1. **Diffeomorphisms**

We want to show that, under infinitesimal diffeomorphisms, a rank $(0,2)$-tensor can be equivalently expressed using either covariant or partial derivatives, i.e.

$$\delta F_{\alpha\beta} = F_{\alpha\beta;\gamma}\,\xi^\gamma + F_{\gamma\beta}\,\xi^\gamma{}_{;\alpha} + F_{\alpha\gamma}\,\xi^\gamma{}_{;\beta} = F_{\alpha\beta,\gamma}\,\xi^\gamma + F_{\gamma\beta}\,\xi^\gamma{}_{,\alpha} + F_{\alpha\gamma}\,\xi^\gamma{}_{,\beta}$$

To do so, we express the covariant derivatives in terms of partial derivatives and Christoffel symbols

$$\begin{aligned} F_{\alpha\beta;\gamma} &= F_{\alpha\beta,\gamma} - \Gamma^\delta{}_{\alpha\gamma} F_{\delta\beta} - \Gamma^\delta{}_{\beta\gamma} F_{\alpha\delta}, \\ \xi_{\gamma;\alpha} &= \xi_{\gamma,\alpha} - \Gamma^\delta{}_{\gamma\alpha}\xi_\delta \\ \xi^\gamma{}_{;\alpha} &= \xi^\gamma{}_{,\alpha} + \Gamma^\gamma{}_{\delta\alpha}\xi^\delta. \end{aligned}$$

Then the variation can be expressed as

$$\begin{aligned} \delta F_{\alpha\beta} &= F_{\alpha\beta,\gamma}\,\xi^\gamma + F_{\gamma\beta}\,\xi^\gamma{}_{,\alpha} + F_{\alpha\gamma}\,\xi^\gamma{}_{,\beta} - \Gamma^\delta{}_{\alpha\gamma} F_{\delta\beta}\,\xi^\gamma - \Gamma^\delta{}_{\beta\delta} F_{\alpha\delta}\,\xi^\gamma + F_{\gamma\beta}\Gamma^\gamma{}_{\delta\alpha}\,\xi^\delta + F_{\alpha\gamma}\Gamma^\gamma{}_{\delta\beta}\,\xi^\delta \\ &= F_{\alpha\beta,\gamma}\,\xi^\gamma + F_{\gamma\beta}\,\xi^\gamma{}_{,\alpha} + F_{\alpha\gamma}\,\xi^\gamma{}_{,\beta} + (F_{\gamma\beta}\Gamma^\gamma{}_{\delta\alpha}\,\xi^\delta - \Gamma^\delta{}_{\alpha\gamma} F_{\delta\beta}\,\xi^\gamma) + (F_{\alpha\gamma}\Gamma^\gamma{}_{\delta\beta}\,\xi^\delta - \Gamma^\delta{}_{\beta\delta} F_{\alpha\delta}\,\xi^\gamma). \end{aligned}$$

After using the symmetry of the Christoffel symbols and renaming dummy indices, the terms in parenthesis cancel; demonstrating that using covariant derivatives in infinitesimal diffeomorphisms is equivalent to using partial derivatives. If we specialize to the case where $F_{\alpha\beta}$ is the metric tensor $g_{\alpha\beta}$, and use metric compatibility of the connection $g_{\alpha\beta;\gamma} = 0$, we find

$$\delta g_{\alpha\beta} = g_{\alpha\beta;\gamma}\xi^\gamma + g_{\gamma\beta}\xi^\gamma{}_{;\alpha} + g_{\alpha\gamma}\xi^\gamma{}_{;\beta} = \xi_{\alpha;\beta} + \xi_{\beta;\alpha}.$$

2. **EM in curved space**

a. **Field Strength**

We want to see that, much like the previous problem, we can replace the partial derivatives in the definition of the field strength tensor, $F_{\alpha\beta} = A_{\alpha,\beta} - A_{\beta,\alpha}$, with covariant derivatives. We can use the above expression for the covariant derivative acting on a rank $(0,1)$-tensor to see that

$$A_{\alpha;\beta} - A_{\beta;\alpha} = A_{\alpha,\beta} - \Gamma^\gamma{}_{\alpha\beta} A_\delta - A_{\beta,\alpha} + \Gamma^\gamma{}_{\beta\alpha} A_\gamma.$$

