

Principles of Ideal Fluid Flow; The Bernoulli and Continuity Equations

Some Key Definitions

We next begin our consideration of the behavior of fluid *dynamics*, i.e., of fluids that are in motion. Initially, we consider *ideal fluids*, defined as those that have zero viscosity (they are *inviscid*). Inviscid fluids experience no resistance to movement, either past solid objects or past adjacent portions of the fluid that are moving at different velocities.

The analysis of such systems starts with a differential volume of fluid that is envisioned to remain as a coherent unit of mass as it moves through the system of interest; such a unit is referred to as a “particle” of fluid. Different particles of fluid in a given system are likely to be moving in different directions and at different velocities at any instant, and a single fluid particle is likely to change directions and speeds over time. Imagine that we took a picture that showed the velocity vectors of all fluid particles in the system at some instant, t_1 . Now imagine that we drew an arrow in the direction of the velocity vector for a particle at the point where it enters the system. Then, we could move a differential distance in the direction of the velocity vector, and draw another arrow in the direction of the velocity vector for the fluid particle that was at that point at t_1 . If we continued this process until we reached the outlet of the system and connected all the differential-length arrows, we would have a continuous line describing the velocity vectors from the given inlet point to a point at the outlet. Such a line is called a *streamline*. A different streamline originates from every point at the system inlet; most streamlines continue to a point at the system outlet, but some can dead-end in the middle of the system by intersecting a surface a solid or a fluid layer that is perpendicular to the fluid particle’s velocity vector and is immobile; such dead-ends are called *stagnation points*.

Streamlines characterize the velocity distribution in the system at a given instant. If a fluid particle started at the inlet end of a streamline, it might not follow that streamline through the system, because by the time it reached some downstream location, the streamline might have shifted from its original position. The path followed by a given fluid particle over time is called a *pathline* (this term is defined in Chapter 4 of Munson).

If streamlines do not change shape or location over time, then the velocity at a given point in the system remains the same from one time to the next. In that case, all fluid particles that pass through a given location will have the same velocity, and particles that enter the system at a given location will proceed along the same pathline, no matter when they enter. Such a system is said to have *steady flow*. In a system with steady flow, the streamlines and pathlines coincide.

The Bernoulli Equation: Concept and Derivation

Now, consider the movement of a particle along a pathline in an ideal fluid, and define distance along the pathline by a coordinate s . A particle of fluid that is at any point on the pathline (s_1) at one instant is at a different point on the same pathline (s_2) at some later time. As it moves from s_1 to s_2 , this particle might undergo changes in three parameters that affect the

amount of energy associated with it: it might accelerate or decelerate (change in v)¹, it might move vertically upward or downward (change in z), or it might experience an increase or decrease in the local pressure (change in p). Specifically, a change in velocity changes the particle's kinetic energy, a change in elevation changes its gravitational potential energy, and a change in pressure changes its "mechanical potential energy." (This latter statement is based on the idea that pressure can be interpreted as the mechanical potential energy of an object per unit volume that it occupies. Note that, if the fluid is compressible, the change in pressure will be accompanied by a change in its volume.) These three forms of energy can be quantified as follows:

$$\text{KE} = \frac{1}{2}mv^2 \quad (1)$$

$$\text{PE}_{\text{grav}} = mgz \quad (2)$$

$$\text{PE}_{\text{mech}} = p\mathcal{V} \quad (3)$$

where m is the mass of the particle and \mathcal{V} is its volume. It is convenient to normalize these energy values to the particle mass, yielding:

$$\frac{\text{KE}}{m} = \frac{1}{2}v^2 \quad (4)$$

$$\frac{\text{PE}_{\text{grav}}}{m} = gz \quad (5)$$

$$\frac{\text{PE}_{\text{mech}}}{m} = \frac{p\mathcal{V}}{m} \quad (6)$$

Because we have specified that the fluid is ideal, it has no viscosity, and therefore no energy is converted to heat (via friction) as the particle travels from s_1 to s_2 . Therefore, if no other process occurring between points s_1 and s_2 adds or removes energy from the particle, the principle of conservation of energy tells us that the total energy of the particle must remain constant between the two locations. Put another way, any increase in one "form" of energy carried by the particle must be balanced by decreases in the other "forms." Writing the sum of the three forms of energy identified above as E_{tot} , we can express the energy conservation principle in either of the following ways:

$$\frac{E_{\text{tot}}}{m} = \frac{p\mathcal{V}}{m} + \frac{1}{2}v^2 + gz = \text{Constant} \quad (7)$$

¹ By convention, a lower case v is used to represent the velocity of a fluid particle or an infinitesimal amount of fluid at a given location, whereas an upper case V is used to represent the average velocity of a macroscopic amount of fluid (e.g., the average velocity of the fluid at the outlet of a pipe).

$$\frac{p_1 \cancel{V_1}}{m} + \frac{1}{2} v_1^2 + g z_1 = \frac{p_2 \cancel{V_2}}{m} + \frac{1}{2} v_2^2 + g z_2 \quad (8)$$

where '1' and '2' refer to locations corresponding to s_1 and s_2 on a single pathline.

