Lecture 22. Nonlinear Phase-Plane Analysis and Stability

We are finally at a point where we can start to study systems which are much more realistic in nature: *Nonlinear Systems.* The methods we will utilize in describing such systems, which take the form:

$$\frac{dx}{dt} = F(x, y, t)$$
$$\frac{dy}{dt} = G(x, y, t)$$

with $F(x, y, t)$ and $G(x, y, t)$ being some nonlinear functions of x and y, are qualitative rather than quantitative. Further, they rely on understanding two primary parts of the problem:

- Equilibrium Points (also known as *Critical Points*)
- Stability of Critical Points

Provided these two issues are understood, we can develop a variety of approaches to solving very complicated problems. It should be noted that although our analytic techniques provide a tremendous amount of insight into the problem, quantitative understanding comes largely through numerical simulations of the governing equations and application of the various perturbation methods developed so far.

Since the behavior in linear systems is completely characterized in Lectures 5 and 6, we move on to consider a classical problem which is actually *nonlinear.* Such is the case of the pendulum. The schematic of the pendulum is shown in Fig. 1. By conservation of angular momentum, we find that the swing of the pendulum can be described by the equation:

$$mL^2\frac{d^2\Theta}{dt^2} = -cL\frac{d\Theta}{dt} - mgL\sin\Theta$$

where m is the pendulum mass, L is the length, g is the acceleration due to gravity, and c measures the frictional/damping forces acting on the pendulum. We can rewrite this equation as

$$\Theta'' + \gamma\Theta' + \omega^2\sin\Theta = 0$$

where $\gamma = c/mL$ and $\omega^2 = g/L$.

To convert this into a system of equations, we define

$$x = \Theta \qquad \text{and} \qquad y = \frac{d\Theta}{dt}$$

which then results in the *nonlinear* system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} y \\ -\omega^2\sin x - \gamma y \end{pmatrix}.$$

Equilibrium solutions are found by letting $x' = y' = 0$ which yields

$$y = 0 \qquad \text{and} \qquad \sin x = 0$$

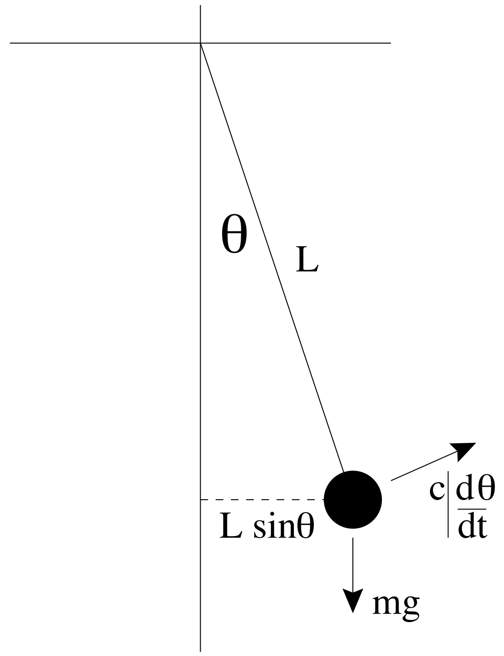


FIG. 1. *Schematic of pendulum oscillating from a fixed support subject to forces of gravity and damping.*

so that the critical points are

$$y = 0 \qquad \text{and} \qquad x = \pm n\pi \quad n = 0, 1, 2, \dots .$$

Thus unlike a linear system, we have more than one equilibrium point. In fact, we have an infinity of them which lie at multiples of π along the the x-axis.

To simplify the analysis, we begin by considering the case in which there is no damping so that $\gamma = 0$ and our governing equations are

$$\begin{aligned} x' &= y \\ y' &= -\omega^2 \sin x \, . \end{aligned}$$

Note that the critical points (equilibrium) above are independent of the damping parameter. The idea now is to look very close to one of the equilibrium points using the ideas of *perturbation theory*. Therefore we let

$$\begin{aligned} x &= \pm n\pi + \tilde{x} \\ y &= 0 + \tilde{y} \end{aligned}$$

where $\tilde{x}$ and $\tilde{y}$ are both very small. Thus this implies we are very near one of the fixed points. Plugging this into our governing equations for no damping yields the system:

$$\begin{aligned} \tilde{x}' &= \tilde{y} \\ \tilde{y}' &= -\omega^2 \sin(\pm n\pi + \tilde{x}) \, . \end{aligned}$$

