CEE 483 Winter 2009 HW#3 Solutions

1. (a) For each type of particle, the mass concentration c_i (mg/L) can be converted into the number concentration N_i (#/L) by dividing by the mass per particle m_i, i.e., $N_i = c_i / m_i$. The mass per particle is the product of the particle's density and its volume; assuming the particles are spherical, this mass is $m_i = \pi d_i^3 \rho_i / 6$. Therefore, $N_i = \frac{c_i}{m_i} = \frac{6c_i}{\pi d_i^3 \rho_i}$. The results for the given particle collection are:

	A	B	C	D
Concentration, #/L	$4.42\text{x}10^7$	$7.21\text{x}10^5$	$1.52\text{x}10^6$	$8.26\text{x}10^3$
% of total particles	95.2	1.6	3.3	0.02

Note that the overwhelming *number* of particles are of Type A, even though these particles contribute only 10% of the total mass.

(b) This part of the question can be answered using either a numerical or analytical approach. Both approaches are presented here, with the numerical approach first.

Numerical approach. Using this approach, we can calculate the initial force on the (stationary) particle and use that force to compute the initial acceleration. We then assume that the acceleration remains steady at the computed value for a short period of time and compute a new velocity at the end of that time. Then, the whole process is repeated, over and over, until the computed velocity is 90% of the terminal velocity.

The terminal velocity can be estimated simply by carrying out the numerical simulation until the change in the particle's velocity is negligible. Alternatively, it can be computed separately, by computing the drag force that must that must exist when the particle is traveling at its terminal velocity (in order that the total force be zero); we designate this force as $F_{D,\text{known}}$. We then determine the velocity that yields that drag force, using an iterative approach. For instance, we can guess a value of v_{term}, use that value to compute the Reynolds number (Re), and then use Re to compute the drag coefficient (C_D). C_D is then used to compute the drag force (F_D) by:

$$F_D = -C_D A_p \rho_w \left(\frac{v_p^2}{2} \right)$$

This value, which we can call $F_{D,\text{est}}$, is the drag force that would apply if the particle was indeed traveling at the assumed terminal velocity. The trial and error solution involves adjusting v_{term} until $F_{D,\text{est}}$ equals $F_{D,\text{known}}$.

Yet another alternative is to assume that the terminal velocity is small enough that the particle is characterized by creeping flow, so that the drag coefficient is approximately Re/24. In that case, we can calculate v_{term} as $\frac{g\left(\rho_p - \rho_w\right) d_p^2}{18\mu}$. Then, once we obtain an estimate for v_{term}, we must check to see if the assumption that Re<<1 is satisfied. In this problem, that assumption is satisfied extremely well for particle types A, B, and C, and quite well even for particle type D.

Using any of these approaches, the results are:

Type A: $v_{term} = 3.30 \times 10^{-4}$ cm/s

Type B: $v_{term} = 1.01 \times 10^{-2}$ cm/s

Type C: $v_{term} = 2.44 \times 10^{-3}$ cm/s

Type D: $v_{term} = 1.26 \times 10^{-1}$ cm/s

Thus, 90% of v_{term} for a Particle C is 2.20 cm/s. We now return to the problem of simulating the movement of the particle from an initial velocity of 0 to this target velocity.

Because the initial particle velocity is zero, the only forces on the particle initially are imposed by gravity and buoyancy. These forces can be calculated as the product of the gravitational constant with the mass of the particle and the mass of the water that it displaces, respectively. For the 20-μm particles, these values are:

$$F_G = \left(\frac{\pi d_p^3}{6}\right)\rho_p g = \frac{\pi\left(0.002\,\text{cm}\right)^3}{6}\left(1.1\frac{\text{g}}{\text{cm}^3}\right)\left(980\frac{\text{cm}}{\text{s}^2}\right) = 4.52\text{x}10^{-6}\text{ dyne}$$

$$F_B = -\left(\frac{\pi d_p^3}{6}\right)\rho_w g = -\frac{\pi\left(0.002\,\text{cm}\right)^3}{6}\left(1.0\frac{\text{g}}{\text{cm}^3}\right)\left(980\frac{\text{cm}}{\text{s}^2}\right) = -4.11\text{x}10^{-6}\text{ dyne}$$

