Fuzzy Topologies Induced by Fuzzy Relation Based Modifiers

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Abstract. In this paper we highlight the specific meaning of images of fuzzy sets under fuzzy relations in the context of fuzzy topology. More precisely we show that fuzzy modifiers taking direct and superdirect images of fuzzy sets under fuzzy pre-orderings are respectively closure and interior operators, inducing fuzzy topologies. Furthermore we investigate under which conditions the same applies to the recently introduced general closure and opening operators based on arbitrary fuzzy relations.

1 Introduction

Images of fuzzy sets under fuzzy relations prove to be very powerful tools in a wide range of applications, varying from fuzzy databases, over fuzzy morphology, fuzzy rough set theory, and the representation of linguistic modifiers, to approximate reasoning schemes [5]. In this paper we will show that they have a specific meaning in the context of fuzzy topology as well. More specifically in Section 2 we will show that fuzzy modifiers taking direct and superdirect images of fuzzy sets under fuzzy pre-orderings are respectively closure and interior operators inducing Chang fuzzy topologies. In Section 3 we will investigate under which conditions the same holds for the general closure and opening operators introduced recently by Bodenhofer [2].

Throughout the paper T will denote a triangular norm with left-continuous partial mappings $T(x, .)$ for all x in $[0, 1]$. Furthermore $\vec{T}$ will denote its residual implication defined by

$$\vec{T}(x, y) = \sup\{\lambda \in [0, 1] | T(\lambda, x) \leq y\}$$

It can be verified then that $\vec{T}$ is non-increasing and left-continuous in the first argument while non-decreasing and right-continuous in the second. Furthermore for all x, y, and z in $[0, 1]$, and $\{y_i \mid i \in I\}$ a family in $[0, 1]$:

(1) $T(0, x) = 0$
(2) $T(x, \sup_{i\in I} y_i) = \sup_{i\in I} T(x, y_i)$
(3) $\vec{T}(x, y) = 1$ iff $x \leq y$

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(4) $\vec{T}(1, x) = x$
(5) $\vec{T}(x, \inf_{i \in I} y_i) = \inf_{i \in I} \vec{T}(x, y_i)$
(6) $\vec{T}(T(x, y), z) \leq \vec{T}(x, \vec{T}(y, z))$

Let X be a non-empty set. The class of all fuzzy sets on X will be denoted $\mathcal{F}(X)$. As usual union, intersection and complement of fuzzy sets are defined by, for x in X,

$$\bigcup_{i \in I} A_i(x) = \sup_{i \in I} A_i(x)$$

$$\bigcap_{i \in I} A_i(x) = \inf_{i \in I} A_i(x)$$

$$(co\ A)(x) = 1 - A(x)$$

in which $\{A_i | i \in I\}$ is a family of fuzzy sets on X, and A in $\mathcal{F}(X)$. Inclusion for two fuzzy sets A and B on X is defined as:

$$A \subseteq B \text{ iff } A(x) \leq B(x) \text{ for all } x \text{ in } X$$

Definition 1 (Fuzzy modifier). *[16] A fuzzy modifier on X is an $\mathcal{F}(X)$-$\mathcal{F}(X)$ mapping.*

In [4] the following class of fuzzy modifiers based on fuzzy relations is defined:

Definition 2 (Fuzzy relation based modifiers). *Let R be a fuzzy relation on X, i.e. $R \in \mathcal{F}(X \times X)$. The fuzzy modifiers $R\!\uparrow$ and $R\!\downarrow$ on X are defined by, for A in $\mathcal{F}(X)$ and y in X:*

$$R\!\uparrow\! A(y) = \sup_{x \in X} T(A(x), R(x, y))$$

$$R\!\downarrow\! A(y) = \inf_{x \in X} \vec{T}(R(y, x), A(x))$$

$R\!\uparrow\! A$ is also called the direct image of A under R, while $R\!\downarrow\! A$ is the superdirect image of A under R^{-1} (i.e. the inverse relation of R) [12]. In fuzzy set theoretical settings, fuzzy modifiers are usually associated with the representation of linguistic hedges such as very, more or less, rather,... (see e.g. [7], [13], [16]). The class of fuzzy modifiers based on fuzzy relations as defined above however can be applied to a wide range of other purposes as well. In [4], [10] it is shown that for a suitable fuzzy relation R they correspond to the dilation and erosion operators of fuzzy morphology (used for image processing), while in [6], [10] it is illustrated that they can be used as fuzzy-rough approximators (for dealing with incomplete information). In this paper we will show that for R a fuzzy pre-ordering the R-based fuzzy modifiers defined in Definition 2 also are closure and interior operators inducing fuzzy topologies (in the sense of Chang [3]).