To begin, we consider the well known example of the pendulum which oscillates about the equilibrium $x = 0$ (which corresponds to $\Theta = 0$). In this case $n = 0$ so that we have

$$\sin(\pm n\pi + \tilde{x}) = \sin\tilde{x} = \tilde{x} - \frac{\tilde{x}^3}{3!} + \frac{\tilde{x}^5}{5!} + \cdots \approx \tilde{x}$$

where we have used the Taylor series representation of sine and approximated everything by $\tilde{x}$ since all the other terms are much smaller (provided, of course, that $\tilde{x}$ is very small). This is the standard trick that is used in introductory physics in order to turn the nonlinear system into a linear one. In particular, if we plug this result into the above equation we preceding equation we find.

$$\begin{aligned} \tilde{x}' &= \tilde{y} \\ \tilde{y}' &= -\omega^2 \tilde{x} . \end{aligned}$$

which results in the *linear* system:

$$\vec{x}' = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \vec{x}$$

where $\vec{x} = (\tilde{x}\ \tilde{y})^T$. We learned how to solve this system in Lectures 5 and 6. Thus we let $\vec{x} = \vec{v}e^{\lambda t}$ which yields the eigenvalue problem

$$\begin{pmatrix} -\lambda & 1 \\ -\omega^2 & -\lambda \end{pmatrix} \vec{v} = 0 .$$

The eigenvalues are found by taking the determinant of the above matrix to be zero. This then yields

$$\lambda^2 + \omega^2 = 0 \quad \rightarrow \quad \lambda = \pm i\omega$$

which are purely imaginary eigenvalues. Thus the equilibrium point for $n = 0$, i.e. $(x, y) = (0, 0)$ is a center. Thus solutions near the critical point are all elliptic trajectories.

More generally, we can consider all the equilibrium points that are multiples of 2π away from the origin $(x, y) = (0, 0)$. Thus we consider the perturbation theory for these points:

$$\begin{aligned} x &= \pm 2n\pi + \tilde{x} \\ y &= 0 + \tilde{y} \end{aligned}$$

where $\tilde{x}$ and $\tilde{y}$ are both very small. This implies we are very near one of the fixed points located at multiples of 2π from the origin. Plugging this into our governing equations for no damping now yields the system:

$$\begin{aligned} \tilde{x}' &= \tilde{y} \\ \tilde{y}' &= -\omega^2 \sin(\pm 2n\pi + \tilde{x}) . \end{aligned}$$

But since

$$\sin(\pm 2n\pi + \tilde{x}) = \sin\tilde{x} \approx \tilde{x} ,$$

we then arrive at exactly the same linearized equations as before. Therefore, we can conclude that *all* the equilibrium points in multiples of 2π from the origin are centers with periodic solutions near each critical points (see Fig. 2).

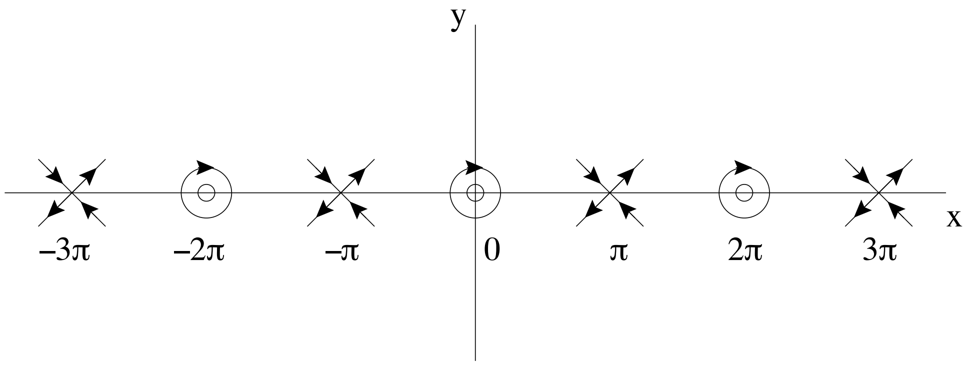


FIG. 2. *Behavior of solutions near each of the fixed points which are multiples of π from the origin. Note that multiples of 2π produce centers while multiples of odd π are saddles.*

