1. **Diffeomorphisms**
We start by asking how the components of a vector field transform under the arbitrary reparameterizations $x^\mu = f^\mu(\tilde{x})$ for a differentiable set of functions $\{f^\mu\}$. We are given that the coordinate basis vectors follow $\tilde{\mathbf{e}}_\alpha(\tilde{x}) = \frac{\partial x^\mu}{\partial \tilde{x}^\alpha}\mathbf{e}_\mu(x)$.

Starting from the definition of a vector field $\mathbf{v} = v^\alpha \mathbf{e}_\alpha$, we see that since the abstract vector field is ignorant of the particular coordinatization used on the manifold:

$$\mathbf{v} = \tilde{\mathbf{v}} = \tilde{v}^\alpha(\tilde{x})\tilde{\mathbf{e}}_\alpha(\tilde{x}) = \Big(\frac{\partial x^\mu}{\partial \tilde{x}^\alpha}\Big)\tilde{v}^\alpha \mathbf{e}_\mu(x) \quad \Rightarrow \quad \tilde{v}^\alpha(\tilde{x}) = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu}v^\mu(x).$$

From this point we can easily ascertain the transformation law for a 1-form, $\omega \in T^1(\mathcal{M})$. We start with the fact that the dual basis $d\tilde{x}^\alpha = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu}dx^\mu$. The components thus transform as

$$\omega = \tilde{\omega} = \tilde{\omega}_\alpha(\tilde{x})d\tilde{x}^\alpha = \Big(\frac{\partial \tilde{x}^\alpha}{\partial x^\mu}\Big)\tilde{\omega}_\alpha(\tilde{x})dx^\mu \quad \Rightarrow \quad \tilde{\omega}_\alpha(\tilde{x}) = \frac{\partial x^\mu}{\partial \tilde{x}^\alpha}\omega_\mu(x)$$

Moving one step up the tensorial ladder, we can ask how the components of the metric, $g_{\alpha\beta}$, transform. Let us take a lesson from above and write metric $\mathbf{g} = g_{\alpha\beta}dx^\alpha \otimes dx^\beta$. Then from the rules governing the transformation of the dual basis:

$$\mathbf{g} = \tilde{\mathbf{g}} = \tilde{g}_{\alpha\beta}\Big(\frac{\partial \tilde{x}^\alpha}{\partial x^\mu}\Big)\Big(\frac{\partial \tilde{x}^\beta}{\partial x^\nu}\Big)dx^\mu \otimes dx^\nu \quad \Rightarrow \quad \tilde{g}_{\alpha\beta} = \frac{\partial x^\mu}{\partial \tilde{x}^\alpha}\frac{\partial x^\nu}{\partial \tilde{x}^\beta}g_{\mu\nu}$$

What is key here is that the arbitrary relabeling of points on $\mathcal{M}$ does not change $\mathbf{g}$. What does this mean for the change of the determinant of the metric $g = \det ||g_{\mu\nu}||$? We know that the $\det AB = \det A \det B$, and so, we can see that

$$\tilde{g} = \det ||\tilde{g}_{\alpha\beta}|| = \det \Big|\Big|\frac{\partial x^\mu}{\partial \tilde{x}^\alpha}\frac{\partial x^\nu}{\partial \tilde{x}^\beta}g_{\mu\nu}\Big|\Big| \quad \Rightarrow \quad \tilde{g} = J^2 g$$

where the transformation in the last step was written as a product of Jacobians given by

$$J = \Big|\det\Big(\frac{\partial x^\mu}{\partial \tilde{x}^\alpha}\Big)\Big|.$$

