1. Consider an arbitrary change of coordinates (or *diffeomorphism*) from a set $\{x^\mu\}$ to a new set $\{\tilde{x}^\alpha\}$ such that $x^\mu = f^\mu(\tilde{x})$ for some set of (differentiable) functions $\{f^\mu(\tilde{x})\}$. Under such a change of coordinates, a *scalar* function $\phi \to \tilde{\phi}$ where $\phi(\tilde{x}) \equiv \phi(x)$ evaluated at $x = f(y)$. Using coordinate bases, the new and old basis vectors are related via $\tilde{\mathbf{e}}_\alpha(\tilde{x}) = (\partial x^\mu / \partial \tilde{x}^\alpha)\, \mathbf{e}_\mu(x)$ (again evaluated at $x = f(\tilde{x})$). (*i*) How do the components of a vector field transform? (*ii*) How do the components of a one-form transform? (*iii*) How do the components of the metric transform? (*iv*) How does the determinant of the metric transform? (*v*) Explain why $(-g)^{1/2}$ is the spacetime volume element, where $g \equiv \det \|g_{\mu\nu}\|$.

2. With $g \equiv \det \|g_{\alpha\beta}\|$, prove that:
   (*i*) $(\ln g)_{,\alpha} = g^{\beta\gamma} g_{\beta\gamma,\alpha}$
   (*ii*) $\Gamma^\alpha{}_{\alpha\beta} = (\ln |g|^{1/2})_{,\beta}$ in a coordinate frame.
   (*iii*) $A^\alpha{}_{;\alpha} = |g|^{-1/2}(|g|^{1/2} A^\alpha)_{,\alpha}$ in a coordinate frame.
   (*iv*) $\Box\phi \equiv \phi_{;\alpha}{}^{;\alpha} = |g|^{-1/2}(|g|^{1/2} g^{\alpha\beta} \phi_{,\alpha})_{,\beta}$ in a coordinate frame.

3. The metric of S^D, a D-dimensional sphere, may be written in the form

$$ds^2 = d\alpha_1^2 + \sin^2\alpha_1\, d\alpha_2^2 + \sin^2\alpha_1\, \sin^2\alpha_2\, d\alpha_3^2 + \cdots + \Big(\prod_{j=1}^{D-1} \sin^2\alpha_j \Big)\, d\alpha_D^2\,,$$

   where $\{\alpha_1, \cdots, \alpha_D\}$ are the angular coordinates one introduces when defining spherical coordinates in $\mathbb{R}^{D+1}$. (*i*) Justify this assertion. What are the appropriate ranges for each angle? (*ii*) Calculate the connection coefficients and show that great circles are geodesics. Demonstrating this for great circles passing through the "north pole" is sufficient. (*iii*) Calculate the Riemann curvature tensor, the Ricci tensor, and the Ricci scalar. Show that $R_{ijkl} = K\,(g_{ik}\, g_{jl} - g_{il}\, g_{jk})$, $R_{jl} = (D{-}1)\, K\, g_{jl}$, and $R = D(D{-}1)\, K$ for some constant K.

   Hints: Calculating with pencil and paper is, of course, fine. So is using computer assistance. If you choose the later, I recommend becoming acquainted with, and using, the Riemannian Geometry & Tensor Calculus package with Mathematica (MMA). Links to these are on the class web page. When using this package with MMA, the spacetime dimension D must be a number and cannot be left free. Write a MMA function which will let you perform the above calculations for any given value of D, and verify the assertions above for all $D \leq 6$.