This is not the case for critical points which are odd multiples of π from the origin. For these we can perturb around each critical point by letting

$$x = \pm 2n\pi + \pi + \tilde{x}$$
$$y = 0 + \tilde{y}$$

where $\tilde{x}$ and $\tilde{y}$ are both very small. This implies we are very near one of the fixed points located at odd multiples of π from the origin (i.e. $\pm\pi, \pm 3\pi, ...$). Plugging this into our governing equations for no damping now yields the system:

$$\tilde{x}' = \tilde{y}$$
$$\tilde{y}' = -\omega^2 \sin(\pm 2n\pi + \pi + \tilde{x}) \, .$$

But since

$$\sin(\pm 2n\pi + \pi + \tilde{x}) = \sin(\pi + \tilde{x}) = \sin\pi\cos\tilde{x} + \cos\pi\sin\tilde{x} = -\sin\tilde{x} \approx -\tilde{x} \, ,$$

we then arrive at a slightly different set of linearized equations. In matrix form, this can be written as

$$\vec{x}' = \begin{pmatrix} 0 & 1 \\ \omega^2 & 0 \end{pmatrix} \vec{x}$$

where the only difference now is in the sign of ω^2. Letting $\vec{x} = \vec{v}e^{\lambda t}$ yields the eigenvalue problem

$$\begin{pmatrix} -\lambda & 1 \\ \omega^2 & -\lambda \end{pmatrix} \vec{v} = 0 \, .$$

whose eigenvalues are

$$\lambda^2 - \omega^2 = 0 \quad \rightarrow \quad \lambda = \pm\omega$$

which are purely real eigenvalues or opposite sign. Thus the equilibrium point for odd multiples of π are saddles. The eigenvectors can then be found:

$$\lambda = \omega : \quad \begin{pmatrix} -\omega & 1 \\ \omega^2 & -\omega \end{pmatrix} \vec{v} = 0 \ \rightarrow \ -\omega v_1 + v_2 = 0 \ \rightarrow \ \vec{v}^{(1)} = \begin{pmatrix} 1 \\ \omega \end{pmatrix}$$
$$\lambda = -\omega : \quad \begin{pmatrix} \omega & 1 \\ \omega^2 & \omega \end{pmatrix} \vec{v} = 0 \ \rightarrow \ \omega v_1 + v_2 = 0 \ \rightarrow \ \vec{v}^{(2)} = \begin{pmatrix} 1 \\ -\omega \end{pmatrix} .$$

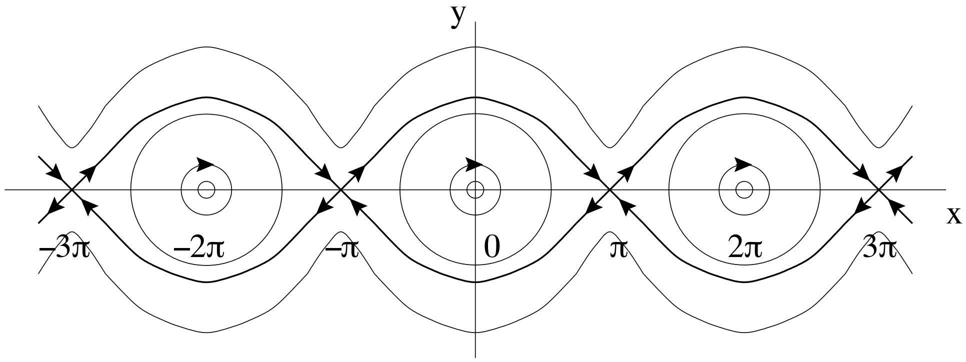


FIG. 3. *Full nonlinear behavior of solutions near each of the fixed points and beyond. This behavior is deducted directly from Fig. 2 and is the only consistent behavior since all the critical points are on the line $y = 0$. The bolded lines are the* separatrix *which separate the oscillatory behavior from the behavior above it corresponding to a pendulum which continuously rotates around.*

Thus a complete description of the saddle is given near each critical point in odd multiples of π from the origin. The resulting dynamics is depicted in Fig. 2 which shows the results of our perturbation calculations locally near each of the fixed points.