Now using the symmetry of the Christoffel symbols, we see that the terms involving $\Gamma^\gamma{}_{\alpha\beta}$ cancel. Thus,

$$F_{\alpha\beta} = A_{\alpha,\beta} - A_{\beta,\alpha} = A_{\alpha;\beta} - A_{\beta;\alpha}\,.$$

b. **Action and Maxwell's equations**

Given the Lagrangian density $\mathcal{L} = -\frac{1}{4} F_{\alpha\beta}F^{\alpha\beta}$, the action can be written as (for concreteness specifying a Lorentzian $d = 4$ manifold),

$$S = -\tfrac{1}{4}\int d^4x\,\sqrt{-g}\,F_{\alpha\beta}F^{\alpha\beta}\,.$$

To derive Maxwell's equations in curved space, we vary the above action:

$$\begin{aligned} \delta S &= -\tfrac{1}{2}\int d^4x\,\sqrt{-g}\,F^{\alpha\beta}\,\delta F_{\alpha\beta} = -\tfrac{1}{2}\int d^4x\,\sqrt{-g}\,F^{\alpha\beta}\,(\delta A_{\beta;\alpha} - \delta A_{\alpha;\beta}) \\ &= -\int d^4x\,\sqrt{-g}\,F^{\alpha\beta}\,\nabla_\alpha \delta A_\beta = \int d^4x\,\sqrt{-g}\,\nabla_\alpha F^{\alpha\beta}\,\delta A_\beta\,. \end{aligned}$$

Here we used the result of part (a) to write the variation of the field strength in terms of the variation of the gauge field (which is unconstrained). In the last step, we integrated the covariant derivative by parts (which implicitly uses the fact that the covariant derivative of the metric vanishes), assuming that the variation δA_μ vanishes sufficiently rapidly so that surface terms vanish. Demanding that the variation of the action vanish for arbitrary δA_μ (of compact support) requires that the integrand vanish identically. Thus, the equations of motion — Maxwell's equations (in vacuum) — are

$$\nabla_\alpha F^{\alpha\beta} = 0\,.$$

c. **Stress-Energy Tensor**

Use the standard definition of the stress-energy tensor as a metric variation,

$$T_{\alpha\beta} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta\sqrt{-g}\,\mathcal{L}}{\delta g^{\alpha\beta}}.$$

Applying this to the Maxwell Lagrangian, and using $\delta\sqrt{-g} = -\frac{1}{2}g_{\alpha\beta}\,\delta g^{\alpha\beta}$ gives

$$T_{\alpha\beta} = \tfrac{1}{2}\left(F_{\mu\nu}F_{\rho\sigma}\frac{\delta(g^{\mu\rho}g^{\nu\sigma})}{\delta g^{\alpha\beta}} - \tfrac{1}{2}g_{\alpha\beta}F_{\delta\gamma}F^{\delta\gamma}\right).$$

Computing the remaining variations produces

$$T_{\alpha\beta} = F_{\alpha\gamma}F_\beta{}^\gamma - \tfrac{1}{4}g_{\alpha\beta}F_{\delta\gamma}F^{\delta\gamma}.$$

To establish covariant conservation, we want to see that $\nabla_\alpha T^{\alpha\beta} = 0$. Computing the divergence, using metric compatibility

$$\nabla_\alpha T^{\alpha\beta} = (\nabla_\alpha F^\alpha_{\ \gamma})F^{\beta\gamma} + F^\alpha_{\ \gamma}(\nabla_\alpha F^{\beta\gamma}) - \tfrac{1}{2}g^{\alpha\beta}F_{\gamma\delta}(\nabla_\alpha F^{\gamma\delta})\,.$$

The first term on the right hand side vanishes by the equations of motion. Suitably renaming indices and using the antisymmetry of the field strength, the remaining terms can be written

$$\begin{aligned}\nabla_\alpha T^{\alpha\beta} &= F_{\alpha\gamma}F^{\beta\gamma;\alpha} - \tfrac{1}{2}F_{\gamma\delta}F^{\gamma\delta;\beta} = F_{\alpha\gamma}F^{\beta\gamma;\alpha} + \tfrac{1}{2}F_{\alpha\gamma}F^{\gamma\alpha;\beta} \\ &= \tfrac{1}{2}\,F_{\alpha\gamma}(F^{\beta\gamma;\alpha} - F^{\beta\alpha;\gamma} + F^{\gamma\alpha;\beta}) = \tfrac{1}{2}\,F_{\alpha\gamma}(F^{\beta\gamma;\alpha} + F^{\alpha\beta;\gamma} + F^{\gamma\alpha;\beta}) = 0\,,\end{aligned}$$

where the last step is the Bianchi identity.

