1. **Null Infalling Coordinates**
We start with the metric in the form

$$ds^2 = \tilde{g}_{\mu\nu}(x,r)dx^\mu dx^\nu + 2dx^0 dr.$$

We want to see that curves parameterized by r are null. The obvious place to start is looking at the line element and asking what the proper distance along such curves would be. Since $g_{\mu\nu}$ contains no rr-component and the only place dr enters is in the off-diagonal term $dx^0 dr$, we can conclude that if we hold fixed the values of all other coordinates and parameterize curves by how they vary along r, then $ds^2 = 0$. So radial curves are indeed null. To put it another way, the tangent vector $\xi \propto e_r = \partial/\partial r$ is null since the only non-zero component of this tangent vector is ξ^r and $\xi^2 = \xi^M \xi_M = g_{rr}\,\xi^r\xi^r = 0$. (Here and below, capital letters are being used as spacetime indices running from 0 to $D{-}1$, to distinguish them from "radial slice" indices μ,ν running from 0 to $D{-}2$.) To show that these null curves are geodesics, one needs to solve (for some affine parameter λ along the curve),

$$\frac{d\xi^M}{d\lambda} + \Gamma^M_{NP}\,\xi^N\,\xi^P = 0,$$

with $\xi^M = dx^M/d\lambda$. If the only non-zero component of the tangent vector is ξ^r, then this becomes

$$\frac{d\xi^M}{d\lambda} + \Gamma^M_{rr}\,(\xi^r)^2 = 0.$$

So one only needs to evaluate connection coefficients with the last two indices equal to r. We have

$$\Gamma^M_{rr} = \tfrac{1}{2} g^{MN}\big(2g_{Nr,r} - g_{rr,N}\big) = g^{M0}\partial_r(g_{r0}) = 0\,,$$

since g_{rr} vanishes and $g_{r0} = 1$. Thus, the geodesic equation reduces to $d\xi^M/d\lambda = 0$, which is trivially solved by $\xi^r = \text{const.}$, $\xi^\mu = 0$, and $\lambda = r$. So the curves $x^M(\lambda)$, with $x^r = A + B\lambda$ and all other components constant, are geodesics with r itself as an affine parameter.

If instead one holds fixed the coordinate x^0, how should the resulting hypersurface be described? Going back to the line element, fixing x^0 means

$$ds^2\big|_{const.\ x^0} = \tilde{g}_{ab}(x,r)dx^a dx^b,$$

where the indices a, $b = 1, \cdots, D{-}2$. The radial-slice metric $\tilde{g} = \tilde{g}_{\mu\nu}\,dx^\mu\,dx^\nu$ is, by assumption, a Lorentzian signature $D{-}1$ dimensional metric — meaning that (at every point) one may define a basis with one basis vector timelike and the other $D{-}2$ are spacelike. Although not stated explicitly, assume that $e_0 = \partial/\partial t$ is always timelike. Then the $D{-}2$ spacelike basis vectors $e_a = \partial/\partial x^a$ are all tangent to the $x_0 = \text{const.}$ surface. The full tangent space to any point on the surface $x_0 = const.$ is spanned by these $D{-}2$ basis vectors *plus* the vector $e_r = \partial/\partial r$. As noted above, e_r is null, not spacelike. Hence, the $x_0 = const.$ surface is not a spacelike surface (for which all tangent vectors are spacelike) but rather a null surface. An appropriate definition of a null surface is a surface for which, at every point, there exists *one* tangent vector to the surface which is null, with all other linearly independent tangent vectors spacelike. (And an appropriate definition of a timelike surface is a surface for which, at every point, one may choose an orthogonal basis for the tangent space with one timelike basis vector and all orthogonal vectors spacelike.) For a null surface, note that the null tangent vector, e_r

in our example here, is *also* a normal vector to the surface, since this vector is orthogonal to *all* tangent vectors to the surface! This is a special peculiarity associated with null surfaces. At every point on a $D{-}1$ dimensional null surface (lying within a D dimensional geometry) the space of normal vectors is 2 dimensional, while the tangent space is $D{-}1$ dimensional!

2. **Static, Spherically Symmetric Geometry in Infalling Coordinates**
Our general metric ansatz for a D-dimensional, static, spherically symmetric spacetime described by in-falling (or "Eddington-Finklestein") coordinates is

$$ds^2 = -A(r)dt^2 + \Sigma(r)^2 d\Omega_{D-2}^2 + 2dt\,dr.$$

The condition of a static metric means that there is no explicit t dependence in the components of our metric tensor, or in other words $t \to t + \text{const.}$ leaves the above metric invariant. Spherical symmetry implies no angular dependence in any of the components of the metric tensor, and hence the unknown metric functions A and Σ can only depend on the radial coordinate r. The infalling coordinate system manifests itself in the same way discussed (more generally) in the previous problem: curves along which r varies, with t and the $D{-}2$ angles of the $D{-}2$ sphere fixed, are null geodesics. Fixing t and r, we are left with the metric of just a $D{-}2$ unit sphere, scaled by Σ^2. So the volume of the hypersurface at fixed t, r is just that of a sphere of radius $\Sigma(r)$, namely $\Sigma(r)^{D-2}\mathcal{S}_{D-2}$ where $\mathcal{S}_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$ is the surface area of a unit $n{-}1$ sphere (equal to $2\pi, 4\pi, 2\pi^2, \cdots$ for $n = 2, 3, 4, \cdots$).