Although the perturbation results are insightful, their full power is not realized until we generalize our thinking of Fig. 2. In particular, since trajectories cannot intersect, we can utilize the picture in Fig. 2 to develop a full qualitative understanding of the dynamics. Specifically, we can describe the behavior far from the critical points by their behavior near the critical points. In Fig. 3, we develop the full nonlinear qualitative behavior by simply taking the phase-plane picture and generalizing it in the only way possible. This results in a dynamical picture which makes a great deal of sense. Note that near the critical points in multiples of 2π (which corresponds to the rest position of the pendulum), the behavior is exactly as expected: oscillatory. Whereas for odd π values (which corresponds to a pendulum sticking straight up), the behavior is given by a saddle and is unstable. Note that the trajectory separating the oscillatory behavior from the trajectories above it is called the *separatrix*. The separatrix projects along the unstable eigenvector of one saddle into the stable eigenvector of a neighboring eigenvector. The behavior above this corresponds to the undamped pendulum swinging around and around its support. This is the case if we give it a strong enough initial speed. And since there is no damping in this model, it will continue to circle around and around and will never fall into the oscillatory back and forth motion predicted near the center equilibrium.

We now generalize our treatment in order to treat the case of the damped pendulum. Recall that the system in this case is given by

$$\begin{aligned} x' &= y \\ y' &= -\omega^2 \sin x - \gamma y \,. \end{aligned}$$

As in the undamped case, the key now is to find the critical points and their stability. The critical points (equilibrium) are determined for $x' = y' = 0$ so that

$$y = 0 \qquad \text{and} \qquad x = \pm n\pi \;\; n = 0, 1, 2, ...$$

exactly as in the undamped case. We once again perturb about these equilibrium

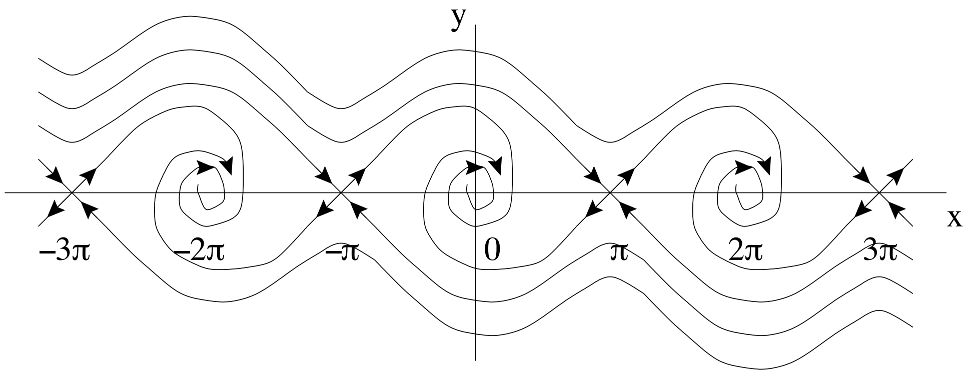


FIG. 4. *Full nonlinear behavior of solutions near each of the fixed points when under-damping is applied to the pendulum. There are no separatrix in this case.*

points to determine stability. Therefore we let

$$\begin{aligned} x &= \pm n\pi + \tilde{x} \\ y &= 0 + \tilde{y} \end{aligned}$$

where $\tilde{x}$ and $\tilde{y}$ are both very small. Plugging this into our damped equations yields the system:

$$\begin{aligned} \tilde{x}' &= \tilde{y} \\ \tilde{y}' &= -\omega^2 \sin(\pm n\pi + \tilde{x}) - \gamma \tilde{y}\,. \end{aligned}$$

As in the undamped pendulum case, there are two interesting cases to consider. The first is when the critical point is at the origin or at multiples of 2π from it. Thus we have

$$\sin(\pm 2n\pi + \tilde{x}) = \sin \tilde{x} \approx \tilde{x}$$

where we have again approximated sine by $\tilde{x}$ since it is small. Plugging this result into the linear damped equation above results in the system:

$$\vec{x}' = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -\gamma \end{pmatrix} \vec{x}$$

where $\vec{x} = (\tilde{x} \;\; \tilde{y})^T$. Letting $\vec{x} = \vec{v}e^{\lambda t}$ yields the eigenvalue problem

$$\begin{pmatrix} -\lambda & 1 \\ -\omega^2 & -\gamma - \lambda \end{pmatrix} \vec{v} = 0\,.$$

The eigenvalues are found from the determinant to be

$$-\lambda(-\gamma - \lambda) + \omega^2 = \lambda^2 + \gamma\lambda + \omega^2 = 0 \quad \rightarrow \quad \lambda = -\frac{\gamma}{2} \pm i\sqrt{\omega^2 - \frac{\gamma^2}{4}}\,.$$

So depending on the quantity $\omega^2 - \gamma^2/4$, the equilibrium is either a spiral ($\omega^2 > \gamma^2/4$), an improper node ($\omega^2 = \gamma^2/4$), or a node ($\omega^2 < \gamma^2/4$). The three different cases are referred to as underdamped, critically damped, and overdamped respectively. In any case, the real part of the eigenvalue is negative so that the equilibrium point is asymptotically stable. In what we depict in Fig. 4, we assume that $\omega^2 > \gamma^2/4$ so that the equilibrium points at multiples of 2π are all spirals.

