Topic 9: The Impulse-Momentum Principle

To summarize what we've done thus far…

We first considered fluid statics, in which case a mass balance is of little value – it would simply tell us that the amount of mass in a static system remains constant. However, we did find a force balance useful to understand the relationship between pressure and depth in a liquid. Looking back, we now understand that relationship to also be an energy balance, reflecting the fact that, in static fluids, energy is converted between two "forms" (gravitational potential energy and pressure-based energy) as a function of the vertical location. We used this understanding to evaluate the forces of objects and static fluids on one another, considering flat surfaces initially, and then curved or bent surfaces. Finally, we used that force analysis to compute the buoyancy of partially or fully submerged objects.

We then extended the analysis to systems with moving fluids. The key feature of this extension was the inclusion of another form in which fluids could acquire and hold energy – as kinetic energy. As before, we recognized that energy could be converted among different forms, and we concluded that the sum of pressure-based, gravitational, and kinetic energy would remain constant in an ideal fluid; this conclusion led to the Bernoulli equation.

Recognizing that the initial derivation of the Bernoulli equation is strictly applicable only to fluid particles traveling along streamlines, but that it is difficult to keep track of all such particles in a fluid system, we developed a general approach for analyzing fluid behavior from an Eulerian rather than a Lagrangian perspective. This analysis led to the RTT which, when applied to mass, yields the continuity equation. In this section, we apply the RTT first to momentum and then energy. As we will see, using the RTT simultaneously to analyze mass, momentum, and energy changes in fluid systems allows us to solve an impressive number of complex fluids problems. We begin with the analysis of momentum, as follows.

Consider a parcel of mass with center of gravity moving with velocity vector $\vec{\mathbf{V}}$. If the parcel is subjected to a force $\vec{\mathbf{F}}$, it will accelerate in the direction of the force. The acceleration is $d\vec{\mathbf{V}}/dt$, and, by Newton's second law, is related to the force by:

$$\vec{\mathbf{F}} = m\vec{\mathbf{a}} = m\frac{d\vec{\mathbf{V}}}{dt} = \frac{d\left(m\vec{\mathbf{V}}\right)}{dt}$$

The product $m\vec{\mathbf{V}}$ is the linear momentum of the parcel, so the force can be equated with the rate of change of linear momentum. Thus, a force in any direction exerted on the fluid causes the linear momentum of the fluid to increase in the direction of the force, and, correspondingly, any change in the linear momentum of the fluid reflects the result of a force acting on it.

The linear momentum of a parcel of fluid is an extensive property. We can convert that quantity into an intensive property by normalizing with respect to mass, yielding the term we have defined in the context of the RTT as i_{mom}. In this case, i_{mom} turns out to be the velocity vector, i.e.:

$$\vec{i}_{mom} = \frac{m\vec{V}}{m} = \vec{V} \qquad (1)$$

Momentum can be carried into or out of a CV by fluids that cross the control surface (advective momentum transport), or it can be added to or taken out of the CV by non-advective processes. Formally, the RTT tells us that the rate at which momentum accumulates in the CV equals the sum of the net rates of advective transport of momentum into the CV and non-advective momentum transport into the CV. In the RTT, the advective input is actually written as the negative of the advective output, so the equation has the following form:

$$\frac{d}{dt} \int_{CV} \vec{i}_{mom} \rho \, d\mathcal{V} = -\int_{CS} \vec{i}_{mom} \rho \vec{V} \bullet d\vec{A} + \dot{\boldsymbol{E}}_{mom} \qquad (2)$$

Note that $\vec{i}$ and $\boldsymbol{E}$ are shown in bold because momentum is a vector. (I don't know how to put both a dot and an arrow over the $\boldsymbol{E}$!) Substituting $\vec{i}_{mom} = \vec{V}$, and making the assumption of steady flow so that the accumulation term is zero, we obtain:

$$0 = -\int_{CS} \vec{V} \rho \vec{V} \bullet d\vec{A} + \dot{\boldsymbol{E}}_{mom} \qquad (3)$$