If the fluid is incompressible, $\cancel{V_1} = \cancel{V_2} = \cancel{V}$. In that case, we can multiply through by the (fixed) density of the fluid to obtain Equation 9, and if we divide the resulting equation by the specific weight, we obtain Equation 10:

$$\frac{E_{\text{tot}}}{\cancel{V}} = -p_1 + \frac{1}{2} \rho v_1^2 + \gamma z_1 = p_2 + \frac{1}{2} \rho v_2^2 + \gamma z_2 \quad (9)$$

$$\frac{E_{\text{tot}}}{mg} = \frac{p_1}{\gamma} + \frac{1}{2g} v_1^2 + z_1 = \frac{p_2}{\gamma} + \frac{1}{2g} v_2^2 + z_2 \quad (10)$$

Equations 9 and 10 are statements of the principle of conservation of energy as it applies to a single fluid particle at two different locations and two different times. The two equations express the same idea and differ only in that the terms in Equation 9 have dimensions of pressure (or, equivalently, energy per unit volume of the particle), whereas in Equation 10, the terms have dimensions of length (energy per unit weight of the particle).

Certain terms in Equation 9 are referred to by special names: p is called the *static pressure*, γz is called the *hydrostatic pressure*, and the sum $p + \rho v^2/2$ is called the *dynamic pressure*. The static pressure is the pressure one measures if the measuring device moves with the fluid particle or, equivalently, is placed in the flow path in a way that does not cause any kinetic energy to be converted to mechanical energy at the point of measurement. The dynamic pressure, on the other hand, is the pressure that one measures if all the kinetic energy is converted to mechanical energy at the measurement point; this pressure is the largest pressure that can be generated in the fluid at the given elevation.

The terms in Equation 10 are also given special names and are commonly referred to as various forms of *head*: p/γ is called the *pressure head*, $v^2/2g$ is called the *velocity head*, and z is called the *elevation head*. Also, the sum of the elevation head and the pressure head is called the *piezometric head*, and the sum of all three forms of head is called the *total head*. Conceptually, the head represents the elevation gain that could be induced in the fluid if the energy associated with that head were completely converted to gravitational potential energy. Thus, for example, a velocity head of 6 m means that, if the kinetic energy were completely converted to potential energy, the fluid would rise by $\Delta z = 6$ m.

While Equations 9 and 10 are useful conceptually, they are inconvenient to use in a practical way because they require that we keep track of the location of individual fluid particles over time. We can eliminate this requirement by restricting the application of the equations to systems with steady flow. In that case, all the fluid particles that pass through s_1 at any time have identical properties, and the same is true for the particles that pass through s_2 . As a result, an analysis that compares the energy of the particle at s_1 now with the energy of the particle at s_2 now is equivalent to the analysis that would apply to any particle that has flowed from s_1 to s_2 . In

other words, if a system has steady flow, Equations 9 and 10 apply to the (different) particles that are at s_1 and s_2 at any instant, as long as s_1 and s_2 are on the same pathline. For steady flow, the constraint that the particles be on the same pathlines is identical to a constraint that they be on the same streamline. When applied to particles on a single streamline in steady flow, Equations 9 and 10 are both known as the **Bernoulli equation**, and the corresponding, constant value of $E_{\text{tot}} / \mathcal{V}$ or E_{tot} / mg is referred to as the **Bernoulli constant**.

Note that the Bernoulli equation can be used without any knowledge about the detailed path that the fluid particle follows as it travels from point 1 to point 2; all that is required is that both points be on the same streamline in a system with steady flow. Of course, the equation also applies if the distance between points 1 and 2 is differential, i.e., if $s_2 - s_1 = ds$. In that case, the form of the Bernoulli equation shown in Equation 9 can be written as follows:

$$\frac{d}{ds} \left(p + \frac{1}{2} \rho v^2 + \gamma z \right) = 0 \quad (11)$$

Carrying out the differentiation, and applying the fact that v can be written as ds/dt , we find:

$$0 = \frac{dp}{ds} + \rho v \frac{dv}{ds} + \gamma \frac{dz}{ds} = \frac{dp}{ds} + \rho \frac{ds}{dt} \frac{dv}{ds} + \gamma \frac{dz}{ds} = \frac{dp}{ds} + \rho a_s + \gamma \frac{dz}{ds} \quad (12)$$

$$\rho a_s = - \frac{dp}{ds} - \gamma \frac{dz}{ds} \quad (13)$$

where a_s is the acceleration of the particle in the s direction. The term ρa_s is ma_s / \mathcal{V} , so it can be written as F_s / \mathcal{V} , where F_s is the force exerted on the particle in the s direction. Also, dz/ds can be written as $\sin \theta$, where θ is the angle of the streamline with the horizontal. Therefore, we can write:

$$\frac{F_s}{\mathcal{V}} = - \frac{dp}{ds} - \gamma \sin \theta \quad (14)$$

Equation 14 shows that Bernoulli equation can be interpreted as a force balance on the fluid particle, expressing the idea that the net force per unit volume in the s direction (i.e., the force causing any acceleration that occurs along the streamline) equals the sum of the gradients in the static and hydrostatic pressures in that direction.