d. **Sources**

If we couple the gauge field to a background current density, $j^\alpha(x)$, the change at the level of the action is

$$S \to S_j = S + \int d^4x\sqrt{-g}\,j^\alpha A_\alpha.$$

The resultant change in the equations of motion for A_α are found from using the results in part [b.] for the variation of S as

$$\frac{\delta S_j}{\delta A_\gamma} = \frac{\delta S}{\delta A_\gamma} + \int d^4x\sqrt{-g}\,j^\gamma = 0 \quad \Rightarrow \quad \nabla_\beta F^{\gamma\beta} = j^\gamma\,.$$

As long as the current density is treated as a fixed background field, with some prescribed spacetime dependence, then it is natural to use the original (sourceless) definition of $T^{\alpha\beta}$ and just recompute its divergence. The only change from the result in part [c.] is that the field strength is no longer divergence-free. Consequently, the stress-energy tensor also acquires a non-zero divergence,

$$\nabla_\alpha T^{\alpha\beta} = F^{\beta\gamma} F^{\alpha}{}_{\gamma;\alpha} = j_\gamma F^{\gamma\beta} \,. \tag{1}$$

The right-hand side can (and should) be interpreted as the electromagnetic *force density* — i.e., the force per unit volume exerted by the electromagnetic field on the whatever charges are producing the given current density. The time component of this force density is the power (density), or the rate at which the EM field is doing work (per unit volume) on the external current.

If one were doing a more complete treatment where the charges giving rise to the current density are treated as fully dynamical, with the action containing both EM and matter terms, then it would be appropriate to reapply the definition $T_{\alpha\beta} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta\sqrt{-g}\,\mathcal{L}}{\delta g^{\alpha\beta}}$. to the complete action, leading to a stress-energy tensor with both EM field contributions and matter contributions. This complete theory is, once again, diffeomorphism invariant which implies that the complete stress-energy tensor will be divergence free.

3. **Deflection of light by Sol**

We want to analyze how a massive body deforms incoming null geodesics. Even though the Sun is massive, it still weakly curves the space around it, and so we want to consider the deviation of the path of a photon incoming into the solar system from a distant source. To simplify the model, we consider the Sun to be a static, spherically symmetric object with mass M and radius R such that the external geometry is described by the isotropic Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 + \frac{2M}{r}\right)(dx^2 + dy^2 + dz^2),$$

where $r \equiv |\vec{x}|$. The photon trajectory follows a null geodesic with affine parameter s and tangent vector $p \equiv \frac{dx}{ds}$, which is the 4−momentum of the photon. We orient our coordinate system such that the path of the photon lies on the equatorial plane $z = 0$, and approaches the Sun with impact parameter $y = b$. That is, our initial spatial momentum is in the $\hat{x}$-direction. To measure the deflection of the incoming photon by the gravitational field of the Sun, we want to look at how the geodesic path changes from a straight line trajectory in the $\hat{x}$-direction as measured by an asymptotic observer. The relevant component of the geodesic equation to study is

$$\tfrac{d}{ds}\, p^y + \Gamma^y{}_{\alpha\beta}\, p^\alpha p^\beta = 0\,.$$

From the metric above, we can calculate the connection coefficients as

$$\begin{aligned}
\Gamma^0{}_{0i} &= \frac{Mx_i}{r^2(r-2M)} \approx \frac{Mx_i}{r^3}\,, \\
\Gamma^i{}_{jj} = \Gamma^i{}_{00} &= \frac{Mx_i}{r^2(r+2M)} \approx \frac{Mx_i}{r^3}\,, \qquad i \neq j \\
\Gamma^i{}_{ij} &= -\frac{Mx_j}{r^2(r+2M)} \approx \frac{Mx_j}{r^3}\,,
\end{aligned}$$

where the final forms use the approximation $r \gg 2M$ relevant for physics at (or outside) the surface of the sun. Since the 4-momentum will be dominated by $p^x \approx p^0$ (with $|p^y| \ll p^x$), we may approximate:

$$\tfrac{d}{ds}\, p^y + ({\Gamma^y}_{00} + {\Gamma^y}_{xx})(p^x)^2 \approx 0.$$

where terms like $(p^y)^2$ and $p^x p^y$ were dropped. Using the initial condition $y = b$, and the above listed connection coefficients, we have

$$\tfrac{d}{ds}\, p^y = -\frac{2Mb\,(p^x)^2}{(x^2+b^2)^{3/2}}.$$

For small deflection (which will be the case since $2M_\odot \ll R_\odot$), the x-component of momentum is nearly constant, $p^x \approx \bar{p}$, and hence $x(s) = \bar{p}\, s$. Thus, our equation for p^y becomes

$$p^y(+\infty) = -2Mb\,\bar{p}\int_{-\infty}^{\infty}\frac{dx}{(x^2+b^2)^{3/2}} \qquad \Rightarrow \qquad p^y(+\infty) = -\frac{4\bar{p}M}{b}.$$