Definition 3 (Fuzzy T-preordering). *A fuzzy relation R on X is called a fuzzy T-preordering w.r.t. a t-norm T iff for all x, y and z in X:*

(FP.1) $R(x, x) = 1$ *(reflexivity)*
(FP.2) $R(x, z) \leq T(R(x, y), R(y, z))$ *(T-transitivity)*

Definition 4 (Fuzzy topology). *A subset τ of $\mathcal{F}(X)$ is called a fuzzy topology on X iff*

(FT.1) $\emptyset \in \tau \wedge X \in \tau$
(FT.2) $O_1 \in \tau \wedge O_2 \in \tau$ *implies* $O_1 \cap O_2 \in \tau$
(FT.3) $(\forall i \in I)(O_i \in \tau)$ *implies* $\bigcup_{i \in I} O_i \in \tau$

Furthermore the class τ' is defined by

$$F \in \tau' \text{ iff } co(F) \in \tau$$

Every element of τ is called an open fuzzy set*; every element of τ' is called a* closed fuzzy set.

Definition 5 (Interior, closure). *Let τ denote a fuzzy topology on X. For A a fuzzy set on X the interior and the closure of A w.r.t. τ are the fuzzy sets $int(A)$ and $cl(A)$ on X defined by*

$$int(A) = \cup\{O | O \in \tau \wedge O \subseteq A\}$$

$$cl(A) = \cap\{F | F \in \tau' \wedge A \subseteq F\}$$

I.e. $int(A)$ is the largest open fuzzy set contained in A, while $cl(A)$ is the smallest closed fuzzy set containing A. int and cl are called the interior operator *and the* closure operator *respectively.*

It can be verified that they satisfy the following properties, for all A and B in $\mathcal{F}(X)$ [11]:

(1) $int(X) = X$ (1') $cl(\emptyset) = \emptyset$
(2) $int(A) \subseteq A$ (2') $A \subseteq cl(A)$
(3) $int(A \cap B) = int(A) \cap int(B)$ (3') $cl(A \cup B) = cl(A) \cup cl(B)$
(4) $int(int(A)) = int(A)$ (4') $cl(cl(A)) = cl(A)$

2 Induced Fuzzy Topologies

To show that the fuzzy relation based modifiers of Definition 2 are closure and interior operators inducing fuzzy topologies on X, we rely on the following theorem [11]:

Theorem 1. *Let f be a fuzzy modifier on X.*

Part I *If for all A and B in $\mathcal{F}(X)$*

(C1) $f(X) = X$
(C2) $f(A) \subseteq A$
(C3) $f(A \cap B) = f(A) \cap f(B)$

then $\tau = \{O|O \in \mathcal{F}(X) \wedge f(O) = O\}$ *is a fuzzy topology on* X*. Furthermore if also*

(C4) $f(f(A)) = f(A)$

for all A *in* $\mathcal{F}(X)$*, then* f *is the interior operator corresponding to* τ*.*

Part II *If for all* A *and* B *in* $\mathcal{F}(X)$

(C1') $f(\emptyset) = \emptyset$
(C2') $A \subseteq f(A)$
(C3') $f(A \cup B) = f(A) \cup f(B)$

then $\tau = \{O|O \in \mathcal{F}(X) \wedge f(co\ O) = co\ O\}$ *is a fuzzy topology on* X*. Furthermore if also*

(C4') $f(f(A)) = f(A)$

for all A *in* $\mathcal{F}(X)$*, then* f *is the closure operator corresponding to* τ*.*

Theorem 2. *Let* R *be a reflexive fuzzy relation on* X*, then*

$$\tau = \{O|O \in \mathcal{F}(X) \wedge R{\downarrow}O = O\}$$

is a fuzzy topology on X*. Moreover if* R *is a fuzzy* T*-preordering on* X *then* $R{\downarrow}$ *is the interior operator corresponding to* τ*.*