At this point, it is worth reviewing the assumptions of the derivation that might limit the application of the Bernoulli equation. First, the equation applies only along a pathline, or equivalently, a streamline in a system with steady flow. A second assumption is that no energy is transferred into or out of the particle between s_1 and s_2 , and that the only transformations of energy in the particle between those two points involve conversions among kinetic, mechanical (pressure-based), and gravitational energy. The absence of energy transfers into the particle means that no pumps or other energy-delivering devices are present between s_1 and s_2 , and the absence of energy losses means that no energy-removing devices (e.g., turbines) are between the

two locations and also that there is no energy loss due to friction (i.e., the fluid is inviscid, or ideal). Finally, it was assumed that the fluid was incompressible; this assumption was used when we equated \mathcal{V}_1 to \mathcal{V}_2 in going from Equation 8 to Equations 9 and 10; if the fluid were compressible, we could still carry out a similar analysis, but the mathematics would be more complicated because the mechanical energy would depend on both the pressure and the (variable) particle volume.

As noted above, the Bernoulli equation applies strictly only along a streamline. In general, the total energy per unit volume (as expressed by Equation 9) or per unit weight (as expressed by Equation 10) can change from one streamline to the next, making it inappropriate to apply the equation to the total flow along a collection of streamlines. Most often, of course, we are more interested in macroscopic flow than in the flow of individual fluid particles, so it is useful to identify conditions under which the Bernoulli equation applies at a macroscopic scale. Those conditions are, in fact, relatively easy to identify: if the Bernoulli constant is identical for all the streamlines in the system, then all the fluid particles anywhere in the system have the same energy per unit volume or weight, and the Bernoulli equation can be applied to a collection of streamlines equally as well as to a single streamline.

A physical system in which such a condition applies (approximately) is one in which, at some point, the fluid is contained in a reservoir and has negligible velocity (more precisely, if the velocity head is negligible compared to the piezometric head). In that case, the changes in pressure and elevation anywhere in the reservoir correspond to those for a static fluid

($\Delta p = -\Delta[\gamma z]$), so the Bernoulli constant has the same value (equal to $p + \frac{1}{2}\rho v^2 + \gamma z$) for all

the fluid (Figure 1). Other physical arrangements that lead to the same outcome are discussed below. In this course, we consider only systems in which all the fluid entering the system has equal energy per unit volume or weight (or, if fluid enters the system at multiple points, the energy per unit volume or weight is equal for all the fluid entering at each location).

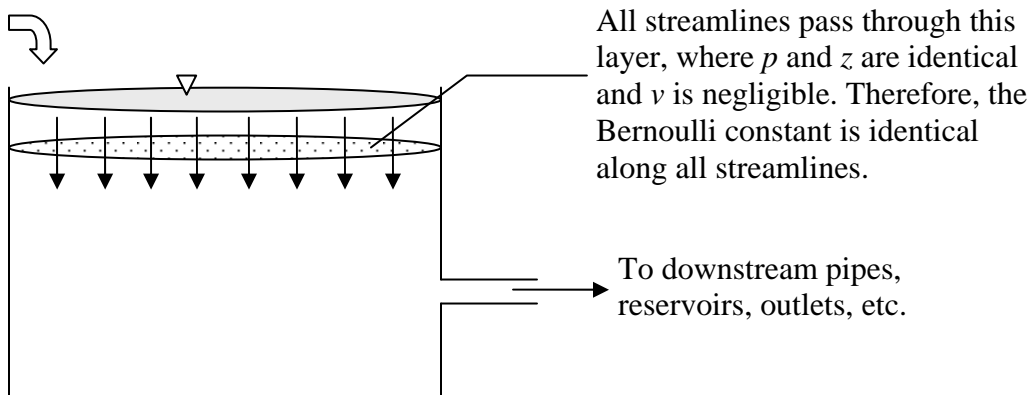


Figure 1. A model system in which the Bernoulli constant is identical for all streamlines.

Piezometers and Pitot Tubes

A **piezometer tube** is a tube that is placed in a fluid system such that the opening in the tube is tangent to the flow. Fluid will rise in the tube until the pressure exerted from the inside of the tube into the system equals that exerted from the system into the tube. That pressure equals the static pressure of the fluid in the system (p). Correspondingly, the height to which the fluid will rise equals the pressure head in the system at that location (p/γ). Also, the elevation of the fluid in the tube corresponds to the value of z at the location where the tube is inserted, plus the value of p/γ at that point; i.e., it equals the value of the piezometric head at that location in the system. A curve drawn through the locus of points representing the piezometric head at every point in the system is referred to as the **hydraulic grade line (HGL)**; a curve drawn through the locus of points representing the total head is called the **energy line (EL)**.

On the other hand, if the opening of the tube is directed into the flow at the same location, a stagnation point will develop at the entrance to the tube. Such a tube is called a **Pitot tube**. The pressure at the stagnation point will be the dynamic pressure of the fluid, because at that point the kinetic energy of the upstream fluid will all have been converted to mechanical potential energy. Correspondingly, fluid will rise in the Pitot tube to a height equal to the sum of the pressure head and the velocity head of the fluid that is passing by the tube (far enough away that it is not affected by the tube). The difference in fluid height between a piezometer tube and a Pitot tube inserted into the system at the same location thus indicates the local velocity head, from which the fluid velocity can be determined.