The net force is the sum of F_G and F_B, and the initial acceleration is the ratio of the initial net force to the particle mass:

$$a = \frac{F_{net}}{m_p} = \frac{\left(4.52-4.11\right)\text{x}10^{-6}\text{ dyne}}{\frac{\pi\left(0.002\,\text{cm}\right)^3}{6}\left(1.1\frac{\text{g}}{\text{cm}^3}\right)} = 89\frac{\text{cm}}{\text{s}^2}$$

We can make the approximation that this initial value of acceleration applies over some short period of time; I used $\Delta t = 10^{-6}$ s. The velocity at the end of the time step and the distance traveled are then computed as follows:

$v_{\text{fin}} = v_{\text{init}} + a\,\Delta t = 0 + (89\text{ cm/s}^2)(10^{-6}\text{ s}) = 8.9\text{x}10^{-5}$ cm/s

$\Delta x = v_{\text{init}}\Delta t + 0.5\,a(\Delta t)^2 = 0(10^{-6}\text{ s}) + 0.5\,(89\text{ cm/s}^2)(10^{-6}\text{ s})^2 = 4.45\text{x}10^{-11}$ cm

We can use the velocity at the end of the first time step to compute the Reynolds number, the drag coefficient, and the drag force at that time. Since we know that the approximation $C_D =$

24/Re applies even at the end of the simulation (when the particle is settling at 0.9 v_{term}), the approximation must apply before it reaches that velocity. Therefore:

$$\text{Re} = \frac{d_p v_p \rho_w}{\mu_w} = \frac{(0.002\text{cm})(8.9\text{x}10^{-5}\,\text{cm/s})(1.0\text{g/cm}^3)}{0.0089\text{g/cm-s}} = 2.00\text{x}10^{-5}$$

$$C_D = \frac{24}{\text{Re}} = 1.20\text{x}10^{6}$$

$$F_D = \frac{1}{2} C_D \frac{\pi d_p^2}{4} \rho_w v_p^2 = 1.50\text{x}10^{-8}\,\text{dyne}$$

Finally, we can re-compute the net force on the particle as $F_{\text{net}} = F_{\text{G}} - F_{\text{B}} - F_{\text{D}}$, and use that value to compute the acceleration during the second time step. The whole process is then repeated until the velocity is 90% of the terminal velocity. Using this approach, the terminal velocity is computed to be 0.00244 cm/s, and 90% of this velocity is attained extremely quickly (in about 0.06 ms), as shown in Figure 1 below. The corresponding distance traveled by the particle, shown in Figure 2, is less than 1 nm!