Our first goal is to understand the residual diffeomorphism freedom after imposing the above form on the metric. Let θ_a $(a = 1, \cdots, D{-}2)$ denote some set of angular coordinates for the $D{-}2$ dimensional unit sphere, say standard spherical coordinates as in the previous problem set. If one introduces new coordinates $\tilde t, \tilde\theta_a, \tilde r$ related to the original coordinates in some fashion, $t = t(\tilde t, \tilde\theta_a, \tilde r)$, $\theta_b = \theta_b(\tilde t, \tilde\theta_a, \tilde r)$, $r = r(\tilde t, \tilde\theta_a, \tilde r)$, under what conditions will the transformed metric have the form

$$ds^2 = -\widetilde{A}(\tilde r)d\tilde t^2 + \widetilde{\Sigma}(\tilde r)^2\, d\Omega_{D-2}^2 + 2d\tilde t\, d\tilde r,$$

for some new functions $\widetilde{A}$ and $\widetilde{\Sigma}$ depending only on $\tilde r$? Note that if, for example, $\partial t/\partial \tilde r$ is non-zero then dt, when reexpressed in terms of the new variables, will contain a $d\tilde r$ piece. And that means that the dt^2 term in the original metric will induce $d\tilde t\, d\tilde r$ and $d\tilde r^2$ terms in the transformed metric. Preserving the given form of the metric means that there can be no $d\tilde r^2$ term in the result, and the coefficient of the $d\tilde t d\tilde r$ must remain fixed at 2. Any dependence of t on $\tilde r$ will mess this up. Similarly, the metric ansatz contains no $dt\, d\theta_a$ cross-terms, and this will be messed up if $\partial t/\partial\tilde\theta_a$ is non-zero. And if $\partial t/\partial \tilde t$ is not a constant, then this will mess-up the fixed $d\tilde t d\tilde r$ coefficient (and will also be inconsistent with rigid shifts $\tilde t \to \tilde t + \text{const.}$ being a symmetry of the transformed metric). So t and $\tilde t$ must be related by a linear transformation, independent of $\tilde\theta_a$ and $\tilde r$. Continuing with this style of reasoning, one ends up with the following changes of coordinates which *do* preserve the form of the ansatz:

(a) Rescalings of t and r which are inversely related: $t = \kappa\tilde t$, $r = \tilde r/\kappa$;

(b) Constant shifts of t and r: $t = \tilde t + \delta t$, $r = \tilde r + \delta r$;

(c) Static, r-independent transformations of the angular coordinates which correspond to rotations of the sphere.

Now examine the differential equations which must be satisfied by A and Σ in order for this metric to describe a vacuum solution to Einstein's equations. Vacuum solutions are defined such that $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$. Given the highly restrictive spherical and time translational symmetry, one can greatly reduce the number of independent components which must be considered. At this point, one can perform the needed calculations by hand in all generality, but equally acceptable is looking at a sequence of test cases with $D = 3, 4, 5, \cdots$ in your favorite differential geometry package and establishing a pattern. One can quickly see that no matter the value of $D \geq 3$,

$$G_{rr} = -(D-2)\, \Sigma''/\Sigma = 0$$

where primes ($\prime$) denote radial derivatives. Thus, $\Sigma(r)$ must be a linear function of r and one may, without loss of generality, set $\Sigma(r) = r$ as constant radial shifts and rescalings are trivial reparameterizations. Feeding $\Sigma = r$ into the rest of the equations, one finds that

$$\begin{aligned} G_{tt} &= -\tfrac{1}{2}(D{-}2)\, r^{-2}\, A\big[(D{-}3)(A-1) + rA'\big] \\ G_{tr} &= -A^{-1}\, G_{tt} \\ G_{aa} &= \tfrac{1}{2}\, r^{-2}\, g_{aa} \left[(D{-}3)(D{-}4)(A-1) + 2(D{-}3) rA' + r^2 A''\right] \end{aligned}$$

where a runs over the directions on the S^{D-2}. This isn't so bad, thanks to spherical symmetry! Setting $G_{tt} = 0$ and solving for A yields either $A = 0$ (physically unacceptable, as this would make the metric degenerate) or $A(r) = 1 - C/r^{D-3}$. For later convenience, let's rename the constant of integration: $C \to 2M$. Inserting this form of A into G_{aa} yields zero, so we have succeeded in finding a vacuum solution to Einstein's equations:

$$A(r) = 1 - 2M/r^{D-3}, \qquad \Sigma(r) = r.$$

(One may check that the G_{aa} equation is a linear combination of the G_{tt} equation and the radial derivative of this equation.)