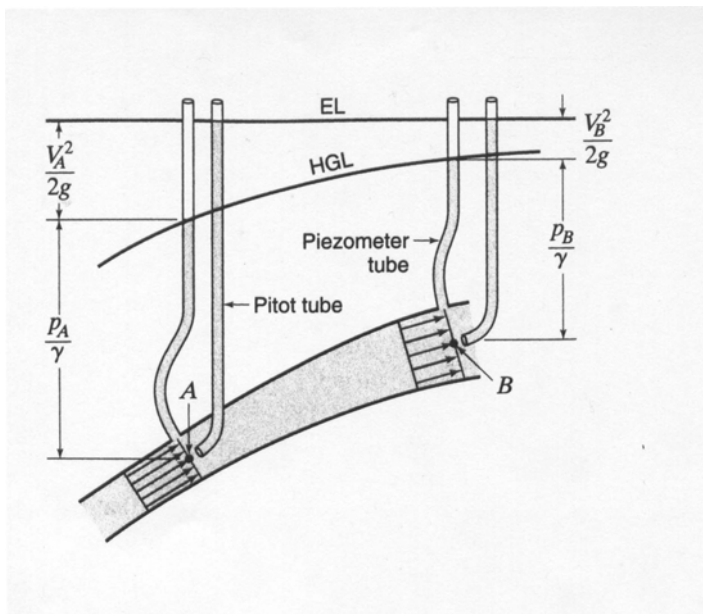


Figure 2. Pitot tubes can be used to determine the height of the HGL and the EL at a given location in a fluid.

Steady Flow with Straight, Parallel Streamlines

Consider a system in which the streamlines are straight and parallel, such as in a section of a pipe with a constant cross-section, or flow between two flat plates aligned parallel to one

another. A schematic of a finite fluid element that is aligned with the cross-section of the flow in such a system is shown in Figure 3. The dimensions of this element are δs in the direction of flow, δl in the direction perpendicular to the boundaries above and below the fluid, and δw in the third coordinate direction (into the page, in the figure).

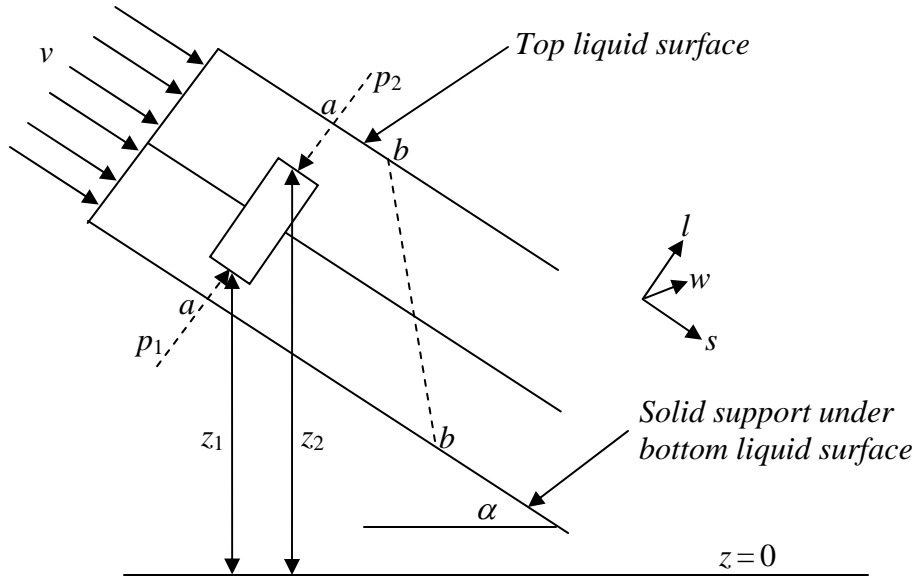


Figure 3. Definition diagram for a system with straight parallel streamlines.

Because the streamlines are straight and parallel, the element remains a fixed distance from the lower boundary as it moves, so the acceleration, and therefore the force, on the element in the l direction must be zero. Now consider a section with the same dimensions as the element in the s and w directions, but only differentially long in the l direction (i.e., an element with dimensions $\delta s * \delta w * dl$). The force balance in the $+l$ direction on this section is as follows:

$$-\left(p + \frac{\partial p}{\partial l} dl\right)(\delta s \delta w) + p(\delta s \delta w) - \gamma(\delta s \delta w dl) \cos \alpha = 0$$

$$-dp = \gamma(dl) \cos \alpha$$

$$dp = -\gamma dz$$

Then, integrating from point 1 to point 2, we find:

$$p_2 - p_1 = \gamma(z_1 - z_2) \quad (15)$$

The result indicates that the pressure distribution across the cross-section is the same as it would be in a static fluid. That is, if we follow a line perpendicular to the streamlines, the pressure increases with the vertical distance from the top of the liquid in proportion to γ . For this reason, the pressure distribution across the cross-section is referred to as *hydrostatic*, even though it applies to a fluid in motion. Rearranging Equation 15 and dividing through by γ , we see that it implies that the sum of the pressure and elevation heads is a constant across the cross-section of flow:

$$\frac{p_1}{\gamma} + z_1 = \frac{p_2}{\gamma} + z_2 \quad (16)$$

Equation 16 implies that the piezometric head is identical for all the streamlines across the system cross-section. However, as noted previously, an implicit assumption applicable to all systems we will consider is that the Bernoulli constant is the same for streamlines in a system. Given that the Bernoulli constant is the sum of the piezometric head and the velocity head, we conclude that the velocity head must be identical for all the streamlines in Figure 3, i.e., *if the Bernoulli equation applies to a system, then over any section of the system where the streamlines are straight and parallel, the velocity must be the same along every streamline.*

Now, consider another section through the flow shown in Figure 3, but this time consider a section that is not perpendicular to the boundaries (e.g., section *b-b*). That section cuts across all the streamlines in the system, all of which have the same Bernoulli constant. Furthermore, the velocity (and the velocity head) is identical along all the streamlines. Therefore, the piezometric head must also be identical for all the streamlines. Thus, we can conclude that *for systems in which the Bernoulli equation applies and in which the streamlines are straight and parallel, the piezometric head is the same everywhere in the system; equivalently, the pressure distribution is hydrostatic everywhere in the system.* Because the piezometric head is identical everywhere in such systems, we expect the fluid in a piezometer tube to rise to the same height if the tube is inserted into a region where the streamlines are straight and parallel, regardless of where in that region the tube is inserted. This is a useful result, because it means that we can insert a piezometer tube anywhere in a pipe with a constant cross-section, and the measured piezometric head will be the same as would be found at any other location in that pipe, i.e., we need not worry about placing the tube in a particular location.

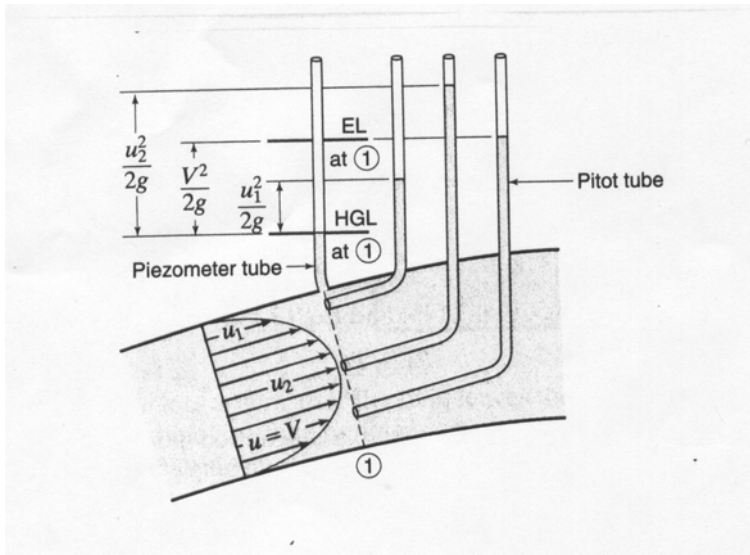


Figure 4. The pressure distribution across the cross-section of flow in a straight section of pipe is hydrostatic, so the HGL is the same at every point in that cross-section. For ideal fluids, the velocity is also constant across the cross-section, so the EL is the same at every point as well. However, for real fluids, the velocity varies across the cross-section, as shown in this figure.

The Continuity Equation for an Incompressible Fluid in Steady Flow

Often, it is useful to combine an equation for conservation of mass with the Bernoulli equation (which characterizes conservation of energy) to analyze systems with flow. To do so, consider the volume occupied by all the fluid in a system of interest between locations 1 and 2, with all the flow entering that volume at point 1 and leaving at point 2. The mass flow rate (mass/time) entering this volume is $\rho_1 Q_1$, and that leaving is $\rho_2 Q_2$, where Q is the volumetric flow rate (volume/time). Also, designating the cross-sectional area as A , we can define the average velocity perpendicular to A as $V = Q/A$.

If the Bernoulli equation applies between these two locations, the fluid must be incompressible, and the flow must be steady. To meet these constraints, the fluid density must be constant ($\rho_1 = \rho_2 = \rho$), and the mass flow rate into the volume at 1 must be matched by the mass flow rate out at 2. Thus:

$$\rho Q_1 = \rho Q_2$$

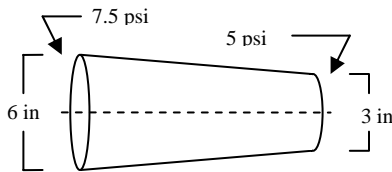
$$V_1 A_1 = V_2 A_2 \tag{17}$$

Equation 17 indicates that, in a system to which the Bernoulli equation applies, the product of the average velocity of the fluid perpendicular to the system boundary and the cross-sectional area of that boundary must be the same at the entry and the exit of the system. As noted above, this equation, known as the *continuity equation*, is a statement of the principle of conservation of mass. It is derived here for somewhat restrictive conditions; we will derive a more general statement of the same idea later in the course. The main consequence of Equation 17 is that,

between two locations that meet the requirements for the Bernoulli equation to apply, the cross-sectional area of the flow must increase in proportion to any decrease in the flow velocity, and vice versa.

