d is a $[0,1]^3 \times [0,1]^3 - [0,1]$ mapping defined by

$$d((x_1, y_1, z_1), (x_2, y_2, z_2)) = max(|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|)$$

for all (x_1, y_1, z_1) and (x_2, y_2, z_2) in $[0,1]^3$. Now we can model approximate equality by the (g, d)-resemblance relation E on X with (see Proposition 6)

$$E = \text{co}_{\mathscr{N}_s}\, d \circ (g \times g).$$

The matrix representation of E is

E	*Snowwhite*	*Witch*	*Wolf*	*Dwarf*	*Prince*	*Red-hood*
Snowwhite	1.00	0.00	0.00	0.10	0.80	0.50
Witch	0.00	1.00	0.70	0.50	0.20	0.30
Wolf	0.00	0.70	1.00	0.20	0.00	0.00
Dwarf	0.10	0.50	0.20	1.00	0.30	0.60
Prince	0.80	0.20	0.00	0.30	1.00	0.70
Red-hood	0.50	0.30	0.00	0.60	0.70	1.00

The elements on the diagonal are 1 because of the reflexivity of E. The fact that E is symmetric causes the matrix to be symmetric. Furthermore, E respects the separation characteristic, i.e. $E(x,y)=1$ iff $x=y$. By applying a contrast intensifier to E we can construct a resemblance relation E_1 that is not separated, i.e. two different fairy tale characters can be approximately equal in beauty to degree 1. In general a *contrast intensifier* on the unit interval $[0,1]$ is a $[0,1]-[0,1]$ mapping f satisfying for x in $[0,1]$

$$x \leqslant \tfrac{1}{2} \Rightarrow f(x) \leqslant x,$$

$$x \geqslant \tfrac{1}{2} \Rightarrow f(x) \geqslant x.$$

A possible contrast intensifier on $[0,1]$ is the mapping L depending on a parameter α: for $\alpha \in [0, \frac{1}{2}[$:

$$L(.;\alpha) : [0,1] \to [0,1]$$

$$x \mapsto 0, \quad \forall x \in [0,\alpha],$$

$$x \mapsto 2\left(\frac{x-\alpha}{1-2\alpha}\right)^2, \quad \forall x \in \left[\alpha, \frac{1}{2}\right],$$

$$x \mapsto 1-2\left(\frac{x-1+\alpha}{1-2\alpha}\right)^2, \quad \forall x \in \left[\frac{1}{2}, 1-\alpha\right],$$

$$x \mapsto 1, \quad \forall x \in [1-\alpha, 1].$$

$L(.;0.3)$ is increasing and satisfies $L(1;0.3)=1$, hence we find with Proposition 8 that the fuzzy relation E_1 defined by

$$E_1 = L(.;0.3) \circ E$$

is also a (g,d)-resemblance relation on X.

E_1	*Snowwhite*	*Witch*	*Wolf*	*Dwarf*	*Prince*	*Red-hood*
Snowwhite	1.00	0.00	0.00	0.00	1.00	0.50
Witch	0.00	1.00	1.00	0.50	0.00	0.00
Wolf	0.00	1.00	1.00	0.00	0.00	0.00
Dwarf	0.00	0.50	0.00	1.00	0.00	0.88
Prince	1.00	0.00	0.00	0.00	1.00	1.00
Red-hood	0.50	0.00	0.00	0.88	1.00	1.00

It is easy to see that $\mathscr{T}$-transitivity does not hold, for any t-norm $\mathscr{T}$. For e.g. snowwhite and the prince are approximately equal to degree 1, the prince and little-red-riding-hood are approximately equal to degree 1, but snowwhite and little-red-riding-hood are approximately equal only to degree 0.5.

6. Conclusion

In this paper we showed that the representation of approximate equality by means of a fuzzy equivalence relation gives raise to paradoxical situations. In this perspective some fuzzy equivalence relations are even worse than others. Similarity relations are generally useless, while likeness relations are clearly better. It is however preferable to use another framework that corresponds more to intuition. For this purpose we introduced the concept of a (g,d)-resemblance relation. We stated some properties of this new type of relations and we illustrated the concept in several kinds of universa.

Acknowledgements

Martine De Cock would like to thank the Fund For Scientific Research-Flanders for funding the research reported on in this paper. The authors would like to thank the anonymous referees for their helpful comments.

References

[1] W. Bandler, L.J. Kohout, Fuzzy relational products as a tool for analysis and synthesis of the behaviour of complex natural and artificial systems, in: S.K. Wang, P.P. Chang (Eds.), Fuzzy Sets: Theory and Application to Policy Analysis and Information Systems, Plenum Press, New York and London, 1980, pp. 341–367.
[2] U. Bodenhofer, A similarity-based generalization of fuzzy orderings, Schriftenreihe der Johannes-Kepler-Universität Linz, C 26, Universitätsverlag Rudolf Trauner, 1999.
[3] B. De Baets, R. Mesiar, Pseudo-metrics and $\mathscr{T}$-equivalences, J. Fuzzy Math. 5 (2) (1997) 471–481.
[4] J. Fodor, M. Roubens, Fuzzy Preference Modelling and Multicriteria Decision Support, Kluwer Academic Publishers, Dordrecht/Boston/London, 1994.
[5] U. Höhle, The Poincaré Paradox and non-classical logics, in: D. Dubois, H. Prade, E.P. Klement (Eds.), Fuzzy Sets, Logics and Reasoning about Knowledge, Kluwer Academic Publishers, Dordrecht/Boston/London, 1999, pp. 7–16.
[6] E.E. Kerre, Introduction to the basic principles of fuzzy set theory and some of its applications, Communication and Cognition, Gent, 1993.
[7] F. Klawonn, R. Kruse, Equality relations as a basis for fuzzy control, Fuzzy Sets and Systems 54 (1993) 147–156.
[8] S. Kundu, The min–max composition rule and its superiority over the usual max–min composition rule, Fuzzy Sets and Systems 93 (1998) 319–329.
[9] H. Poincaré, La Science et l'Hypothèse, Flammarion, Paris, 1902.
[10] H. Poincaré, La Valeur de la Science, Flammarion, Paris, 1904.
[11] L. Valverde, On the structure of F-indistinguishability operators, Fuzzy Sets and Systems 17 (1985) 313–328.
[12] L. Wittgenstein, Philosophical Investigations, Blackwell, Oxford, 1953.
[13] L.A. Zadeh, Similarity relations and fuzzy orderings, Inform. Sci. 3 (1971) 177–200.
[14] L.A. Zadeh, A fuzzy-set-theoretic interpretation of linguistic hedges, J. Cybernet. 2 and 3 (1972) 4–34.
[15] L.A. Zadeh, Calculus of fuzzy restrictions, in: L.A. Zadeh, K.-S. Fu, K. Tanaka, M. Shimura (Eds.), Fuzzy Sets and their Applications to Cognitive and Decision Processes, Academic Press, New York, 1975, pp. 1–40.