So the deflection angle is given by

$$\Delta\phi = \frac{p^y(+\infty)}{p^x(+\infty)} \approx \frac{p^y(+\infty)}{\bar{p}} = -\frac{4M}{b},$$

which was the desired result. (The minus sign indicates deflection *toward* the sun.) In our geometrized units, $M_\odot/R_\odot \approx 2.1\times 10^{-6}$, and so

$$|\Delta\phi| \approx 8.4\times 10^{-6}\,\frac{R_\odot}{b}\,.$$

4. **Gravitational red shift**

To derive the gravitational redshift from the surface of the sun, we first wish to establish that $p_0 = \mathbf{p}\cdot\mathbf{e}_0$ is a constant along the null geodesic followed by a solar photon. Consider the one-form $\tilde{\mathbf{p}}$ with components $p_\mu = g_{\mu\nu}\, p^\nu$. With an affine parameter s defined such that $\mathbf{p} = d\mathbf{x}/ds$, the vector $\mathbf{p}$ satisfies the geodesic equation $\nabla_{\mathbf{p}}\mathbf{p} = 0$, and hence the one-form $\tilde{\mathbf{p}}$ satisfies $\nabla_{\mathbf{p}}\tilde{\mathbf{p}} = 0$ which, in components, reads

$$\frac{dp_\mu}{ds} - {\Gamma^\alpha}_{\mu\beta}\, p_\alpha\, p^\beta = 0\,.$$

So $dp_0/ds = {\Gamma^\alpha}_{0\beta}\, p_\alpha\, p^\beta = \Gamma_{\alpha 0\beta}\, p^\alpha\, p^\beta$. Now use the Christoffel symbol definition, $\Gamma_{\alpha 0\beta} = \frac{1}{2}(g_{\alpha 0,\beta} + g_{\alpha\beta,0} - g_{0\beta,\alpha})$, and the weak field approximation to the Schwarzschild geometry outside the solar surface (again choosing isotropic coordinates),

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1+\frac{2M}{r}\right)(dx^2+dy^2+dz^2)\,,$$

where $r \equiv |\vec{x}|$. The off-diagonal metric components g_{0i} vanish identically, and all components are time-independent. Hence, each term in $\Gamma_{\alpha 0\beta}$ vanishes, proving that $dp_0/ds = 0$.

So the (lowered) component of a photon momentum, $p_0 = \mathbf{p}\cdot\mathbf{e}_0$, is unchanged from when the photon is emitted at the solar surface to when it is detected by an asymptotic observer. The frequency of photon as measured by an observer with 4-velocity $\mathbf{u}$ is given by $\omega = -\mathbf{u}\cdot\mathbf{p}$. This assumes, of course, that the 4-velocity satisfies its proper normalization, $\mathbf{u}\cdot\mathbf{u} = -1$.

The 4-velocity of a static observer at infinity is just $\mathbf{u}_\infty = \mathbf{e}_0 = \partial/\partial t$ while the 4-velocity of a static observer at the solar surface is $\mathbf{u}_{R_\odot} = (1 - \frac{2M_\odot}{R_\odot})^{-1/2}\,\mathbf{e}_0 \approx (1 + \frac{M_\odot}{R_\odot})\,\mathbf{e}_0$. Hence

$$\frac{\lambda_{\rm obs}}{\lambda_{\rm emit}} = \frac{\omega_{\rm emit}}{\lambda_{\rm obs}} = \frac{\mathbf{u}_{R_\odot}\cdot\mathbf{p}}{\mathbf{u}_\infty\cdot\mathbf{p}} = (1 + M_\odot/R_\odot)\frac{\mathbf{e}_0\cdot\mathbf{p}}{\mathbf{e}_0\cdot\mathbf{p}} = (1 + M_\odot/R_\odot)\,.$$

So the redshift

$$z \equiv \frac{\lambda_{\rm obs} - \lambda_{\rm emit}}{\lambda_{\rm emit}} = \frac{M_\odot}{R_\odot}\,,$$

and numerically this is about 2×10^{-6}. This fractional loss of energy may be viewed as the conversion of photon kinetic energy into gravitational potential energy as the photon climbs out of the gravitational potential well of the sun and escapes to the asymptotic detector.