1. Show that ordinary derivatives may be replaced by covariant derivatives, without affecting the result, in the variation of a tensor under an arbitrary infinitesimal diffeomorphism, $\delta F_{\alpha\beta}(x) = F_{\alpha\beta,\nu}\,\xi^\nu + F_{\nu\beta}\,\xi^\nu{}_{,\alpha} + F_{\alpha\nu}\,\xi^\nu{}_{,\beta}$, and thereby demonstrate that for the metric $\delta g_{\alpha\beta} = \xi_{\alpha;\beta} + \xi_{\beta;\alpha}$.

2. The Lagrange density of a Maxwell field (i.e., the electromagentic field) is given by $-\mathcal{L} = \frac{1}{4}\,F_{\mu\nu}\,F^{\mu\nu} = \frac{1}{4}\,g^{\mu\alpha}\,g^{\nu\beta}\,F_{\mu\nu}\,F_{\alpha\beta}$, with $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$.

   (a) Show that $F_{\mu\nu} = A_{\nu;\mu} - A_{\mu;\nu}$ is equally correct, as the affine connection cancels.

   (b) What is the corresponding action? Derive Maxwell's equations in curved (but empty) spacetime from this action and the definition of the field strength tensor.

   (c) Vary the action with respect to the metric to find the stress-energy tensor. Show that it is conserved when the field strength satisfies the vacuum Maxwell equations.

   (d) Now suppose that some current density $j^\mu(x)$ is also present. How does this change the Lagrange density and action? How does this change Maxwell's equations? How does this change the stress-energy tensor? How does this change the divergence of the stress-energy tensor?

3. To very good approximation, the sun is static and spherical, and only weakly distorts the spacetime geometry. Hence, the exterior line element has the form $ds^2 = -(1-2M/r)\,dt^2 + (1+2M/r)(dx^2+dy^2+dz^2)$, where $r \equiv |\vec{x}|$. Consider a photon which passes by the sun with an impact parameter b (greater than the solar radius). Let s be an affine parameter for the null geodesic of the photon, scaled such that the tangent vector $p = dx/ds$ is the photon 4-momentum, and orient coordinates such that the photon trajectory lies in the $x{-}y$ plane, with incoming spatial momentum in the $\hat{x}$ direction. Starting from the geodesic equation $dp^\alpha/ds = \Gamma^\alpha_{\beta\gamma}\,p^\beta\,p^\gamma = 0$, calculate the connection coefficients in the equatorial plane ($z = 0$), insert appropriate approximate values for the big components of the photon momentum, $p^0 \approx p^x$, determine dp^y/ds and integrate to find $p^y(s=+\infty)$. Show that the resulting deflection angle $\Delta\phi = 4M/b$. If R denotes the solar radius, what is M/R (as a dimensionless number)? What is $\Delta\phi$ in units of R/b?

4. Given a metric of the form $ds^2 = -(1-2M/r)\,dt^2 + (1+2M/r)(dx^2+dy^2+dz^2)$, show (from the equation for null geodesics) that $p_0 \equiv p\cdot e_0$ is constant along the photon's worldline. Suppose some atom, at rest on the sun's surface, emits a photon which travels outward and, far from the sun, has its wavelength measured by a diffraction spectrometer. The photon comes from a particular spectral line of some atom which, when measured in a terrestrial laboratory, is found to lie at wavelength λ_e. Use the above result to show that the observed wavelength λ_o of photons in this spectral line, when emitted from the solar surface, will be redshifted by the amount $z \equiv (\lambda_o - \lambda_e)/\lambda_e = M/R$, where M and R are the solar mass and radius. What is the numerical value of this redshift? Clearly explain your logic.