Proof. According to Theorem 1 the following conditions have to be fulfilled in order for τ to be a fuzzy topology on X: for all A and B in $\mathcal{F}(X)$:

(C1) $R{\downarrow}X = X$
(C2) $R{\downarrow}A \subseteq A$
(C3) $R{\downarrow}(A \cap B) = R{\downarrow}A \cap R{\downarrow}B$

Furthermore if also (C4) $R{\downarrow}(R{\downarrow}A) = R{\downarrow}A$ for all A in $\mathcal{F}(X)$ then $int = R{\downarrow}$. We will now prove that (C1), (C2) and (C3) hold for a reflexive fuzzy relation R, while (C4) holds for a fuzzy T-preordering. For all y in X:

$$R{\downarrow}X(y) = \inf_{x \in X} \vec{T}(R(y,x), X(x)) = \inf_{x \in X} \vec{T}(R(y,x), 1) = \inf_{x \in X} 1 = 1 = X(y)$$

hence (C1) is valid. Furthermore since R is reflexive

$$R{\downarrow}A(y) = \inf_{x \in X} \vec{T}(R(y,x), A(x)) \leq \vec{T}(R(y,y), A(y)) = \vec{T}(1, A(y)) = A(y)$$

which proves (C2). (C3) can be verified as follows:

$$\begin{aligned} R{\downarrow}(A \cap B)(y) &= \inf_{x \in X} \vec{T}(R(y,x), \min(A(x), B(x))) \\ &= \inf_{x \in X} \min(\vec{T}(R(y,x), A(x)), \vec{T}(R(y,x), B(x))) \\ &= \min(\inf_{x \in X} \vec{T}(R(y,x), A(x)), \inf_{x \in X} \vec{T}(R(y,x), B(x))) \\ &= \min(R{\downarrow}A(y), R{\downarrow}B(y)) \\ &= (R{\downarrow}A \cap R{\downarrow}B)(y) \end{aligned}$$

From (C2) we know that $R{\downarrow}(R{\downarrow}A) \subseteq R{\downarrow}A$ for a reflexive R. To complete the proof for (C4) we show that the other inclusion holds as well for T-transitive R ($\vec{T}$ being the residual implication of T):

$$\begin{aligned} R{\downarrow}(R{\downarrow}A)(y) &= \inf_{x\in X} \vec{T}(R(y,x), \inf_{z\in X} \vec{T}(R(x,z), A(z))) \\ &= \inf_{x\in X} \inf_{z\in X} \vec{T}(R(y,x), \vec{T}(R(x,z), A(z))) \\ &\geq \inf_{x\in X} \inf_{z\in X} \vec{T}(T(R(y,x), R(x,z)), A(z))) \\ &\geq \inf_{x\in X} \inf_{z\in X} \vec{T}(R(y,z), A(z)) \\ &= R{\downarrow}A(y) \end{aligned}$$

□

Theorem 3. *Let R be a reflexive fuzzy relation on X, then*

$$\tau = \{O | O \in \mathcal{F}(X) \wedge R{\uparrow}(co\ O) = co\ O\}$$

is a fuzzy topology on X. Moreover if R is a fuzzy T-preordering on X then $R{\uparrow}$ is the closure operator corresponding to τ.

Proof. According to Theorem 1 the following conditions have to be fulfilled in order for τ to be a fuzzy topology on X: for all A and B in $\mathcal{F}(X)$:

(C1') $R{\uparrow}\emptyset = \emptyset$
(C2') $A \subseteq R{\uparrow}A$
(C3') $R{\uparrow}(A \cup B) = R{\uparrow}A \cup R{\uparrow}B$

