Due Tuesday, November 29, 2016

1. A static, spherical black hole of mass M and Schwarzschild radius $R = 2M$ lies at a distance $d \gg R$ away from an observer. Hot gas in the vicinity of the black hole emits electromagnetic radiation, some of which propagates in directions which eventually reach the observer. Your task is to calculate the image which the observer sees and, in particular, to understand the nature of the "shadow" imprinted on this image by the black hole. Assume that the distribution of gas (and its properties) are symmetric under rotations about the line of sight to the center of the black hole. For convenience, regard this direction as the observer's polar direction. The resulting image will have rotational symmetry, with an intensity $f(\theta)$ which depends only on the observer's polar angle θ. Find this function. Plot it and also describe it qualitatively. (Does it have any interesting structure? Is there a well-defined "shadow"? Are there any sharp features? How does the intensity fall off with angle?)

   To simplify the analysis, make a highly unrealistic but convenient model for the illuminating gas. Assume that it lies only on the far side of the black hole, and is flowing steadily toward the black hole in a thin stream along the polar direction. Consider a bit of gas which, at some instant, is located at radial coordinate r (in standard Schwarzschild coordinates) away from the black hole. This bit of gas emits light isotropically, but at any given radius r only light which is emitted at certain angle(s) $\theta(r)$ (relative to the stream of gas) will eventually reach the observer. Neglect, purely for simplicity, the difference between isotropic emission in a local Lorentz frame which is co-moving with the gas, and isotropy in a local Lorentz frame which is stationary relative to the black hole. (In other words, ignore the flow velocity of the gas and pretend that all the gas in the illuminating jet is static in Schwarzschild coordinates.) The essence of the problem is then to calculate which light rays — null geodesics — can connect the observer to some point on the illuminating jet, and to find the mapping between the observer's angle of observation θ and the emission radius r. Hints: work backwards. Imagine light emitted by the observer at some polar angle θ. Does such a light ray ever reach some point on the polar axis ($\theta = 0$) behind the black hole, and if so at what radial position r (relative to the black hole) and intersecting the polar axis at what angle ϕ? Suitably use the functions $r(\theta)$ and $\phi(\theta)$ to relate the actual observed intensity $f(\theta)$ with the source distribution (say uniform emission all along the illuminating jet).

   Apply your results to Sgr A*, the compact radio source at the center of our galaxy which is believed to be a supermassive black hole whose mass M is approximately four million solar masses. What angular resolution is needed to image this source? What angular resolution might be achievable with millimeter wavelength interferometry (on the surface of the Earth)?