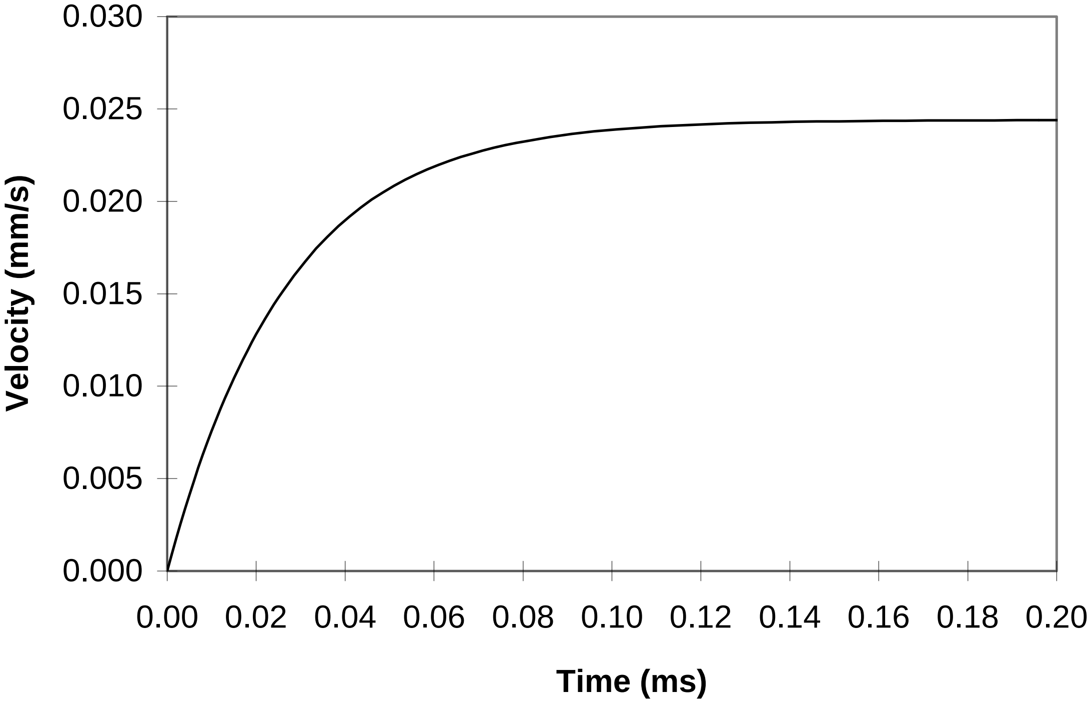


Figure 1. Settling velocity of type C particles as a function of time in the basin.

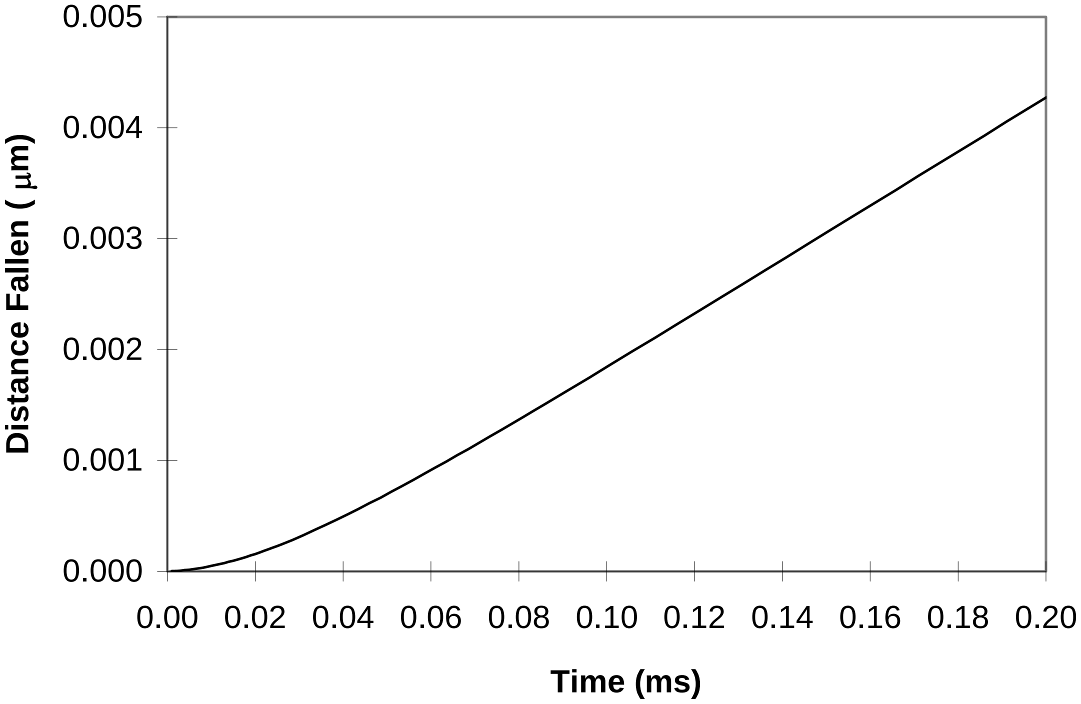


Figure 2. Distance fallen by type C particles as a function of time in the basin.

