1. **Commutators and Connections**

If we consider the set of basis vectors $\mathbf{e}_\alpha$ on an arbitrary manifold, $\mathcal{M}$, equipped with coordinates x^μ, then the commutator of $[\mathbf{e}_\alpha, \mathbf{e}_\beta]$, taking the basis such that $\mathbf{e}_\alpha$ simply point along a direction with unit weight, can w.l.o.g. be expressed as

$$[\mathbf{e}_\alpha, \mathbf{e}_\beta] = [\partial_\alpha, \partial_\beta] = (\partial_\alpha\partial_\beta - \partial_\beta\partial_\alpha) = 0.$$

The last equality follows from standard properties of mixed ∂'s.

If instead, we were to take an arbitrary vector field $\mathbf{u}$ in the coordinate basis, which can be expanded on the above basis $\mathbf{u}(f) = u^\alpha\partial_\alpha(f)$ for any scalar function $f \in C^\infty$. Then we find that the commutator $[\mathbf{u}, \mathbf{v}]\mathbf{f}$ is given by

$$[\mathbf{u}, \mathbf{v}]\mathbf{f} = \mathbf{u}(\mathbf{v}(f)) - \mathbf{v}(\mathbf{u}(f)) = (u^\alpha\partial_\alpha(v^\beta\partial_\beta f) - v^\beta\partial_\beta(u^\alpha\partial_\alpha f)),$$

which defines for us a new vector field $\mathbf{w}(f) = w^\alpha\partial_\alpha f$ defined by

$$\mathbf{w}(f) = (u^\alpha\partial_\alpha(v^\beta) - v^\alpha\partial_\alpha(u^\beta))\partial_\beta f.$$

In components, this evaluates to

$$w^\beta = (u^\alpha\partial_\alpha(v^\beta) - v^\alpha\partial_\alpha(u^\beta)).$$

What does this mean for the symmetry properties of the components of the connect? By definition

$$\nabla_\alpha\mathbf{e}_\beta = \Gamma^\lambda_{\alpha\beta}\mathbf{e}_\lambda.$$

Now, consider $f \in C^\infty(\mathcal{M})$ and let us compute the action of $[\mathbf{e}_\alpha, \mathbf{e}_\beta] \to [\nabla_\alpha, \nabla_\beta]$ on f

$$\begin{aligned}[\nabla_\alpha, \nabla_\beta]f &= \partial_\alpha(\nabla_\beta f) + \Gamma^\delta_{\alpha\beta}\nabla_\delta f - \partial_\beta(\nabla_\alpha f) - \Gamma^\delta_{\beta\alpha}\nabla_\delta f \\ &= [\partial_\alpha, \partial_\beta]f + (\Gamma^\delta_{\alpha\beta} - \Gamma^\delta_{\beta\alpha})\partial_\delta f = 2\Gamma^\delta_{[\alpha\beta]}\partial_\delta f\end{aligned}$$

In the last line, we used that for a scalar function $\nabla f = \partial f$ and $[\partial_\alpha, \partial_\beta] = 0$. Then we see that, $\Gamma^\delta_{[\alpha\beta]}$ vanishes which implies that the connection in the coordinate basis is symmetric in its lower indices

$$\Gamma^a_{[bc]} = 0 \quad \Rightarrow \Gamma^a_{bc} = \Gamma^a_{cb}$$

In a spacetime with torsion, this is certainly not the case as explicitly $\Gamma^a_{[bc]} \equiv K^a_{bc} \neq 0$.

2. **Geodesics**

   a. **Variational Calculus**

   We start with the coordinatization of $\mathbb{R}^2$ by $\{r, \phi\}$ and parameterize a curve in the plane by s with a functional defined on the curve as

$$I = \int_{\xi_0}^{\xi_1} ds \frac{1}{2}((r'(s))^2 + r^2(\phi'(s))^2)$$

   where $' \equiv \frac{d}{ds}$ and the fixed endpoints of the curve are $\{\xi_0, \xi_1\}$. This functional, I, describes the free motion of a particle in $\mathbb{R}^2$ with the parameter s playing the role of time.