Given this solution, is the metric describing asymptotically flat spacetime? For $D > 3$, sending $r \to \infty$ makes $A \to -1$, so consider the large-r form of the line element:

$$ds^2 = -dt^2 + r^2 d\Omega^2_{D-2} + 2\, dt\, dr\,.$$

One may directly calculate the Riemann curvature for this asymptotic form of the metric, and find that it vanishes. Alternatively, one can eliminate the $dt\, dr$ term by making a simple change of variables, $t = \tilde{t} + r$. Using this redefined time coordinate, the asymptotic form of the line element becomes

$$ds^2 = -d\tilde{t}^2 + r^2 d\Omega^2_{D-2} + dr^2\,,$$

which is immediately recognizable as describing flat Minkowski space (with spherical spatial coordinates). For our complete solution, one may check that all components of the Riemann curvature tensor R_{MNPQ} vanish as $r \to \infty$ at least as fast as $1/r^{D-3}$, but this is a coordinate-dependent statement. A much better, coordinate independent test is to calculate the simplest non-vanishing scalar built out of the curvature, $R_{MNPQ}\, R^{MNPQ}$. One finds

$$R_{MNPQ}\, R^{MNPQ} = 4(D{-}1)(D{-}2)^2(D{-}3)/r^{2(D-1)},$$

which does indeed vanish as $r \to \infty$.

The case of three dimensions is special. If $D = 3$, then the metric function A is an arbitrary constant. For any value of A the Riemann curvature vanishes identically. The change of variables $t = A^{-1/2}\, \tilde{t} + A^{-1}\, r$ turns our D=3 line element, $ds^2 = -A dt^2 + 2dt\, dr + r^2 d\theta^2$, into

$$ds^2 = -d\tilde{t}^2 + A^{-1} dr^2 + r^2 d\theta^2\,.$$

If $A \neq 1$, then this is the metric of a cone — a locally flat space with a conical singularity at $r = 0$ (so this is not a smooth manifold). To see this, note that the circumference of the circle $r = r_0$ is $2\pi r_0$ (given that θ is a periodic angle running from 0 to 2π), but the radius of this circle, defined by integrating the line element from $r = 0$ to $r = r_0$, is $r_0/\sqrt{A}$. So if $A \neq 1$, then the circumference does not equal 2π times the radius, exactly as occurs when you curl a flat sheet of paper up into a cone.

Moving on to the question of radial geodesics, we start with the line element for fixed values of the angular coordinates,

$$ds^2 = -A(r)\, dt^2 + 2dt\, dr,$$

with $A(r)$ determined above. Null curves are solutions to $ds^2 = 0$. There are two roots,

$$dt = 0, \qquad \frac{dt}{dr} = \frac{2}{A(r)}.$$

The first solution, $t =$ const., gives the same infalling null geodesics as we saw in the previous problem. The second solution describes curves moving outward with a slope which varies according to the inverse of the metric function $A(r)$. Integrating, for arbitrary D gives a (not very illuminating) expression involving a hypergeometric function,

$$t(r) = 2r\, {}_2F_1\big(1, \frac{1}{3{-}D}, \frac{D{-}4}{D{-}3}, \frac{2M}{r^{D-3}}\big)\,,$$

up to an additive constant. In the specific case of $D = 4$ this becomes

$$t(r)\big|_{D=4} = 2r + 4M \log(r - 2M).$$

Finally, one may check that these null curves are, in fact, geodesics. For the infalling curves with $dt/dr = 0$, this was already done in the previous problem. Given the spherical symmetry, it should be clear (or at least plausible) that there will exist null geodesics ("radial null geodesics") for which every point on the geodesic lies at the same position on the unit $D{-}2$ sphere, and that such geodesics must be the just discussed null curves with $dt/dr = 2/A(r)$. To test this analytically, construct the tangent vector to these outgoing curves (parameterized by r), namely $\xi = (dt/dr)e_t + e_r = 2A(r)^{-1}e_t = e_r$. A short exercise verifies that $d\xi^M/dr + \Gamma^M_{PQ}\, \xi^P\, \xi^Q = 0$, showing that these curves are geodesics with r as an affine parameter. Finally, the only integration constant in the solution for the metric is the parameter M which was introduced in such a way that, for $D = 4$, this is precisely the total mass of the spacetime as discussed in lecture.