Analytical approach. The analytical solution to the problem can be developed most easily if we assume that the Reynolds' number is always small enough to allow the drag coefficient to be approximated as 24/Re. In such a case, we can substitute into the force balance and carry out some algebra to obtain:

$$ma = F_G - F_B - F_D$$

$$\frac{\pi}{6}\rho_p d_p^3 \frac{dv_p}{dt} = \frac{\pi}{6} d_p^3 g\left(\rho_p - \rho_w\right) - \frac{24}{\mathrm{Re}} A_p \rho_w \frac{v_p^2}{2}$$

$$\frac{\pi}{6}\rho_p d_p^3 \frac{dv_p}{dt} = \frac{\pi}{6} d_p^3 g\left(\rho_p - \rho_w\right) - \frac{24}{d_p v_p \rho_w / \mu_w}\left(\pi \frac{d_p^2}{4}\right)\rho_w \frac{v_p^2}{2}$$

$$\rho_p d_p^2 \frac{dv_p}{dt} = d_p^2 g\left(\rho_p - \rho_w\right) - 18\mu_w v_p$$

Substituting known values for the ρ_p, d_p, g, ρ_w, and μ_w, the equation can be simplified and then integrated as follows (all units cgs):

$$4.40\text{x}10^{-6}\frac{dv_p}{dt}=3.92\text{x}10^{-4}-0.160v_p$$

$$\frac{dv_p}{dt}=89.1-3.64\text{x}10^4 v_p$$

$$\int_{v_{p,init}}^{v_{p,final}}\frac{dv_p}{89.1-3.64\text{x}10^4 v_p}=\int_{t_{init}}^{t_{final}}dt$$

$$-\frac{1}{3.64\text{x}10^4}\ln\frac{89.1-3.64\text{x}10^4 v_{p,final}}{89.1-3.64\text{x}10^4 v_{p,init}}=t_{final}-t_{init}$$

$v_{p,init}$ and t_{init} are zero. Therefore, designating $v_{p,fin}$ and t_{fin} simply as v_p and t:

$$-\frac{1}{3.64\text{x}10^4}\ln\frac{89.1-3.64\text{x}10^4 v_p}{89.1}=t$$

$$-2.75\text{x}10^{-5}\ln\left(1-408.5v_p\right)=t$$

For a v_p of 0.0022 cm/s (90% of v_{term}), we find:

$$t=-\frac{1}{3.64\text{x}10^4}\ln\frac{89.1-3.64\text{x}10^4\left(0.0022\right)}{89.1}=-\frac{1}{3.64\text{x}10^4}\ln\left(0.101\right)=6.30\text{x}10^{-5}$$

Thus, using this approach, the time computed to reach a velocity of 0.0022 cm/s is 0.063 ms, which is very close to the value computed numerically. This result is obtained because the Reynolds number throughout the period of interest is <<1, consistent with our assumption. (Re when v equals 0.9 v_{term} is ~5 x 10^{-4}.

The distance fallen by the particle can be assessed by integrating the equation developed above for velocity over time. That is, solving Equation ** for v_p, we find:

$$1-408.5v_p=\exp\left(-\frac{t}{2.75\text{x}10^{-5}}\right)$$

$$v_p=\frac{1}{408.5}\left[1-\exp\left(-\frac{t}{2.75\text{x}10^{-5}}\right)\right]$$

Then, noting that the distance fallen, x, is the product of velocity and time:

$$x=\int_0^t v_p dt=\frac{1}{408.5}\int_0^t\left[1-\exp\left(-\frac{t}{2.75\text{x}10^{-5}}\right)\right]dt$$

$$x = \frac{1}{408.5}\left[t + \frac{1}{2.75\text{x}10^{-5}}\exp\left(-\frac{t}{2.75\text{x}10^{-5}}\right)\right]_0^{6.30\text{x}10^{-5}}$$

$$x = 9.38\text{x}10^{-8} \text{ cm} = 0.938 \text{ nm}$$

where the units on *x* are based on the fact that all the calculations were carried out with cgs units. The result is essentially the same as that obtained numerically.