Extremizing the functional:

$$\delta I = \int_{\xi_0}^{\xi_1} ds\big((r(\phi')^2 - r'')\delta r - (r^2\phi'' + 2rr'\phi')\delta\phi) + (r'\delta r + r^2\phi'\delta\phi)\big|_{\xi_0}^{\xi_1}$$

With the endpoints fixed, the boundary terms vanish, and thus stationarity of the functional implies the following equations

$$\begin{aligned} r'' - r(\phi')^2 &= 0 \\ r\phi'' + 2r'\phi' &= 0 \end{aligned}$$

b. **Straight Lines**
If we return to the original Cartesian coordinate system by $r(s)^2 = x(s)^2 + y(s)^2$ and $\phi(s) = \tan^{-1}\left(\frac{y(s)}{x(s)}\right)$, then the equations of motion become

$$\begin{aligned} xx'' + yy'' &= 0 \\ yx'' - xy'' &= 0 \end{aligned}$$

Which after simple manipulation gives

$$\begin{aligned} x'' &= \frac{xy''}{y} \\ (x^2+y^2)y'' &= 0 \end{aligned}$$

The solution to this system is simply $y(s) = a_y s + b_y$ and $x = a_x s + b_x$.

c. **Geodesics**
We want to compare the equations of motion from part (a) with the geodesic equation in affine parameterization by s

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds} = 0.$$

Mercifully, for $\mathbb{R}^2$ in polar coordinates the only non-vanishing components of the connection are $\Gamma^r_{\phi\phi} = -r$ and $\Gamma^\phi_{r\phi} = r^{-1}$. This means that,

$$\begin{aligned} r'' + \Gamma^r_{\phi\phi}(\phi')^2 = 0 \quad &\Rightarrow \quad r'' - r(\phi')^2 = 0 \quad \checkmark \\ \phi'' + 2\Gamma^\phi_{r\phi}r'\phi' = 0 \quad &\Rightarrow \quad r\phi'' + 2r'\phi' = 0 \quad \checkmark \end{aligned}$$

3. **Structure constants**
Note there was a typo in the assignment: $e_2^\theta = \sin\psi$.
We can easily compute these commutators given results from the first question. The easiest are obviously $[\mathbf{e_i},\, \mathbf{e_3}]$:

$$\begin{aligned} [\mathbf{e}_1, \mathbf{e_3}] &= \cos\psi\cancelto{0}{[\partial_\theta, \partial_\psi]} - \sin\psi(\cot\theta\cancelto{0}{[\partial_\psi, \partial_\psi]} - \csc\psi\cancelto{0}{[\partial_\phi, \partial_\psi]}) - \partial_\psi\cos\psi\partial_\theta + \partial_\psi\sin\psi(\cot\theta\partial_\psi - \csc\psi\partial_\phi) \\ &= \sin\psi\partial_\theta + \cos\psi(\cot\theta\partial_\psi - \csc\psi\partial_\phi) = f^2_{\ 13}\mathbf{e}_2 \\ [\mathbf{e}_1, \mathbf{e_3}] &= -\cos\psi\partial_\theta + \sin\psi(\cot\theta\partial_\psi - \csc\psi\partial_\phi) = f^1_{\ 23}\mathbf{e}_1, \end{aligned}$$

where $f^1_{\ 23} = -1$ and $f^2_{\ 13} = 1$. A similar calculation gives

$$[\mathbf{e}_1,\, \mathbf{e}_2] = -\partial_\psi = f^3_{\ 12}\mathbf{e}_3$$

These $f^i_{\ jk}$ are just the structure constants of $SU(2)$, but this should come as no surprise since the isometry group of the S^3 is $SO(4) \simeq SU(2)\times SU(2)$.