Lastly, we want to know why the spacetime volume element on $\mathcal{M}$ is $\sqrt{|g|}$. First, we note that what we desire is that the volume of $\mathcal{M}$ is invariant under reparameterizations

$$\tilde{V}_\mathcal{M} = V_\mathcal{M} = \int_\mathcal{M} d^dx \cdot (?)$$

What should fill in for '?' is something built out of the metric, since we need a notion of the 'size' of regions of $\mathcal{M}$, and can compensate for how the dual basis transforms. We know from basic multivariate calculus that

$$d^dx = Jd^d\tilde{x}$$

for J given above. Now given the transformation rules above, we see that what should fill in for the volume element must have a transformation like J^{-1} to keep $V_\mathcal{M}$ invariant under diffeomorphisms. The previous calculation shows that

$$\sqrt{|g|} = \sqrt{|J^{-2}\tilde{g}|} = J^{-1}\sqrt{|\tilde{g}|},$$

which is precisely what we want. Thus, demanding that the volume element be built out of the metric and that the total volume is diffeomorphism invariant leads to $\sqrt{|g|}$ being the unique candidate for the volume element. Simply demanding the transformation of J^{-1} is not sufficient as any the square root of the determinant of any $(0,2)$-tensor transforms this way.

This above discussion can be rephrased in the language of differential forms. The components of a k-form are totally anti-symmetric covariant tensors of rank k. On a d-dimensional manifold $\mathcal{M}$, differentiable functions f are 0-forms and d-forms, or top forms, are all proportional to the Levi-Civita tensor

$$\epsilon = \frac{1}{d!}\epsilon_{\alpha_1\dots\alpha_d}dx^{\alpha_1}\wedge\cdots\wedge dx^{\alpha_d}$$

and

$$\epsilon_{\alpha_1\dots\alpha_d} = \sqrt{|g|}[\alpha_1\dots\alpha_d].$$

The beauty of the language of forms is that k-forms can be viewed as integrands of k-dimensional integrals. Thus, ϵ is the volume form, and integrating a scalar function (0-form), f, over the manifold is the same as integrating against ϵ

$$\int_{\mathcal{M}} d^dx\sqrt{|g|}f = \int_{\mathcal{M}} f\epsilon$$

Put in other language seen with differential forms, the volume form is given in terms of the Hodge-$\star$ operation, which maps p-forms to $(d-p)$-forms on a d-dimensional manifold, by

$$\star 1 = \frac{\sqrt{|g|}}{d!}[\alpha_1\dots\alpha_d]dx^{\alpha_1}\wedge\dots\wedge dx^{\alpha_d}$$

2. **Derivatives and Such**

a. $\partial_\alpha \log g$:

For calculations involving variations of the metric determinant, it is convenient to use the identity $\det A = \exp(\mathrm{Tr}\,\log A)$. As such, it becomes clear that

$$\partial_a(\log\det A) = \partial_a \mathrm{Tr}\,\log A = \mathrm{Tr}\,A^{-1}\partial_a A,$$

and so affecting the trace, we see

$$\partial_\alpha(\log g) = g^{\beta\delta}g_{\beta\delta,\,\alpha}.$$

b. $\Gamma^\alpha{}_{\alpha\beta} = \partial_\beta \log\sqrt{|g|}$:

In a coordinate basis, lets recall the definition of the Christoffel symbols

$$\Gamma^\alpha{}_{\beta\delta} = \frac{g^{\alpha\gamma}}{2}\left(g_{\beta\gamma,\,\delta} + g_{\delta\gamma,\,\beta} - g_{\beta\delta,\,\gamma}\right),$$

and trace over the first two indices

$$\Gamma^{\alpha}_{\ \alpha\beta} = \frac{g^{\alpha\gamma}}{2}\left(g_{\gamma\alpha,\,\beta} + g_{\beta\gamma,\alpha} - g_{\gamma\beta,\,\alpha}\right) = \frac{1}{2} g^{\alpha\gamma} g_{\alpha\gamma,\,\beta}.$$

The last equality used the symmetry of the metric to cancel the last two terms. From part [a.],

$$\partial_\beta \log\sqrt{|g|} = \frac{1}{2} g^{\alpha\gamma} g_{\alpha\gamma,\,\beta}.$$

c. $\nabla_\alpha A^\alpha$:

Starting with the definition of the covariant derivative of a vector

$$\nabla_\beta A^\alpha = \partial_\beta A^\alpha + \Gamma^{\alpha}_{\ \beta}\gamma A^\gamma.$$

Affecting the trace, over α, β

$$\nabla_\alpha A^\alpha = \partial_\alpha A^\alpha + \Gamma^{\alpha}_{\ \alpha\gamma} A^\gamma,$$

and using the result in part [b.]

$$\nabla_\alpha A^\alpha = \partial_\alpha A^\alpha + \partial_\gamma(\log\sqrt{g})A^\gamma = \partial_\alpha A^\alpha + \frac{1}{\sqrt{g}} A^\alpha \partial_\alpha \sqrt{g},$$

which can be rearranged to give

$$\nabla_\alpha A^\alpha = |g|^{-\frac{1}{2}}(\sqrt{g}A^\alpha)_{,\alpha}.$$

d. $\Box\phi$:

First, we rewrite $\Box \equiv \nabla_\alpha \nabla^\alpha$, and use the results from part [c.]

$$\nabla_\alpha\Big(\nabla^\alpha\phi\Big) = g^{-\frac{1}{2}}\Big(\sqrt{g}\nabla^\alpha\phi)_{,\alpha} = g^{-\frac{1}{2}}\Big(\sqrt{g}g^{\alpha\beta}\partial_\beta\phi)_{,\alpha}$$

where in the last equality we used metric compatibility of the Christoffel connection $\nabla_a g_{bc} = 0$ and that $\nabla_a f = \partial_a f$ for any $f \in \Omega^0(\mathcal{M})$.

3. **Fun with S^d**

To begin, we want to understand the particular coordinatization, i.e. the round metric, of the unit d-dimensional sphere, S^d, such that the angles $\{\alpha_i\}$ cover the sphere with

$$ds^2 = d\alpha_1^2 + \sin^2\alpha_1 d\alpha_2^2 + \ldots + \prod_{i=1}^{d-1}\sin^2\alpha_i d\alpha_d^2. \tag{1}$$

That is we want to study the embedding $\mathrm{S}^d \hookrightarrow \mathbb{R}^{d+1}$. Let us define coordinates on $\mathbb{R}^{d+1}$, $\{x_0, \ldots, x_d\}$ with the flat Euclidean metric

$$ds^2_{\mathbb{R}} = \delta_{ab} dx^a dx^b.$$

We want to find the unit S^d embedded in this space, which can be found by defining the co-dimension 1 hypersurface in $\mathbb{R}^{d+1}$ such that each point on the surface is unit distance away from the origin $\{0, \ldots, 0\}$. This is found by the restriction

$$x_0^2 + x_1^2 + \ldots + x_d^2 = 1.$$

Any coordinatization of a unit $S^d \hookrightarrow \mathbb{R}^{d+1}$ must satisfy this relation. We can use a generalization of the familiar parameterization of the $S^2 \hookrightarrow \mathbb{R}^3$:

$$x_0 = \cos\alpha_1, \quad x_1 = \sin\alpha_1 \cos\alpha_2, \quad \ldots, x_{d-1} = \prod_{i=1}^{d-1} \sin\alpha_i \cos\alpha_d, \quad x_d = \prod_{i=1}^{d} \sin\alpha_i.$$