(c) The settling basin has a critical velocity of (200 cm)/(3 h), corresponding to 66.7 cm/h, or $1.85\text{x}10^{-2}$ cm/s. The removal efficiency for each type of particle is v_{term}/v_{crit} or 100%, whichever is greater. Therefore, the removal efficiency for each type of particle is as shown below:

Type	$v_{\text{term}}/v_{\text{crit}}$	% Removal	% of Particles
A	0.018	1.8	10
B	0.543	54.3	10
C	0.132	13.2	70
D	4.53	100	10

The overall removal can then be computed as:

$$(10\%)(1.8\%) + (10\%)(54.3\%) + (70\%)(13.2\%) + (10\%)(100\%) = 24.8\%$$

The influent concentration was 10 mg/L, so the particle concentration in the effluent would be 0.752 (10 mg/L), or 7.52 mg/L.

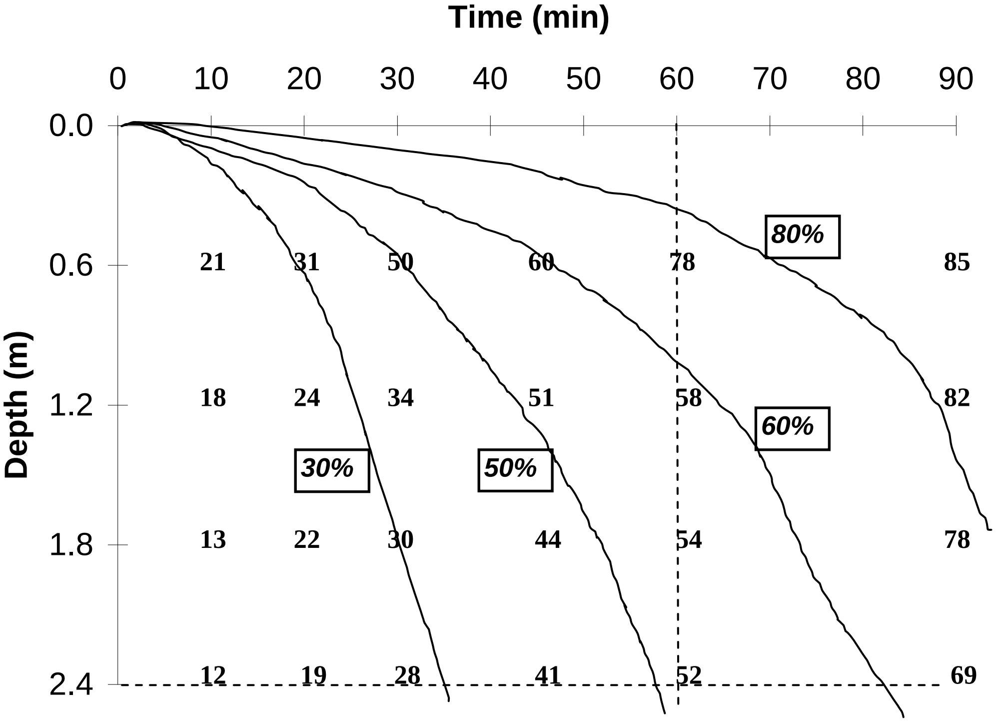
2. (a) Define f_v as the fraction of particles with settling velocities slower than velocity *v*. For $v = 0.25$ mm/s, $f_{v,1} = \sim 1.0$ and $f_{v,2} = 0.2$. Therefore, River 2 has a larger fraction of particles which settle faster than 0.25 mm/s.

(b) Because the particles settle independently of one another (Type 1 settling), we can evaluate the fractional removal for each group of particles and then determine the overall removal based on a weighted average of the two results.