Furthermore if also (C4') $R{\uparrow}(R{\uparrow}A) = R{\uparrow}A$ for all A in $\mathcal{F}(X)$ then $cl = R{\uparrow}$. We will now prove that (C1'), (C2') and (C3') hold for a reflexive fuzzy relation R, while (C4') holds for a fuzzy T-preordering. For all y in X:

$$R{\uparrow}\emptyset(y) = \sup_{x\in X} T(\emptyset(x), R(x,y)) = \sup_{x\in X} T(0, R(x,y)) = \sup_{x\in X} 0 = 0 = \emptyset(y)$$

hence (C1') is valid. Furthermore since R is reflexive

$$R{\uparrow}A(y) = \sup_{x\in X} T(A(x), R(x,y)) \geq T(A(y), R(y,y)) = T(A(y), 1) = A(y)$$

which proves (C2'). (C3') can be verified as follows:

$$\begin{aligned} R{\uparrow}(A \cup B)(y) &= \sup_{x\in X} T(\max(A(x), B(x)), R(x,y)) \\ &= \sup_{x\in X} \max(T(A(x), R(x,y)), T(B(x), R(x,y))) \\ &= \max(\sup_{x\in X} T(A(x), R(x,y)), \sup_{x\in X} T(B(x), R(x,y))) \\ &= \max(R{\uparrow}A(y), R{\uparrow}B(y)) \\ &= (R{\uparrow}A \cup R{\uparrow}B)(y) \end{aligned}$$

From (C2') we know that $R{\uparrow}A \subseteq R{\uparrow}(R{\uparrow}A)$ for a reflexive R. To complete the proof for (C4') we show that the other inclusion holds as well for a T-transitive R (relying on the associativity of T):

$$\begin{aligned}
R{\uparrow}(R{\uparrow}A)(y) &= \sup_{x\in X} T(\sup_{z\in X} T(R(z,x),A(z)),R(x,y))\\
&= \sup_{x\in X}\sup_{z\in X} T(T(A(z),R(z,x)),R(x,y))\\
&= \sup_{x\in X}\sup_{z\in X} T(A(z),T(R(x,y),R(z,x)))\\
&\leq \sup_{x\in X}\sup_{z\in X} T(A(z),R(z,y))\\
&= R{\uparrow}A(y)
\end{aligned}$$

□

From now on we refer to the fuzzy topology defined in Theorem 2 as the fuzzy topology induced by $R{\downarrow}$ while we say that the fuzzy topology of Theorem 3 is induced by $R{\uparrow}$.

Remark 1. If R is not reflexive then (C2) and (C2') do not necessarily hold. For instance it can be verified that if R is defined as $R(x,y)=0$ for all x and y in X, then $R{\downarrow}A = X$ and $R{\uparrow}A=\emptyset$, for all A in $\mathcal{F}(X)$. Therefore for $A\neq\emptyset$ and $A\neq X$ neither $R{\downarrow}A\subseteq A$ (C2) nor $A\subseteq R{\uparrow}A$ (C2') hold.

Remark 2. From the example given in Remark 1 it is clear that (C4) and (C4') do not hold in general for a non-reflexive fuzzy relation R. Indeed for R the empty relation on X, $R{\downarrow}(R{\downarrow}A)=X$ and $R{\uparrow}(R{\uparrow}A)=\emptyset$ hold for all A in $\mathcal{F}(X)$, implying that in this case (C4) only holds for $A=X$ and (C4') only holds for $A=\emptyset$. However even the reflexivity of R is not sufficient in order for (C4) and (C4') to hold for arbitrary A as shown in the following example: let $X=[0,1]$ and let A be the fuzzy set on X defined by $A(x)=x$, for all x in X. Furthermore let R be the reflexive fuzzy relation on X defined by

$$R(x,y) = \begin{cases} 1\ if\ |x-y|<0.1\\ 0\ otherwise\end{cases}$$

for all x and y in X. Then it can be verified that for y in X:

$$\begin{aligned}
R{\downarrow}A(y) &= \inf_{z\in X} \vec{T}(R(y,z),A(z))\\
&= \inf\{z \mid z\in X \wedge z\in]y-0.1,y+0.1[\}\\
&= \max(0,y-0.1)
\end{aligned}$$

Hence $R{\downarrow}A(1)=0.9$. Furthermore

$$\begin{aligned}
R{\downarrow}(R{\downarrow}A)(1) &= \inf_{z\in X} \vec{T}(R(1,z),max(0,z-0.1))\\
&= \inf\{max(0,z-0.1) \mid z\in]0.9,1]\} = 0.8
\end{aligned}$$