If we are dealing with a system in which the flows across the control surface can be represented as a limited number (k) of discrete 1-D flows, the equation can be simplified further by replacing the integration with a summation over the various inlets and outlets:

$$0 = -\sum_k \left(\vec{V} \rho \vec{V} \bullet \vec{A} \right)_k + \dot{\boldsymbol{E}}_{mom} \qquad (4)$$

$$\dot{\boldsymbol{E}}_{mom} = \sum_k \left(\vec{V} \rho \vec{V} \bullet \vec{A} \right)_k = \sum_k \left(\vec{V} \rho Q_{tot} \cos \theta_{VA} \right)_k \qquad (5)$$

where Q_{tot} and θ_{VA} are the overall volumetric flow rate and the angle of the velocity with the area vector ($\vec{A}$), respectively, at each inlet or outlet k. If the flow is perpendicular to the CV boundary at each inlet and outlet, $\cos \theta_{VA}$ is 1.0 at the outlets and -1.0 at the inlets, yielding:

$$\dot{\boldsymbol{E}}_{mom} = \sum \left[\left(\rho Q \vec{V} \right)_{outlets} - \left(\rho Q \vec{V} \right)_{inlets} \right] \qquad (6)$$

If the fluid is incompressible, ρ can be taken outside the summation signs in any of the preceding equations. Finally, if the fluid is incompressible and the CV has only one inlet and outlet (so that $Q_{in} = Q_{out}$):

$$\dot{\boldsymbol{E}}_{mom} = \rho Q \left(\vec{V}_{out} - \vec{V}_{in} \right) \qquad (7)$$

Although the preceding equations are written in vector notation, they are most often applied to only one directional component of momentum at a time. For instance, applying Equations 5, 6, and 7 to x-directed momentum, we would write:

$$\dot{E}_{mom,x} = \sum_k \left[\rho V_x \left(AV \cos \theta_{VA} \right) \right]_k = \sum_k \left[\rho V_x Q_{tot} \cos \theta_{VA} \right]_k \qquad (8)$$

$$\dot{E}_{mom,x} = \sum \left[\left(\rho V_x Q \right)_{outlets} - \left(\rho V_x Q \right)_{inlets} \right] \qquad (9)$$

$$\dot{E}_{mom,x} = \rho Q \left(V_{out,x} - V_{in,x} \right) \qquad (10)$$

where θ_{VA}, is the angle of the velocity with the area vector ($\vec{\mathbf{A}}$) at location k.

It is worth noting the two different ways that V appears in Equation 8. The term $AV \cos \theta_{VA}$ represents the volumetric flow rate across the boundary of the CV at location k, with the product $V \cos \theta_{VA}$ corresponding to the component of the velocity that is perpendicular to the boundary. On the other hand, the term V_x is the component of the velocity in the x direction at location k and appears in the equation because we are computing the change in x-directed momentum; V_x equals $V \cos \theta_{Vx}$ and thus might be a different component of velocity than $V \cos \theta_{Vx}$. Errors associated with using an incorrect component of velocity or in the sign associated with a particular flow are among the most common errors when using the RTT for momentum.

The above equations all represent ways to compute the rate of advective momentum transport into a specified CV, with some involving application of simplifications that are frequently justified. The key to using these equations in a practical way is recognition of the fact that, by Newton's second law, the force that is applied to a mass equals the rate at which momentum is transferred to that mass. In the situation of interest to us, we don't specify the mass explicitly, but rather just state that it is whatever mass is within the CV at a given instant. Correspondingly, the force applied to the mass in the CV can be identified as the rate at which that mass acquires momentum, i.e., it is $\dot{E}_{mom}$. Therefore, we can write:

$$\sum \vec{\mathbf{F}}_{ext} = \dot{E}_{mom} \qquad (11)$$

where $\sum \vec{\mathbf{F}}_{ext}$ is the sum of all external forces applied to the CV.