The fractional particle removal from River 1 equals the area above the settling curve for River 1 and within the rectangle defined by $0.0 < v < 0.1$ mm/s and $0.0 < f < 1.0$, divided by the full area of the rectangle. A similar statement applies to River 2. The area above the curve for River 1 is approximately 0.005 mm/s, and the area above the curve for River 2 is approximately 0.8 mm/s. The area of the full rectangle is 0.1 mm/s. Therefore, the removal efficiencies are approximately 95% for River 1 and 80% for River 2. Since River 2 has 150% as much particulate matter in the influent as River 1, and the waters are mixed in a 1:1 ratio, the overall particle removal efficiency will be:

$$\eta_{overall} = \frac{\eta_1 + 1.5\eta_2}{2.5} = \frac{0.95 + 1.5(0.80)}{2.5} = 0.86 = 86\%.$$

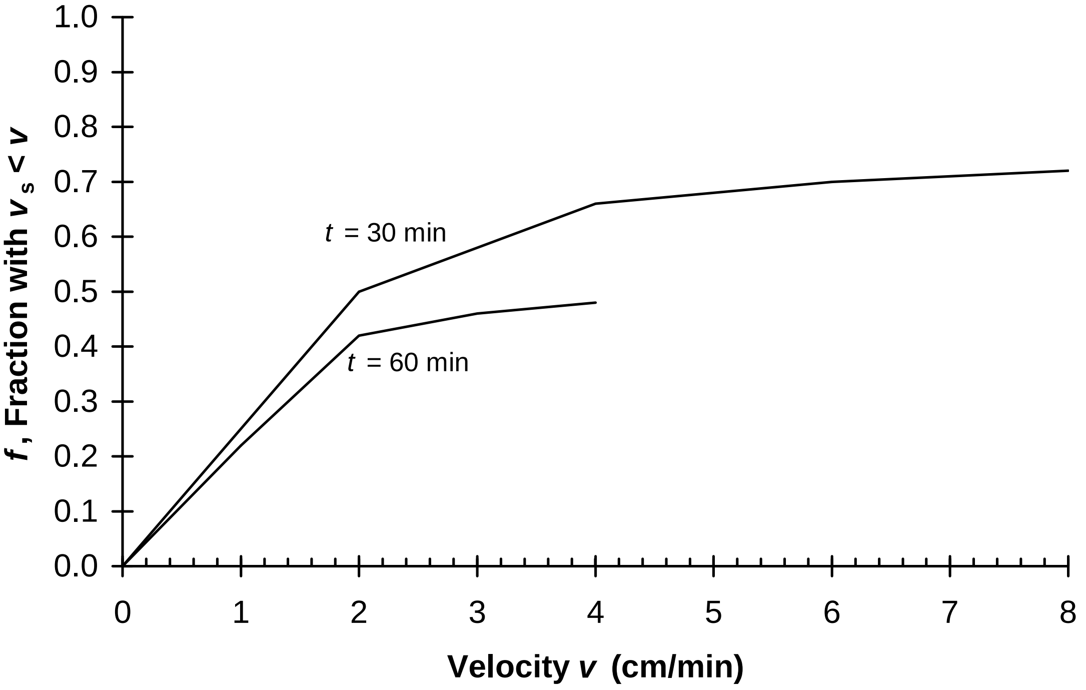
3. A plot with approximate isopleths of percentage removal is shown below. The plot was prepared by writing the given percentage removal values at appropriate depth and time coordinates and then sketching curves that corresponded to fixed percentage removals, based on the plotted values. Also shown are broken lines at $t = 60$ min and $h = 2.4$ m.



The data can also be represented in terms of the fraction of the particle mass that has average velocity less than a given value by recognizing that, if *x*% of the initial solids concentration remains at a given depth (*h*) and time (*t*), then *x*% of the solids must have had average velocities less than *h*/*t* during that time. The logic is that all the particles that fell at velocities greater than *h*/*t* would have fallen below the port where the sample was collected and so would not be detected in the sample.

Thus, for instance, since 24% of the particle mass in the original, well mixed suspension was removed in the sample at $h = 1.2$ m and $t = 20$ min, we can conclude that 24% of the particle mass fell at an average velocity of at least *h*/*t* (6 cm/min) during the first 20 minutes. By carrying out this type of analysis for the samples at all depths after times of 30 and 60 min, we can develop the following table and plot.