Likewise

$$\begin{aligned}
R{\uparrow}A(y) &= \sup_{z\in X} T(A(z),R(z,y))\\
&= \max\{z \mid z\in X\wedge z\in]y-0.1,y+0.1[\}\\
&= \min(1,y+0.1)
\end{aligned}$$

Hence $R{\uparrow}A(0) = 0.1$. Furthermore

$$\begin{aligned} R{\uparrow}(R{\uparrow}A)(0) &= \sup_{z \in X} T(min(1, z + 0.1), R(z, 0)) \\ &= \sup\{min(1, z + 0.1) \mid z \in [0, 0.1[\} = 0.2 \end{aligned}$$

Note that R is not T-transitive.

In rough set theory [14] the universe X of objects is equipped with a crisp equivalence relation R expressing the indiscernibility between objects that can be derived from the available (incomplete) information about them. The lower approximation $\underline{R}A$ of a crisp subset of X is defined as the union of the equivalence classes of R included in A. It is the set of objects that necessarily belong to A considering the available information. The upper approximation $\overline{R}A$ of A is the union of all equivalence classes of X that have a non-empty intersection with A. It is the set of all objects that possibly belong to A.

Indiscernibility between objects might be *vague*: e.g. Alberik (28 years old) and Els (29) are indiscernible in age to a higher degree than Alberik (28) and Mike (25). Furthermore one might also want to make statements concerning the degree to which an object necessarily/possibly belongs to a *fuzzy* set A on X. Hence there is a need for *fuzzy* rough set theory ([8], [9], [15]) in which the universe is equipped with a fuzzy T-equivalence relation R, i.e. asymmetrical fuzzy T-preordering. In this case an object y belongs to the lower approximation of a fuzzy set A on X to the degree to which the fuzzy equivalence class of R containing y is included in A, i.e. $\underline{R}A = R{\downarrow}A$. Likewise y belongs to the upper approximation of A to the degree to which the fuzzy equivalence class of R containing y overlaps with A, in other words $\overline{R}A = R{\uparrow}A$. The lower and upper approximators $\underline{R}$ and $\overline{R}$ can therefore be seen as the interior and closure operator respectively of a fuzzy topology on X.

3 Generalized Opening and Closure Operators

Note that in the theorems of the previous section the reflexivity of R was sufficient to prove (C2) and (C2') while the reflexivity and the T-transitivity were sufficient for (C4) and (C4'). (C1), (C1'), (C3) and (C3') on the other hand hold for arbitrary R. In [2] the following generalized opening and closure operators are introduced:

Definition 6 (Generalized opening and closure operator). *Let R be a fuzzy relation on X. The generalized opening operator R° and the general closure operator $R^\bullet$ on X are defined by, for A in $\mathcal{F}(X)$:*

$$R^\circ A = R{\uparrow}(R{\downarrow}A)$$

$$R^\bullet A = R{\downarrow}(R{\uparrow}A)$$

Both of these operators are of course fuzzy modifiers on X. It is proven in [2] that R° satisfies (C2) and (C4) while $R^\bullet$ fulfills (C2') and (C4'). The beauty of these general operators is that they do not require reflexivity or T-transitivity of R for these properties. Ironically enough the reflexivity of R is a suitable requirement to be able to prove (C1) and (C1'), while we can prove (C3) and (C3') assuming T-transitivity. In fact for reflexive R:

$$\begin{aligned} R^{\circ}X(y) &= R{\uparrow}(R{\downarrow}X)(y) \\ &= R{\uparrow}X(y) \\ &= \sup_{x\in X} T(X(x), R(x,y)) \\ &\geq T(1, R(y,y)) \\ &= 1 = X(y) \end{aligned} \qquad \begin{aligned} R^{\bullet}\emptyset(y) &= R{\downarrow}(R{\uparrow}\emptyset)(y) \\ &= R{\downarrow}\emptyset(y) \\ &= \inf_{x\in X} \vec{T}(R(y,x), \emptyset(x)) \\ &\leq \vec{T}(R(y,y), 0) \\ &= 0 = \emptyset(y) \end{aligned}$$

Note that these properties also hold for an arbitrary fuzzy relation R if the kernels of all of its foresets Ry are not empty[1]. In other words: if all of its foresets are normalized.