Over a short time dt, the amount of momentum transferred into the CV can be expressed as:

$$d \left(m\vec{\mathbf{V}} \right) = \dot{E}_{mom} \, dt = \left(\sum \vec{\mathbf{F}}_{ext} \right) dt \qquad (12)$$

The product of a force and the time over which it is exerted is called an *impulse*, so the two preceding equations are referred to as two versions of the *impulse-momentum principle*. By

3

substituting $\sum \vec{\mathbf{F}}_{ext}$ for $\dot{E}_{mom}$ in the RTT, we obtain expressions that can be used to determine the forces that moving fluids exert on their surroundings, and vice versa. For example, for a system to which Equation 7 applies, we find:

$$\sum \vec{\mathbf{F}}_{ext} = \rho Q \left(\vec{\mathbf{V}}_{out} - \vec{\mathbf{V}}_{in} \right) \tag{13}$$

Equation 13 indicates that, for a system with a single inlet and a single outlet and an incompressible fluid, the external force *on the fluid* equals the product on the right-hand side of the equation. Correspondingly, an equal and opposite force is exerted *by the fluid* on whatever is holding it.

Note that both the force and velocity are vectors and must be dealt with using the usual rules of vector arithmetic. Writing Equation 13 for the components of force and velocity in each coordinate direction, we obtain:

$$\sum F_x = \rho Q \left(V_{out,x} - V_{in,x} \right) \tag{14a}$$

$$\sum F_y = \rho Q \left(V_{out,y} - V_{in,y} \right) \tag{14b}$$

$$\sum F_z = \rho Q \left(V_{out,z} - V_{in,z} \right) \tag{14c}$$

One way to think about the impulse-momentum equation is to consider that forces are like "momentum pumps" – they add momentum into a fluid, in the same way the real pumps can add energy to a fluid. Correspondingly, forces that are applied by a fluid on an external object are like turbines – they represent ways that momentum is extracted from a fluid. The key difference between energy and momentum in this regard is that momentum is a vector, so that forces add or extract momentum in a specific direction, and not just into or out of the fluid in a general sense.

By combining the impulse-momentum equation with the RTT applied to mass (i.e., the continuity equation), we are able to solve many complex fluids problems, often obtaining results that we would be hard-pressed to reach intuitively. Before demonstrating this capability with some examples, one other point is worth noting. If any force is exerted on a fluid, the fluid exerts an equal and opposite reaction force. As a result, the force associated with momentum change of a fluid can be interpreted as either an external force applied *to* the fluid (providing momentum to the fluid in the direction of that force), or a force being applied *by* the fluid on its surroundings, thereby depleting the fluid of momentum in the direction of that force. Since the two forces of interest are equal in magnitude and opposite in direction, these two views are identical, but we cannot take both views at once, or we will be "double-counting." For example, if water in a pipe is initially heading in the $+x$ direction and takes a 180° turn, we can say that the pipe is applying a force, and therefore *adding* momentum, to the water, in the $-x$ direction. This force causes the x component of the water's velocity to decrease and eventually become negative. Equivalently, we could explain the same phenomenon by saying that the water is applying a force to the pipe, and therefore losing momentum, in the $+x$ direction. Either analysis leads to the same result. To minimize confusion, we will attempt to always carry out our analyses by considering the external

forces *on* the fluid of interest, and then, if desirable, interpret the result in terms of the force that the fluid exerts on its surroundings.