One simple application of the continuity equation in conjunction with the Bernoulli equation is in the analysis of two sections of pipe with different diameters. Assuming that the diameter in each section of the pipe is constant, the pressure distribution is hydrostatic at each location. However, by the continuity equation, the velocity in the smaller-diameter section must be greater than that in the larger-diameter section. As a result, the velocity head is also greater in the smaller-diameter section and, since the Bernoulli equation tells us that the total head must be the same in both locations, the piezometric head must be larger in larger-diameter section. Thus, the HGL drops when the diameter of a pipe decreases, and the HGL rises when the diameter increases.

Example. A straight, horizontal water pipe changes diameter from 6 in. at the inlet to 3 in. at the outlet. If the pressures are 7.5 psi at the inlet and 5.0 at the outlet, find the flow rate at 70°F. Ignore friction.



Solution. Because the water is assumed to be frictionless and the flow is completely enclosed, the velocity along all streamlines is the same across any given cross-section. Therefore, we can treat the velocity on any streamline as the average velocity of the whole flow at that location. Designating the inlet as point 1 and the outlet as point 2, we can apply the continuity equation for incompressible flows to obtain:

$$Q = A_1 v_1 = A_2 v_2$$

$$\left(\frac{\pi (6 \text{ in})^2}{4} \right) v_1 = \left(\frac{\pi (3 \text{ in})^2}{4} \right) v_2$$

$$\frac{v_2}{v_1} = 4$$

We can also apply Bernoulli's equation. Noting that, at 70°F, γ_{water} is 62.3 lbs/ft³ and that the pipe is horizontal (so $z_1 = z_2$):

$$\frac{p_1}{\gamma} + z_1 + \frac{v_1^2}{2g} = \frac{p_2}{\gamma} + z_2 + \frac{v_2^2}{2g}$$

$$\frac{p_1 - p_2}{\gamma} = (\cancel{z_2} - \cancel{z_1}) + \frac{v_2^2 - v_1^2}{2g} = \frac{(4v_1)^2 - v_1^2}{2g} = \frac{15v_1^2}{2g}$$

$$v_1 = \sqrt{\frac{2g(p_1 - p_2)}{15\gamma}} = \sqrt{\frac{2(32.2 \text{ ft/s}^2)(2.5 \text{ psi})(144 \text{ in}^2/\text{ft}^2)}{15(62.3 \text{ lbs/ft}^3)}} = 4.98 \text{ ft/s}$$

$$Q = A_1 v_1 = \left(\frac{\pi(0.5 \text{ ft})^2}{4} \right) (4.98 \text{ ft/s}) = 0.978 \text{ ft}^3/\text{s}$$

Free Jets

Free jets represent a different limiting case from flow with straight and parallel streamlines: whereas in systems with straight and parallel streamlines the velocity is constant everywhere and the pressure varies with location, in free jets the pressure is atmospheric everywhere and the velocity varies with location. The characteristics of a free jet are shown schematically in Figure 5.

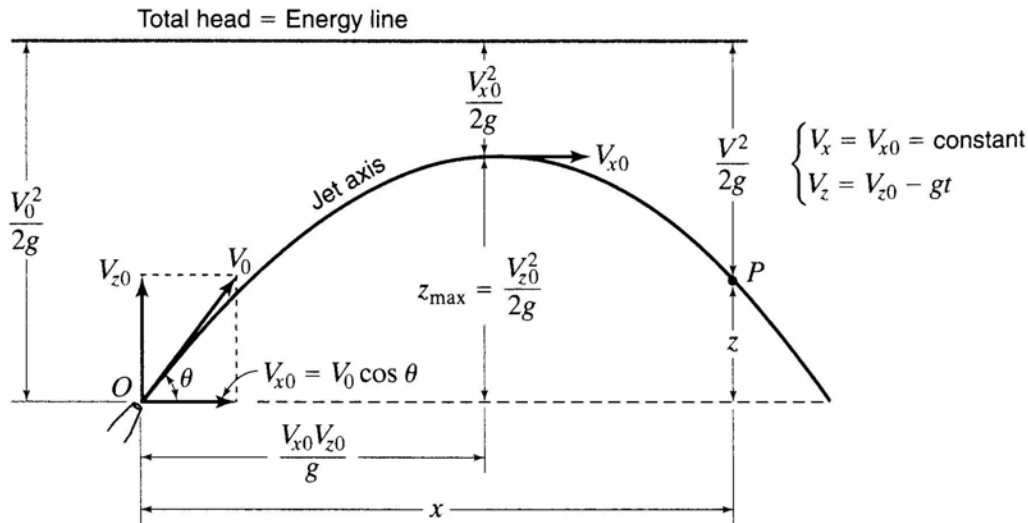


Figure 5. Characteristic behavior of a free jet, based on the Bernoulli equation.