If we compute the line element on this surface, pullback the standard Euclidean metric on $\mathbb{R}^{d+1}$ to get the desired form as in eq. (1). To illustrate this, lets use a low dimensional example, $S^2 \hookrightarrow \mathbb{R}^3$, and compute the line element. Note that, with the above coordinates on the d=2 hypersurface

$$\begin{aligned}
dx_0^2 &= \sin^2\alpha_1 d\alpha_1^2 \\
dx_1^2 &= \cos^2\alpha_1 \cos^2\alpha_2 d\alpha_1^2 + \sin^2\alpha_1 \sin^2\alpha_2 d\alpha_2^2 - 2\cos\alpha_1 \sin\alpha_1 \cos\alpha_2 \sin\alpha_2 d\alpha_1 d\alpha_2 \\
dx_2^2 &= \cos^2\alpha_1 \sin^2\alpha_2 d\alpha_1^2 + \sin^2\alpha_1 \cos^2\alpha_2 d\alpha_2^2 + 2\cos\alpha_1 \sin\alpha_1 \cos\alpha_2 \sin\alpha_2 d\alpha_1 d\alpha_2.
\end{aligned}$$

Plugging this into $\delta_{ab} dx^a dx^b$, we see that

$$ds^2 = \delta_{ab} dx^a dx^b = d\alpha_1^2 + \sin^2\alpha_1 d\alpha_2^2 \qquad \checkmark.$$

What are the ranges for the α_i? To cover the sphere, $\{\alpha_i \in [0, \pi] \big| 1 \le i \le d-1\}$ and $\alpha_d \in [0, 2\pi]$. Note that you can visualize this as a nested fibration of an S^1 over $d-1$ intervals $\mathcal{I} = [0, \pi]$. In this coordinate system if we had chosen $\alpha_d \in [0, \pi]$ and another $\alpha_k \in [0, 2\pi]$, then we would not have covered the entirety of the sphere (check this again with a low dimensional example). Alternatively if we had chosen multiple $\alpha_k \in [0, 2\pi]$, then we would have ended up with a multiple covering of the sphere.

For brevity, label $\Gamma^{\alpha_i}_{\alpha_j \alpha_k} \equiv \Gamma^i{}_{jk}$. Using your favorite package for differential geometry, the calculation of $\Gamma^i{}_{jk}$ over a number of dimensions is done quickly, and from there a pattern quickly emerges:

$$\begin{aligned}
\Gamma^j{}_{ij} &= \Gamma^j{}_{ji} = \cot\alpha_i, \ i < j \\
\Gamma^i{}_{jj} &= -g_{jj} \cot\alpha_i \prod_{k=1}^{i-1} \csc^2\alpha_k, \ i < j
\end{aligned}$$

with the rest vanishing identically. So, the geodesic equation with affine parameter λ becomes, with $\dot\alpha_i = \frac{d\alpha_i(\lambda)}{d\lambda}$,

$$\ddot\alpha^i + 2\Gamma^i{}_{ji} \dot\alpha^j \dot\alpha^i + \Gamma^i{}_{kk} (\dot\alpha^k)^2 = 0, \quad j < i, \, k > i.$$

We can note that the α_d equation can always be written as

$$\dot\alpha_d = c_d \prod_{k=1}^{d-1} \csc^2 \alpha_k(\lambda)$$

for some constant c_d.

How are we going to solve this? We can exploit the symmetry of the background and affect an $SO(d+1)$ rotation such that any solution we wish to find passes through the 'north pole' defined by $\alpha_1(0) = 0$. If from this point we only allow α_1 to vary such that $\dot{\alpha}_j = 0$ for $j > 1$, then the geodesic equations reduce to

$$\ddot{\alpha}_i = 0$$

for all i. The solution to this is obviously $\alpha_1(\lambda) = c_1\lambda$ and $\alpha_j = c_j$ for $j > 1$ and c_i constant, which is the parameterization of a great circle.

Lastly, calculating the Riemann curvature tensor, Ricci tensor, and Ricci scalar for the S^d (see the Mathematica notebook posted along with the solutions), one finds that

$$\begin{aligned} R_{ijkl} &= g_{ik}g_{jl} - g_{il}g_{jk} \\ R_{ij} &= (d-1)g_{ij} \\ R &= d(d-1) \end{aligned}$$

In particular, this shows that the arbitrary constant used in the problem statement $K = 1$. A manifold whose curvatures can be expressed in this way, for any K, is called a 'symmetric space'.