Since $A \cap B \subseteq A$, $A \cap B \subseteq B$, $A \subseteq A \cup B$, $B \subseteq A \cup B$, and the fuzzy modifiers $R{\uparrow}$ and $R{\downarrow}$ are monotonic (cfr. Theorem 2 and 3), it holds that $R^{\circ}(A \cap B) \subseteq R^{\circ}A \cap R^{\circ}B$ and $R^{\bullet}A \cup R^{\bullet}B \subseteq R^{\bullet}(A \cup B)$. The following example shows however that the corresponding equalities do not hold for arbitrary fuzzy relations R.

Example 1. Let $X = [0,1]$. Let A and B be fuzzy sets on X and R a fuzzy relation on X defined by $A(x) = x$, $B(x) = 1 - x$, and

$$R(x,y) = \begin{cases} 1 \; if \; |x-y| < 0.1 \\ 0 \; otherwise \end{cases}$$

for all x and y in X. Then it can be verified that for y in X: $R{\downarrow}A(y) = \max(0, y - 0.1)$ (cfr. Remark 2) and likewise $R{\downarrow}B(y) = \max(0, 0.9 - y)$. Hence

$$\begin{aligned} R^{\circ}A(0.5) &= \sup_{y\in X} T(R{\downarrow}A(y), R(y, 0.5)) \\ &= \sup_{y\in]0.4,0.6[} y - 0.1 = 0.5 \end{aligned}$$

and likewise $R^{\circ}B(0.5) = 0.5$. Therefore $(R^{\circ}A \cap R^{\circ}B)(0.5) = 0.5$. On the other hand

$$\begin{aligned} R{\downarrow}(A \cap B)(y) &= \inf_{z\in X} \vec{T}(R(y,z), \min(A(z), B(z))) \\ &= \inf\{\min(z, 1-z) \mid z \in X \wedge z \in]y-0.1, y+0.1[\} \end{aligned}$$

Hence

$$\begin{aligned} R^{\circ}(A \cap B)(0.5) &= \sup_{y\in X} T(R{\downarrow}(A \cap B)(y), R(y, 0.5)) \\ &= \sup_{y\in]0.4,0.6[} R{\downarrow}(A \cap B)(y) = 0.4 \end{aligned}$$

Quite analogously it can be verified that

$$(R^{\bullet}A \cup R^{\bullet}B)(0.5) = 0.5$$

[1] For y in X the R-foreset of y is the fuzzy set Ry on X defined by $(Ry)(x) = R(x,y)$, for all x in X [1].

while

$$R^{\bullet}(A \cup B)(0.5) = 0.6$$

Note that the relation R in the example above is reflexive and symmetrical, but that the equalities of (C.3) and (C.3') do not hold. They would hold if R was also T-transitive, but as proven in [2], [15], in this case the generalized opening and closure operator coincide with the fuzzy relation based modifiers $R{\downarrow}$ and $R{\uparrow}$, i.e. $R^{\circ}(A) = R{\downarrow}(A)$ and $R^{\bullet}(A) = R{\uparrow}(A)$, for all A in $\mathcal{F}(X)$. As we have shown in the previous section in this case the operators induce a fuzzy topology on X. The generalized opening and closure operators introduced in [2] however are not in general interior and closure operators inducing a fuzzy topology.

4 Conclusion

We have shown that the fuzzy modifiers taking direct and superdirect images of fuzzy sets under reflexive fuzzy relations induce fuzzy topologies of X. Furthermore if the fuzzy relation is also T-transitive, i.e. it is a fuzzy T-preordering, the fuzzy modifiers correspond to the closure and interior operators. If the fuzzy T-preordering relation is also symmetrical, i.e. it is a fuzzy T-equivalence relation, these operators correspond to the lower and upper approximators of fuzzy rough set theory.

Although the generalized opening and closure operators fulfill some properties for arbitrary fuzzy relations that in the case of the fuzzy modifiers mentioned above only hold for reflexive and T-transitive relations, they are not in general interior and closure operators inducing fuzzy topologies. For T-equivalence relations they coincide with the fuzzy modifiers taking images and therefore induce fuzzy topologies.

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