We can apply the Bernoulli equation to free jets as follows. Defining the location where the jet is formed as 1 and any other point on its trajectory as point 2, we find:

$$\frac{p_1}{\gamma} + \frac{1}{2g} v_1^2 + z_1 = \frac{p_2}{\gamma} + \frac{1}{2g} v_2^2 + z_2$$

$$v_2^2 - v_1^2 = 2g(z_1 - z_2) \quad (18)$$

Writing the velocities in terms of their components, and noting that the horizontal component of the velocity, v_x , is constant (so $v_{x,1} = v_{x,2}$):

$$z_1 - z_2 = -\frac{(v_{x,2}^2 + v_{z,2}^2) - (v_{x,1}^2 + v_{z,1}^2)}{2g} = \frac{v_{z,2}^2 - v_{z,1}^2}{2g}$$

At the highest point in the trajectory, the vertical velocity ($v_{z,2}$) is zero, so:

$$\Delta z_{\max} = \frac{v_{z,1}^2}{2g} \quad (19a)$$

$$v_{z,1} = \sqrt{2g\Delta z_{\max}} \quad (19b)$$

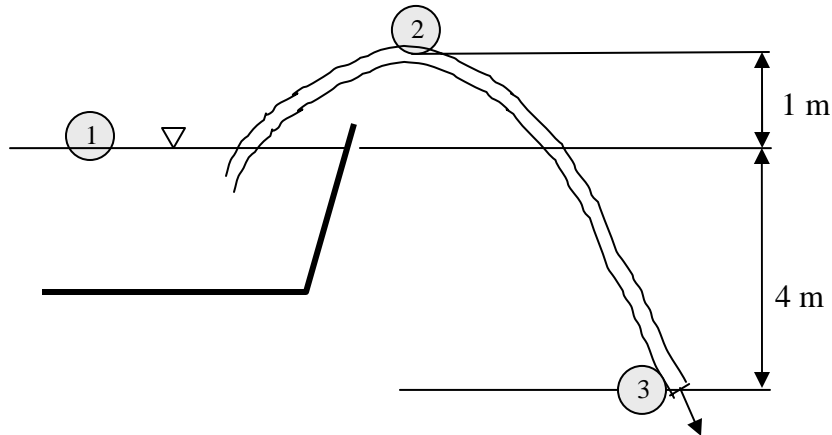
The fluid's vertical velocity is declining steadily at a rate g , so the time required for to reach the apex of its trajectory, and the corresponding horizontal distance traveled, are as follows:

$$t_{\text{apex}} = \frac{v_{z,1}}{g} = \sqrt{\frac{2\Delta z_{\max}}{g}} \quad (20)$$

$$x_{\text{apex}} = v_{x,1} \sqrt{\frac{2\Delta z_{\max}}{g}} = v_x \sqrt{\frac{2\Delta z_{\max}}{g}} \quad (21)$$

The velocity of the fluid in a free jet increases as the fluid falls. By continuity, then, the cross-section of the jet must decrease as the fluid falls (so that VA remains constant).

Example. A siphon operates as shown below. The hose has a uniform diameter of 150 mm. Find the discharge in m^3/s and the pressure head (p/γ) at point 2. Assume that the fluid is frictionless.



Solution. As in the previous example, the assumption that the fluid is frictionless means that we can treat the flow as being identical along all the streamlines. The discharge can be determined by carrying out an energy balance between points 1 and 3. We define the datum for elevation to be at point 3, so that the elevations at points 1 and 2 are 4 m and 5 m, respectively. At point 1, the velocity of the fluid is negligible, and at both points 1 and 3, the pressure is atmospheric. Therefore, writing the equation using gage pressures, we have:

$$\left(\frac{p}{\gamma} + z + \frac{v_{avg}^2}{2g} \right)_1 = \left(\frac{p}{\gamma} + z + \frac{v_{avg}^2}{2g} \right)_3$$

$$0 + 4 \text{ m} + 0 = 0 + 0 + \frac{v_{avg}^2}{2g}$$

$$v_{avg} = \sqrt{2g(4 \text{ m})} = \sqrt{2(9.8 \text{ m/s}^2)(4 \text{ m})} = 8.85 \text{ m/s}$$

$$Q = v_{avg} A = (8.85 \text{ m/s}) \left(\frac{\pi (0.15 \text{ m})^2}{4} \right) = 0.156 \text{ m}^3/\text{s}$$

Note that point 1 is taken away from the hose, where the velocity is negligible and the pressure is atmospheric. If we took point 1 to be in the hose at the level of the reservoir, the velocity would be 8.85 m/s, and the pressure would be less than atmospheric.

The pressure head at point 2 can be determined using a similar energy balance between points 2 and 3. By the continuity equation for incompressible fluids, the velocities must be identical at these two points, so:

$$\left(\frac{p}{\gamma} + z + \frac{v_{avg}^2}{2g} \right)_2 = \left(\frac{p}{\gamma} + z + \frac{v_{avg}^2}{2g} \right)_3$$

$$\frac{p_2}{\gamma} + 5 \text{ m} + \frac{(8.85 \text{ m/s})^2}{2(9.8 \text{ m/s}^2)} = 0 + 0 + \frac{(8.85 \text{ m/s})^2}{2(9.8 \text{ m/s}^2)}$$

$$\frac{p_2}{\gamma} = -5 \text{ m}$$

The negative pressure means that the absolute pressure in the hose at point 2 is less than atmospheric. As a consequence, if a hole were poked in the hose at point 2, air would enter the hose from the atmosphere, rather than water squirting out into the air.