When the critical points are at odd multiples of π from the origin, we once again have

$$\sin(\pm 2n\pi + \pi + \tilde{x}) = \sin(\pi + \tilde{x}) = -\sin\tilde{x} \approx -\tilde{x}$$

Plugging this result into the linear damped equation results in the system:

$$\vec{x}' = \begin{pmatrix} 0 & 1 \\ \omega^2 & -\gamma \end{pmatrix} \vec{x}$$

where $\vec{x} = (\tilde{x} \;\; \tilde{y})^T$. Letting $\vec{x} = \vec{v}e^{\lambda t}$ yields the eigenvalue problem

$$\begin{pmatrix} -\lambda & 1 \\ \omega^2 & -\gamma - \lambda \end{pmatrix} \vec{v} = 0 \, .$$

whose eigenvalues are found from the determinant to be

$$-\lambda(-\gamma - \lambda) - \omega^2 = \lambda^2 + \gamma\lambda - \omega^2 = 0 \quad \rightarrow \quad \lambda = -\frac{\gamma}{2} \pm \sqrt{\omega^2 + \frac{\gamma^2}{4}} \, .$$

This yields two real eigenvalues which are of opposite sign. Thus a saddle is once again generated for all equilibrium points which are odd multiples of π from the origin. The complete nonlinear dynamics is depicted in Fig. 4 where the interaction of the spirals and saddle nodes is shown. Note how in this case, the solutions eventually end up in one of the spiral points.

Lecture 23. Applications of Nonlinear Phase-Plane

We now have enough background material to develop a more general theory and understanding of nonlinear systems. There are two key concepts in nonlinear systems that determine all the resulting dynamics. The two primary issues are:

- Equilibrium (*Critical Points*)
- Stability

Both of these concepts, which were mentioned in the last lecture, are rather intuitive in nature and have been illustrated in the previous lecture.

We begin by considering the following general system of equations

$$\begin{aligned} x' &= F(x, y, t) \\ y' &= G(x, y, t)\,, \end{aligned}$$

where $F(x, y, t)$ and $G(x, y, t)$ are some general functions of x, y, and time t. We will simplify this for the present by considering the *autonomous* system:

$$\begin{aligned} x' &= F(x, y) \\ y' &= G(x, y)\,, \end{aligned}$$

where F and G are not explicitly time dependent.

We begin to analyze this system by considering the concept of equilibrium. Equilibrium occurs when there is no "motion" in the system, i.e. when both $x' = 0$ and $y' = 0$. The point at which this occurs is the equilibrium point (x_0, y_0) which satisfies:

$$\begin{aligned} F(x_0, y_0) &= 0 \\ G(x_0, y_0) &= 0\,, \end{aligned}$$

since $x' = y' = 0$. This is all there is to equilibrium. We simply find the points (there may be more than one) which satisfy the above equations simultaneously. Once this is done, the behavior of the system can be determined entirely from the stability of each equilibrium (critical) point.

The stability of each critical point may be determined by looking very near each individual point. Thus we assume that

$$\begin{aligned} x &= x_0 + \tilde{x} \\ y &= y_0 + \tilde{y}\,, \end{aligned}$$

where $\tilde{x}$ and $\tilde{y}$ are both very small so that they can be considered to be in a very small neighborhood of the critical point. Plugging this into our original equations gives us

$$\begin{aligned} \tilde{x}' &= F(x_0 + \tilde{x}, y_0 + \tilde{y}) \\ \tilde{y}' &= G(x_0 + \tilde{x}, y_0 + \tilde{y})\,, \end{aligned}$$

where we recall that since x_0 and y_0 are constants then $x_0' = y_0' = 0$. The key now is to remember our Taylor expansion formula from the series chapter. Thus to expand about some point, we have

$$f(x_0 + \tilde{x}) = f(x_0) + \tilde{x} f'(x_0) + \frac{\tilde{x}^2}{2!} f''(x_0) + \frac{\tilde{x}^3}{3!} f'''(x_0) + \cdots.$$

Keeping only the first few terms in this approximation is good provided $\tilde{x}$ is small.

