1. Any vector field $\mathbf{u}$ may be regarded as a linear operator acting on scalar fields, with $\mathbf{u}(f) \equiv \partial_{\mathbf{u}} f = u^\alpha \partial_\alpha f = u^\alpha f_{,\alpha}$ for any scalar function f. Given any two vector fields, $\mathbf{u}$ and $\mathbf{v}$, the *commutator* $[\mathbf{u}, \mathbf{v}] \equiv [\partial_{\mathbf{u}}, \partial_{\mathbf{v}}]$. Using a coordinate basis: (i) what is the commutator of basis vectors, $[\mathbf{e}_\alpha, \mathbf{e}_\beta]$, and (ii) what are the components of the commutator $[\mathbf{u}, \mathbf{v}]$? Explain why these results imply that, in any coordinate basis, the connection components are symmetric in the last two indices, $\Gamma^\mu{}_{\alpha\beta} = \Gamma^\mu{}_{\beta\alpha}$.

2. Define polar coordinates on a flat two-dimensional plane as usual: $x \equiv r\cos\phi$ and $y \equiv r\sin\phi$. Let $\{r(s), \phi(s)\}$ describe some curve on the plane parameterized by s. Consider a functional on such curves defined by $I \equiv \int ds\, \frac{1}{2}[(dr/ds)^2 + r^2(d\phi/ds)^2]$.

   (a) Vary the functional I, with fixed endpoints for the curve. What are the resulting differential equations which characterize minima of I?

   (b) Justify the claim that curves (with fixed endpoints) which minize I are necessarily geodesics — i.e., geometrically straight lines on $\mathbb{R}^2$.

   (c) Compare the general form of the geodesic equation, $d^2x^\mu/ds^2 + \Gamma^\mu{}_{\alpha\beta}\,(dx^\alpha/ds)\,(dx^\beta/ds) = 0$, with your result in part (a) and extract the values of all connection coefficients $\{\Gamma^\mu{}_{\alpha\beta}\}$ for polar coordinates (with coordinate basis vectors) on $\mathbb{R}^2$.

3. Let $\{\theta, \phi, \psi\}$ be conventional spherical coordinates for a three-sphere (S^3), related to Cartesian coordinates in $\mathbb{R}^4$ via $x^4 = r\,\cos\theta$, $x^2 = r\,\sin\theta\,\sin\phi\,\cos\psi$, $x^3 = r\,\sin\theta\,\cos\phi$, and $x^1 = r\,\sin\theta\,\sin\phi\,\sin\psi$. Define the vector fields

$$\mathbf{e}_1 \equiv \cos\psi\,\frac{\partial}{\partial\theta} - \sin\psi\left(\cot\theta\,\frac{\partial}{\partial\psi} - \csc\theta\,\frac{\partial}{\partial\phi}\right),$$

$$\mathbf{e}_2 \equiv \sin\psi\,\frac{\partial}{\partial\theta} + \cos\psi\left(\cot\theta\,\frac{\partial}{\partial\psi} - \csc\theta\,\frac{\partial}{\partial\phi}\right),$$

$$\mathbf{e}_3 \equiv \frac{\partial}{\partial\psi}\,.$$

   Compute the commutators $[\mathbf{e}_i, \mathbf{e}_j]$ and show that $[\mathbf{e}_i, \mathbf{e}_j] = f_{ij}{}^k\,\mathbf{e}_k$ for some constant set of coefficients $\{f_{ij}{}^k\}$. What are these coefficients? Can you explain their form (without your explicit calculation)?