Data from $t = 30$ min		Data from $t = 60$ min	
Velocity v (cm/min)	% with $v_s < v$	Velocity v (cm/min)	% with $v_s < v$
0.0	0	0.0	0
2.0	50	1.0	22
4.0	66	2.0	42
6.0	70	3.0	46
8.0	72	4.0	48



The removal efficiency in a basin with a 60-min detention time can be computed by a numerical integration vertically along the depth of the tank at the effluent, corresponding to the $t = 60$ min data in the table of raw data. That is, we can imagine the water column having four, equal vertical portions, each 0.6 m tall, characterized by the following removal efficiencies:

> From the top of the column (100% removed) to a depth of 0.6 m (78% removal), the average removal efficiency is (100% + 78%)/2, or 89%.
>
> From depth 0.6 m (78%) to 1.2 m (58%), the average removal efficiency is (78% + 58%)/2, or 68%.
>
> From depth 1.2 m (58%) to 1.8 m (54%), the average removal efficiency is (58% + 54%)/2, or 56%.

From depth 1. 8 m (54%) to the bottom, at 2.4 m (52%), the average removal efficiency is (54% + 52%)/2, or 53%.

A reasonable estimate of overall efficiency is thus:

$$\eta_{overall} = \sum_{0}^{Z} \eta_i \frac{\Delta z}{Z} = \frac{\sum_{0}^{H} \eta_i \Delta z}{Z}$$

$$= (89\%) \frac{0.60 \text{ m} - 0.0 \text{ m}}{2.4 \text{ m}} + (68\%) \frac{1.20 \text{ m} - 0.60 \text{ m}}{2.4 \text{ m}} +$$

$$(56\%) \frac{1.80 \text{ m} - 1.20 \text{ m}}{2.4 \text{ m}} + (53\%) \frac{2.40 \text{ m} - 1.80 \text{ m}}{2.4 \text{ m}}$$

$$= 66.5\%$$

As an alternative, we can develop a spreadsheet and carry out a numerical integration to compute $\int \eta df_v$. The key values in such a spreadsheet are shown below.

v (cm/min)	% with $v<v_{crit}$	f_v	η [(a)]	Δf_v	η_{avg}	$\eta^* df_v$
0.00	0	0	0			
1.00	22	0.22	0.25	0.22	0.125	0.0275
2.00	42	0.42	0.50	0.20	0.375	0.075
3.00	46	0.46	0.75	0.04	0.625	0.025
4.00	48	0.48	1.00	0.02	0.875	0.0175
10.00	100	1.00	1.00	0.52	1.00	0.52
				sum		**0.665**

(a) For $v_{crit} = 4.0$ cm/min

In the table, the first column is a set of possible settling velocities, with the final row being for an arbitrarily large velocity that is larger than the settling velocity of any particle in the feed. The second column shows the corresponding percentage of the suspended solids in the influent with settling velocities less than the value in the first column, based on the given settling data. The third column is the same as the second column, with values converted to decimal format. The fourth column shows the expected removal efficiency for particles with the given settling velocity, computed as v/v_{crit} or 1.0, whichever is smaller. The fifth column shows the gap in f_v between each two successive rows, and the sixth column shows the average value of η over that Δf_v range. Finally, the seventh column shows the product of η and df_v (or, in this case, Δf_v); the sum of the values in this column is the estimated overall particle removal efficiency. That estimate turns out to be 66.5%.