In the full problem, we now have to expand about both x_0 and y_0. Doing so yields the following:

$$\tilde{x}' = F(x_0, y_0) + \tilde{x}F_x(x_0, y_0) + \tilde{y}F_y(x_0, y_0) + \cdots$$
$$\tilde{y}' = G(x_0, y_0) + \tilde{x}G_x(x_0, y_0) + \tilde{y}G_y(x_0, y_0) + \cdots$$

where we have neglected all terms which are smaller than $\tilde{x}^2$, $\tilde{y}^2$, and $\tilde{x}\tilde{y}$. In matrix form, this *linearized* system can be written as:

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}' = \begin{pmatrix} F_x(x_0, y_0) & F_y(x_0, y_0) \\ G_x(x_0, y_0) & G_y(x_0, y_0) \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}.$$

We call this a *linearized* system since we turned the original *nonlinear* system into a *linear* system near the critical points. The methods developed in the previous chapter and reviewed in the introductory lecture of this chapter are now applicable. Thus we simply need to determine the eigenvalues of the above system and the resulting global, i.e. nonlinear, dynamics can be understood qualitatively.

To make use of these ideas, we turn to some specific examples to help illustrate the key ideas. We begin by considering what are called *predator–prey models*. These models consider the interaction of two species: predators and their prey. It should be obvious that such species will have significant impact on one another. In particular, if there is an abundance of prey, then the predator population will grow due to the surplus of food. Alternatively, if the prey population is low, then the predators may die off due to starvation.

To model the interaction between these species, we begin by considering the predators and prey in the absence of any interaction. Thus the prey population (denoted by $x(t)$) is governed by

$$\frac{dx}{dt} = ax$$

where $a > 0$ is a net growth constant. The solution to this simple differential equation is $x(t) = x(0)\exp(at)$ so that the population grows without bound. We have assumed here that the food supply is essentially unlimited for the prey so that the unlimited growth makes sense since there is nothing to kill off the population.

Likewise, the predators can be modeled in the absence of their prey. In this case, the population (denoted by $y(t)$) is governed by

$$\frac{dy}{dt} = -cy$$

where $c > 0$ is a net decay constant. The reason for the decay is that the population basically starves off since there is no food (prey) to eat.

We now try to model the interaction. Essentially, the interaction must account for the fact the the predators eat the prey. Such an interaction term can result in the following system:

$$\frac{dx}{dt} = ax - \alpha xy$$
$$\frac{dy}{dt} = -cx + \alpha xy$$

where $\alpha > 0$ is the interaction constant. Note that *alpha* acts as a decay to the prey population since the predators will eat them, and as a growth term to the predators since they now have a food supply. These nonlinear and autonomous equations are known as the *Lotka–Volterra equations.*

We rely on the methods introduced in this lecture to study this system. In particular, we consider the equilibrium points and their associated stability in order to determine the qualitative dynamics of the Lotka-Volterra equations.

The critical points are determined by setting $x' = y' = 0$ which gives

$$\begin{aligned} ax - \alpha xy &= x(a - \alpha y) = 0 \\ -cy + \alpha xy &= y(\alpha x - c) = 0\,. \end{aligned}$$

This gives two possible fixed points

$$\begin{aligned} &\text{I. } x = 0 \text{ and } y = 0 \\ &\text{II. } x = c/\alpha \text{ and } y = a/\alpha\,. \end{aligned}$$

Each of these fixed points needs to be investigated separately in order to determine the full (qualitative) nonlinear dynamics.

We begin with the critical point I. $(x_0, y_0) = (0, 0)$. Following the methods outlined above we calculate the following:

$$\begin{array}{llllll} F(x,y) = ax - \alpha xy & \longrightarrow & F_x = a - \alpha y & \longrightarrow & F_x(0,0) = a \\ & & F_y = -\alpha x & & F_y(0,0) = 0 \\ \\ G(x,y) = -cy + \alpha xy & \longrightarrow & G_x = \alpha y & \longrightarrow & G_x(0,0) = 0 \\ & & G_y = -c + \alpha x & & G_y(0,0) = -c\,. \end{array}$$