Example: A flow of 300 L/s of water passes through the vertical 300-to-200-mm diameter reducing pipe bend sketched below. The pressure at the entrance is 70 kPa, the bend has a vertical height of 1.5 m from the centerline of the inlet to the centerline of the outlet, and the volume of the bend is 0.085 m^3. What force (magnitude and direction) is required to anchor the bend in place? Ignore friction.
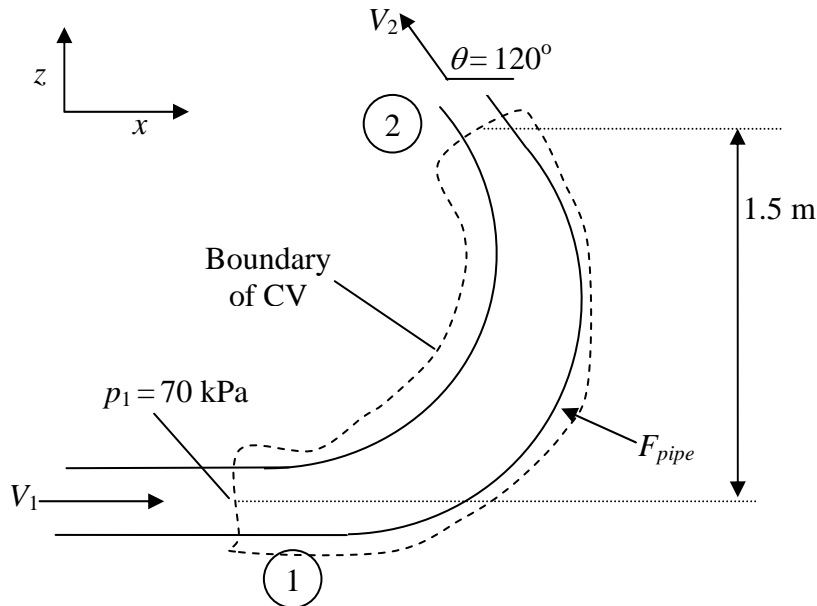
Solution. As the water passes through the reducing pipe, it accelerates in the $-x$ and the $+z$ directions, so it must be experiencing a net force in those directions. The forces on the fluid within the CV, in the horizontal direction, include the horizontal components of the pressure at points 1 and 2 and of the force exerted by the pipe. The pressure at point 1 is given as 70 kPa. The force on the CV at that point can therefore be computed as $p_1 A_1$, and, since the CV is perpendicular to the direction of flow at that point, the force is entirely in the $+x$ direction.

To compute the pressure-based force at point 2, we need to compute the pressure at that point. We expect p_2 to be less than p_1, because between points 1 and 2, the velocity head and elevation head of the water both increase, and these increases in head must be balanced by a corresponding loss of pressure head. The velocity at point 1 can be computed from the flow rate and area, and the velocity at point 2 can then be computed using the continuity equation:

$$V_1 = \frac{Q_1}{A_1} = \frac{0.300 \text{ m}^3/\text{s}}{\pi/4\left(0.300\text{m}\right)^2} = 4.24 \text{ m/s}$$

5

$$V_2 = V_1 \left(\frac{d_1}{d_2}\right)^2 = 4.24 \text{ m/s} \left(\frac{0.300 \text{ m}}{0.200 \text{ m}}\right)^2 = 9.55 \text{ m/s}$$

The pressure at point 2 can then be determined with the Bernoulli equation:

$$\frac{p_1}{\gamma} + z_1 + \frac{V_1^2}{2g} = \frac{p_2}{\gamma} + z_2 + \frac{V_2^2}{2g}$$

$$p_2 = p_1 + \gamma \left[(z_1 - z_2) + \frac{V_1^2 - V_2^2}{2g} \right]$$

$$= 70 \text{ kPa} + 9810 \frac{\text{N}}{\text{m}^3} \left[-1.5 \text{ m} + \frac{\left(4.24\frac{\text{m}}{\text{s}}\right)^2 - \left(9.55\frac{\text{m}}{\text{s}}\right)^2}{2\left(9.81 \text{ m/s}^2\right)} \right] \left(\frac{1 \text{ kPa}}{1000 \text{ N/m}^2}\right)$$

$$= 18.7 \text{ kPa}$$

We can now solve the force balance in the horizontal direction on the water in the CV. The pressure-based force on the CV at point 2 is perpendicular to the surface at that point, so it makes a -60° angle with the $+x$ direction, whereas the velocity at point 2 makes a 120° angle. Applying Equation 14a to this system, we can equate the net, x-directed force with the rate of change in the x-directed momentum:

$$p_1 A_1 \cos\theta_{px,1} + p_2 A_2 \cos\theta_{px,2} + F_{pipe,x} = Q\rho\left(V_2 \cos\theta_{Vx,2} - V_1 \cos\theta_{Vx,1}\right)$$

where the subscripts on θ indicate whether the angle is the one between the velocity or the pressure with the x coordinate. Substituting values:

$$(70,000 \text{ Pa})\left(\frac{\pi(0.300 \text{ m})^2}{4}\right)\cos 0^\circ + (18,700 \text{ Pa})\left(\frac{\pi(0.200 \text{ m})^2}{4}\right)\cos\left(-60^\circ\right) + F_{pipe,x} =$$

$$= \left(0.300\frac{\text{m}^3}{\text{s}}\right)\left(1000\frac{\text{kg}}{\text{m}^3}\right)\left[\left(9.55\frac{\text{m}}{\text{s}}\right)\cos 120^\circ - \left(4.24\frac{\text{m}}{\text{s}}\right)\cos 0^\circ\right]$$

$$F_{pipe,x} = -7946 \text{ N}$$

That is, the force is 7946 N in the $-x$ direction. A similar analysis for the z-directed forces yields:

$$p_1 A_1 \sin\theta_{pz,1} + p_2 A_2 \sin\theta_{pz,2} - \cancel{V}\gamma + F_{pipe,z} = Q\rho\left(V_2 \sin\theta_{Vz,2} - V_1 \sin\theta_{Vz,1}\right)$$

6

$$0+(18,700 \text{ Pa})\left(\frac{\pi(0.200 \text{ m})^2}{4}\right)\sin(-60°)-(0.085 \text{ m}^3)\left(9810\frac{\text{N}}{\text{m}^3}\right)+F_{pipe,x} =$$

$$=\left(0.300\frac{\text{m}^3}{\text{s}}\right)\left(1000\frac{\text{kg}}{\text{m}^3}\right)\left[\left(9.55\frac{\text{m}}{\text{s}}\right)\sin 120° -0\right]$$

$$F_{pipe,z} = 3824 \text{ N}$$

The resultant force and its angle of action can then be computed as follows::

$$F_{pipe} = \sqrt{F_{pipe,x}^2 + F_{pipe,z}^2} = \sqrt{(-7946)^2 + 3824^2}\text{ N} = 8818 \text{ N}$$

$$\alpha_{F_{pipe}} = \arctan\frac{3824}{-7946} = 154.3°$$

Example. The two-dimensional overflow structure shown below (shape and size unspecified) produces the flowfield shown. Calculate the magnitude and direction of the horizontal component of the resultant force the fluid exerts on the structure. Ignore friction.
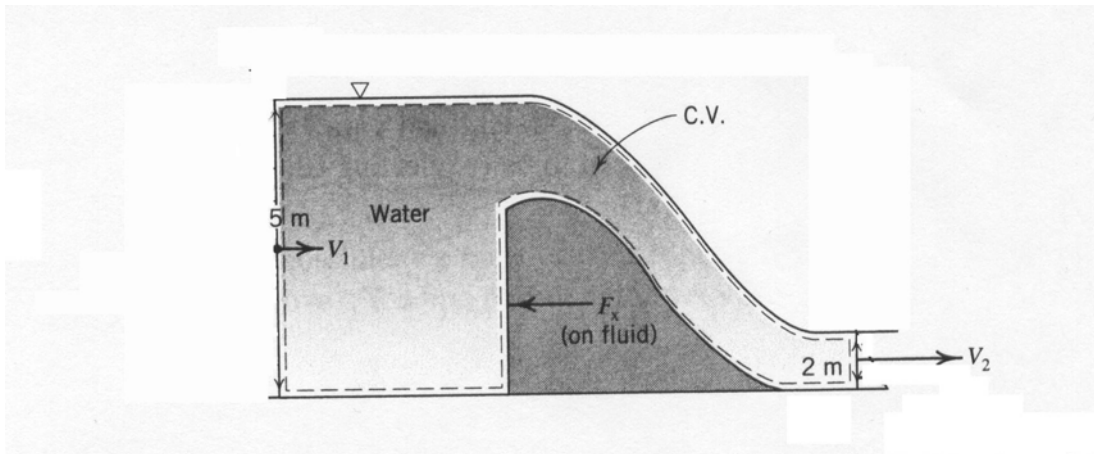
Solution. Applying the Bernoulli equation between the water surface at points sufficiently upstream and downstream of the structure that the flow is horizontal, we can write:

$$\frac{p_1}{\gamma} + z_1 + \frac{V_1^2}{2g} = \frac{p_2}{\gamma} + z_2 + \frac{V_2^2}{2g}$$

$$0 \text{ m} + 5 \text{ m} + \frac{V_1^2}{2g} = 0 \text{ m} + 2 \text{ m} + \frac{V_2^2}{2g}$$

Also, by the continuity equation:

$$V_1\left(5\text{m} * W\right) = V_2\left(2\text{m} * W\right)$$

$$V_2 = 2.5V_1$$

Solving the Bernoulli and continuity equations simultaneously, we obtain:

$$V_1 = 3.33 \text{ m/s} \qquad V_2 = 8.33 \text{ m/s} \qquad \frac{Q}{W} = q = 16.65\frac{\text{m}^3/\text{s}}{\text{m}}$$

Because the streamlines are approximately straight and parallel at the inlet and outlet of the CV, the pressure variation with depth is approximately hydrostatic, so we can compute the pressure-based forces on the two ends of the CV. Noting that the force at the downstream end is in the $-x$ direction, we find:

$$F_{1,x} = \gamma h_{c,1} A_1 = \left(9.81\frac{\text{kN}}{\text{m}^3}\right)\left(\frac{5\text{ m}}{2}\right)\left(5 \text{ m} *W\right)$$

$$\frac{F_{1,x}}{W} = 122.5\frac{\text{kN}}{\text{m}}$$

$$F_{2,x} = -\gamma h_{c,2} A_2 = -\left(9.81\frac{\text{kN}}{\text{m}^3}\right)\left(\frac{2\text{ m}}{2}\right)\left(2 \text{ m} *W\right)$$

$$\frac{F_{2,x}}{W} = -19.6\frac{\text{kN}}{\text{m}}$$

We can now apply the impulse-momentum equation to the water in the CV. The rate of loss of x-directed momentum of water in the CV equals the net x-directed force on that water. The x-directed forces include the pressure-based forces on the ends of the CV and the force of the structure. Thus, normalizing all forces to a unit width:

$$\sum \frac{F_{ext,x}}{W} = \frac{F_{1,x}}{W} + \frac{F_{2,x}}{W} + \frac{F_{x,structure}}{W} = \frac{Q\rho\left(V_{2,x} - V_{1,x}\right)}{W} = q\rho\left(V_{2,x} - V_{1,x}\right)$$

$$122.5\frac{\text{kN}}{\text{m}} - 19.6\frac{\text{kN}}{\text{m}} + \frac{F_{x,structure}}{W} = \left(16.65\frac{\text{m}^3/\text{s}}{\text{m}}\right)\left(1000\frac{\text{kg}}{\text{m}^3}\right)\left(8.33\frac{\text{m}}{\text{s}} - 3.33\frac{\text{m}}{\text{s}}\right)\left(\frac{1\text{ kN}}{1000 \text{ kg-m/s}^2}\right)$$

$$\frac{F_{x,structure}}{W} = -19.65\frac{\text{kN}}{\text{m}}$$

The structure exerts a net force on the water of 19.65 kN/m to the left, so the force of the water on the structure is 19.65 kN/m to the right. The figure characterizing the system, with the additional information derived above and the EL included, is shown below.

$V_1^2/2g_n$ EL

C.V.

$V_2^2/2g_n$

5 m   Water

V_1

F_1
122.5 kN/m

F_x
(on fluid)

2 m

V_2
F_2
19.6 kN/m