1. Commutators and Connections

If we consider the set of basis vectors \( e_\alpha \) on an arbitrary manifold, \( M \), equipped with coordinates \( x^\mu \), then the commutator of \([e_\alpha, e_\beta]\), taking the basis such that \( e_\alpha \) simply point along a direction with unit weight, can w.l.o.g. be expressed as

\[
[e_\alpha, e_\beta] = [\partial_\alpha, \partial_\beta] = (\partial_\alpha \partial_\beta - \partial_\beta \partial_\alpha) = 0.
\]

The last equality follows from standard properties of mixed \( \partial \)'s.

If instead, we were to take an arbitrary vector field \( u \) in the coordinate basis, which can be expanded on the above basis \( u(f) = u^\alpha \partial_\alpha(f) \) for any scalar function \( f \in \mathcal{C}^\infty \). Then we find that the commutator \([u, v]f\) is given by

\[
[u, v]f = u^\alpha \partial_\alpha(v^\beta \partial_\beta f) - v^\beta \partial_\beta(u^\alpha \partial_\alpha f),
\]

which defines for us a new vector field \( w(f) = u^\alpha \partial_\alpha f \) defined by

\[
w(f) = (u^\alpha \partial_\alpha(v^\beta) - v^\beta \partial_\beta(u^\alpha))(\partial_\beta f).
\]

In components, this evaluates to

\[
w_\beta = (u^\alpha \partial_\alpha(v^\beta) - v^\beta \partial_\beta(u^\alpha)).
\]

What does this mean for the symmetry properties of the components of the connect? By definition

\[
\nabla_\alpha e_\beta = \Gamma^\lambda_{\alpha\beta} e_\lambda.
\]

Now, consider \( f \in \mathcal{C}^\infty(M) \) and let us compute the action of \([e_\alpha, e_\beta] \rightarrow [\nabla_\alpha, \nabla_\beta] \) on \( f \)

\[
[\nabla_\alpha, \nabla_\beta]f = \partial_\alpha(\nabla_\beta f) + \Gamma^\delta_{\alpha\beta} \nabla_\delta f - \partial_\beta(\nabla_\alpha f) - \Gamma^\delta_{\beta\alpha} \nabla_\delta f
\]

\[
= [\partial_\alpha, \partial_\beta]f + (\Gamma^\delta_{\alpha\beta} - \Gamma^\delta_{\beta\alpha}) \partial_\delta f = 2\Gamma^\delta_{[\alpha\beta]} \partial_\delta f
\]

In the last line, we used that for a scalar function \( \nabla f = \partial f \) and \([\partial_\alpha, \partial_\beta] = 0 \). Then we see that, \( \Gamma^\delta_{[\alpha\beta]} \) vanishes which implies that the connection in the coordinate basis is symmetric in its lower indices

\[
\Gamma^a_{[bc]} = 0 \quad \Rightarrow \quad \Gamma^a_{bc} = \Gamma^a_{cb}
\]

In a spacetime with torsion, this is certainly not the case as explicitly \( \Gamma^a_{[bc]} \equiv K^a_{bc} \neq 0 \).

2. Geodesics

a. Variational Calculus

We start with the coordinatization of \( \mathbb{R}^2 \) by \( \{r, \phi\} \) and parameterize a curve in the plane by \( s \) with a functional defined on the curve as

\[
I = \int_{\xi_0}^{\xi_1} ds \left( \frac{1}{2} (r'(s))^2 + r^2(\phi'(s))^2 \right)
\]

where \( \phi' \equiv \frac{d}{ds} \) and the fixed endpoints of the curve are \( \{\xi_0, \xi_1\} \). This functional, \( I \), describes the free motion of a particle in \( \mathbb{R}^2 \) with the parameter \( s \) playing the role of time.
Extremizing the functional:

\[ \delta I = \int_{\xi_0}^{\xi_1} ds \left( (r(\phi')^2 - r'') \delta r - (r^2 \phi'' + 2r r' \phi') \delta \phi \right) + \left( r' \delta r + r^2 \phi' \delta \phi \right) \]

With the endpoints fixed, the boundary terms vanish, and thus stationarity of the functional implies the following equations

\[ r'' - r(\phi')^2 = 0 \]
\[ r \phi'' + 2 r' \phi' = 0 \]

b. Straight Lines

If we return to the original Cartesian coordinate system by

\[ r(s) = x(s)^2 + y(s)^2 \quad \text{and} \quad \phi(s) = \tan^{-1} \left( \frac{y(s)}{x(s)} \right) \]

then the equations of motion become

\[ xx'' + yy'' = 0 \]
\[ yx'' - xy'' = 0 \]

Which after simple manipulation gives

\[ x'' = \frac{xy''}{y} \]
\[ (x^2 + y^2)y'' = 0 \]

The solution to this system is simply \( y(s) = a_y s + b_y \) and \( x = a_x s + b_x \).

c. Geodesics

We want to compare the equations of motion from part (a) with the geodesic equation in affine parameterization by \( s \)

\[ \frac{d^2 x^\alpha}{ds^2} + \Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{ds} \frac{dx^\gamma}{ds} = 0. \]

Mercifully, for \( \mathbb{R}^2 \) in polar coordinates the only non-vanishing components of the connection are \( \Gamma^r_{\phi\phi} = -r \) and \( \Gamma^\phi_{r\phi} = r^{-1} \). This means that,

\[ r'' + \Gamma^r_{\phi\phi}(\phi')^2 = 0 \implies r'' - r(\phi')^2 = 0 \quad \checkmark \]
\[ \phi'' + 2 \Gamma^\phi_{r\phi} r' \phi' = 0 \implies r \phi'' + 2r' \phi' = 0 \quad \checkmark \]

3. Structure constants

Note there was a typo in the assignment: \( e_2^\theta = \sin \psi \).

We can easily compute these commutators given results from the first question. The easiest are obviously \( [e_1, e_3] \):

\[ [e_1, e_3] = \cos \psi [\partial_\theta, \partial_\psi] + \sin \psi (\cot \theta \partial_\psi, \partial_\psi) - \csc \psi (\partial_\theta, \partial_\psi) = f_{13}^2 e_2 \]

\[ [e_1, e_3] = -\cos \psi \partial_\theta + \sin \psi (\cot \theta \partial_\psi - \csc \psi \partial_\phi) = f_{23}^1 e_1, \]

where \( f_{23}^1 = -1 \) and \( f_{13}^2 = 1 \). A similar calculation gives

\[ [e_1, e_2] = -\partial_\psi = f_{12}^3 e_3 \]

These \( f_{jk} \) are just the structure constants of \( SU(2) \), but this should come as no surprise since the isometry group of the \( S^3 \) is \( SO(4) \simeq SU(2) \times SU(2) \).