4. The specified alum dose was 30 mg/L, and the MW of alum is 594, so the molar dose of alum is:

$$\left(30\frac{\text{mg}}{\text{L}}\right)\left(\frac{1\text{ mol alum}}{594{,}000\text{ mg}}\right)=5.05\text{x}10^{-5}\,\frac{\text{mol alum}}{\text{L}}$$

The total consumption of Alk when all the aluminum added as alum precipitates as $\text{Al(OH)}_3(s)$ is:

$$\left(5.05\text{x}10^{-5}\,\frac{\text{mol alum}}{\text{L}}\right)\left(2\frac{\text{mol Al}^{3+}}{\text{mol alum}}\right)\left(3\frac{\text{equiv Alk consumed}}{\text{mol Al}\left(\text{OH}\right)_3\left(s\right)\text{ precipitated}}\right)$$
$$=3.03\text{x}10^{-4}\,\frac{\text{equiv Alk consumed}}{\text{L}}$$

The amount of Alk that was initially present was $4.0\text{x}10^{-4}$ equiv/L, so after the precipitation occurs, Alk is $0.97\text{x}10^{-4}$ equiv/L. We can use this value as the approximate molar concentration of HCO_3^-. The concentration of H_2CO_3 in the initial solution, in mol/L, is then computed as:

$$\left(\text{H}_2\text{CO}_3\right)_{init}=\frac{\left(\text{H}^+\right)_{init}\left(\text{HCO}_3^-\right)_{init}}{K_1}=\frac{\left(10^{-7.3}\right)\left(4.0\text{x}10^{-4}\right)}{10^{-6.3}}=4.0\text{x}10^{-5}$$

Thus, the initial value of $TOT\text{CO}_3$ is $4.4\text{x}10^{-4}$ mol/L. Assuming that $TOT\text{CO}_3$ remains constant during the precipitation process, $(\text{H}_2\text{CO}_3)_{fin}$ can then be computed as:

$$\left(\text{H}_2\text{CO}_3\right)_{fin}=TOT\text{CO}_3-\left(\text{HCO}_3^-\right)_{fin}=4.4\text{x}10^{-4}-0.97\text{x}10^{-4}=3.4\text{x}10^{-4}$$

The pH after alum addition can then be found from the known concentrations and K_1:

$$\left(\text{H}^+\right)_{fin}=\frac{\left(\text{H}_2\text{CO}_3\right)_{fin}K_1}{\left(\text{HCO}_3^-\right)_{fin}}=\frac{\left(3.4\text{x}10^{-4}\right)\left(10^{-6.3}\right)}{\left(0.97\text{x}10^{-4}\right)}=1.76\text{x}10^{-6}=10^{-5.76}$$

The pH is thus 5.76. If the pH were increased to 8.2, the ratio of (HCO_3^-) to (H_2CO_3) would be:

$$\frac{\left(\text{HCO}_3^-\right)}{\left(\text{H}_2\text{CO}_3\right)}=\frac{K_1}{\left(\text{H}^+\right)}=\frac{10^{-6.3}}{10^{-8.0}}=10^{1.7}=50$$

The concentration of (HCO_3^-) could then be found as follows:

$$\frac{\left(\text{HCO}_3^-\right)}{TOT\text{CO}_3}=\frac{\left(\text{HCO}_3^-\right)}{\left(\text{HCO}_3^-\right)+\left(\text{H}_2\text{CO}_3\right)}=\frac{50\left(\text{H}_2\text{CO}_3\right)}{50\left(\text{H}_2\text{CO}_3\right)+\left(\text{H}_2\text{CO}_3\right)}=\frac{50}{51}=0.98$$

$$\left(\mathrm{HCO_3}^-\right) = 0.98\left(TOT\mathrm{CO_3}\right) = 0.98\left(4.4\text{x}10^{-4}\right) = 4.31\text{x}10^{-4}$$

We can approximate this value as the Alk of the solution after lime addition. Before lime addition, the Alk was $0.97\text{x}10^{-4}$ equiv/L, so the amount that must be added is $3.34\text{x}10^{-4}$ equiv/L. Each mole of lime contributes two moles of OH^- ions, and hence two equiv of Alk, so the required lime dose is one-half of the required increase in Alk, or $1.17\text{x}10^{-4}$ mol/L. On a mass basis, this addition is:

$$\text{Lime dose} = \left(1.17\text{x}10^{-4}\,\frac{\text{mol}}{\text{L}}\right)\left(\frac{74{,}000\text{ mg lime}}{\text{mol}}\right) = 8.66\,\frac{\text{mg lime}}{\text{L}}$$