Plumes (not covered in class and not required reading – just for interest)

A similar analysis applies to plumes of gases in gases or liquids in liquids, as long as there is a density difference between the two fluids and frictional energy losses are negligible. Also, in the case of liquid-in-liquid plumes, the pressure changes that occur as the plume rises or falls must be taken into account. In that case, the Bernoulli equation can be manipulated as follows:

$$\left(\frac{p_1 - p_0}{\gamma} \right)_{\text{plume}} + (z_1 - z_0) + \frac{v_1^2 - v_0^2}{2g} = 0 \quad (22)$$

where the subscript “plume” has been added to indicate that the term refers to the liquid that is being discharged. Since the pressure in the plume is the same as that of the surrounding fluid, we can manipulate the first term as follows:

$$\left(\frac{p_1 - p_0}{\gamma} \right)_{\text{plume}} = \frac{(p_1 - p_0)_{\text{surround}}}{\gamma_{\text{plume}}} = \frac{\gamma_{\text{surround}} (h_1 - h_0)}{\gamma_{\text{plume}}} = - \frac{\gamma_{\text{surround}} (z_1 - z_0)}{\gamma_{\text{plume}}}$$

where “surround” refers to the surrounding fluid, h is measured downward from the surface of the surrounding fluid, and z is measured upward from the datum. Substituting this result into the Bernoulli equation, we find:

$$\begin{aligned} - \frac{\gamma_{\text{surround}} (z_1 - z_0)}{\gamma_{\text{plume}}} + (z_1 - z_0) + \frac{v_1^2 - v_0^2}{2g} &= 0 \\ \left(1 - \frac{\gamma_{\text{surround}}}{\gamma_{\text{plume}}} \right) (z_1 - z_0) + \frac{v_1^2 - v_0^2}{2g} &= 0 \\ (z_1 - z_0) + \frac{v_1^2 - v_0^2}{2 \left(1 - \frac{\gamma_{\text{surround}}}{\gamma_{\text{plume}}} \right) g} &= 0 \end{aligned} \quad (23)$$

$$(z_1 - z_0) + \frac{v_1^2 - v_0^2}{2g'} = 0 \quad (24)$$

where:

$$g' = \left(1 - \frac{\gamma_{surround}}{\gamma_{plume}} \right) g = \left(1 - \frac{\rho_{plume}}{\rho_{surround}} \right) g = \left(1 - \frac{s \cdot g \cdot plume}{s \cdot g \cdot surround} \right) g \quad (25)$$

Equation 24 is essentially identical to Equation 18, with the exception of having g' instead of g as the acceleration. This makes sense, since the buoyancy of the surrounding fluid counteracts gravity and causes the plume to fall more slowly, or to rise, when the discharge is into a second liquid instead of into a gas phase.

Example. Sewage discharges into the ocean from a horizontal outfall pipe at a depth of 120 ft. When the ocean is still, the plume reaches the surface at a point that is 95 ft horizontally from the end of the pipe. The ocean water has a specific gravity of 1.03. Neglecting mixing and friction, find the velocity of the plume as it exits the pipe.

Solution. The trajectory in this system is like that of water discharging into the atmosphere, flipped vertically. The given information is equivalent to the conditions at the apex, and the desired information corresponds to what would be the velocity at the apex. We can therefore use a modified version of Equation 21, as follows:

$$v_x = x_{apex} / \sqrt{\frac{2\Delta z_{max}}{g'}}$$

$$= 95 \text{ ft} / \sqrt{\frac{2(-120 \text{ ft})}{(1-1.03)(32.2 \text{ ft/s}^2)}} = 6.03 \text{ ft/s}$$

Flow in Open Channels

Flow which is supported from below but is at atmospheric pressure at its upper surface represents an intermediate case between flow in closed pipes and free jets. Such flow is referred to as *open channel flow* and is addressed briefly in the latter part of this course and in CEE 345. In open channel flow, the pressure at the top surface is constant at atmospheric pressure. If the flow is down an incline, then the fluid accelerates as it flows and, by continuity, its cross section must decrease. Thus, as the fluid flows down an open channel, its velocity increases, its cross-section decreases, and the pressure on its upper surface remains constant. The streamlines in such a system cannot be exactly parallel (they must converge to cause the cross-section to decline), so the pressure distribution is not hydrostatic throughout the system, as it is in pipe flow. However, over distances that are short enough that the convergence of the streamlines is negligible, the analysis presented previously for pipe flow is approximately valid. Therefore, it is usually assumed that the pressure distribution perpendicular to the streamlines (e.g., section *a-a* in Figure

3) is hydrostatic, even though that along other sections (e.g., section *b-b*) is not. In open channel flow where the flow is strictly horizontal, there is no convergence of the streamlines, and the pressure distribution is expected to be hydrostatic.