The resulting linearized system is

$$\vec{w}' = \begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix} \vec{w} = 0$$

where $\vec{w} = (\tilde{x}\ \tilde{y})^T$. As with all previous linear systems, we make the substitution $\vec{w} = \vec{v}\exp(\lambda t)$ in order to yield the eigenvalue problem:

$$\begin{pmatrix} a - \lambda & 0 \\ 0 & -c - \lambda \end{pmatrix} \vec{v} = 0\,.$$

Setting the determinant to zero gives the characteristic equation

$$(a - \lambda)(-c - \lambda) = 0$$

whose eigenvalues are

$$\lambda = a \text{ and } \lambda = -c$$

Thus the eigenvalues are real and of opposite sign giving us a saddle at the critical point $(0, 0)$.

The eigenvectors can also be easily determined for this case. They are as follows:

$$\lambda = a: \begin{pmatrix} a - a & 0 \\ 0 & -c - a \end{pmatrix} \vec{v} = \begin{pmatrix} 0 & 0 \\ 0 & -(c + a) \end{pmatrix} \vec{v} = 0 \longrightarrow \vec{v}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

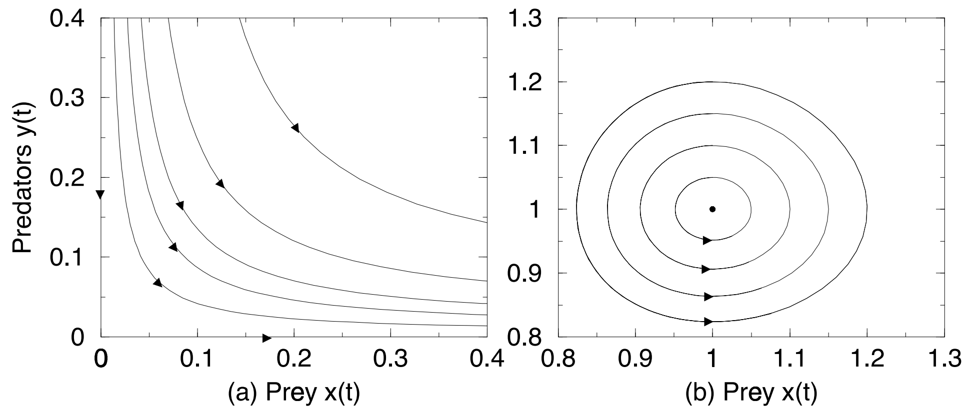


FIG. 1. *Behavior near each of the fixed points. Here we have assumed that* $a = c = \alpha = 1$ *so that the two critical points are the saddle at* $(0, 0)$ *and the center at* $(1, 1)$.

and

$$\lambda = -c: \begin{pmatrix} a-c & 0 \\ 0 & -c+c \end{pmatrix} \vec{v} = \begin{pmatrix} a-c & 0 \\ 0 & 0 \end{pmatrix} \vec{v} = 0 \longrightarrow \vec{v}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The behavior near the critical point at the origin in thus completely determined. Figure 1a depicts the saddle behavior near the origin.

We now consider critical point II. $(x_0, y_0) = (c/\alpha, a/\alpha)$. Following the previous calculation we find:

$$F_x(c/\alpha, a/\alpha) = 0,\ F_y(c/\alpha, a/\alpha) = -c,\ G_x(c/\alpha, a/\alpha) = a,\ G_y(c/\alpha, a/\alpha) = 0\,.$$

The resulting linearized system is

$$\vec{w}' = \begin{pmatrix} 0 & -c \\ a & 0 \end{pmatrix} \vec{w} = 0\,.$$

Making the substitution $\vec{w} = \vec{v}\exp(\lambda t)$ yields the eigenvalue problem:

$$\begin{pmatrix} -\lambda & -c \\ a & -\lambda \end{pmatrix} \vec{v} = 0\,.$$

Setting the determinant to zero gives the characteristic equation

$$\lambda^2 + ac = 0$$

whose eigenvalues are

$$\lambda_{\pm} = \pm i\sqrt{ac}$$

Thus the eigenvalues are imaginary giving a center at the critical point $(c/\alpha, a/\alpha)$. Before calculating the eigenvectors for this case, we note that the periodic behavior goes counter–clockwise in order to be consistent with the flow of the critical point I. The behavior near the critical point is depicted in Fig. 1b. The full nonlinear dynamics is depicted in Fig. 2 which shows the saddle behavior near critical point I. and periodic motion around the critical point II.

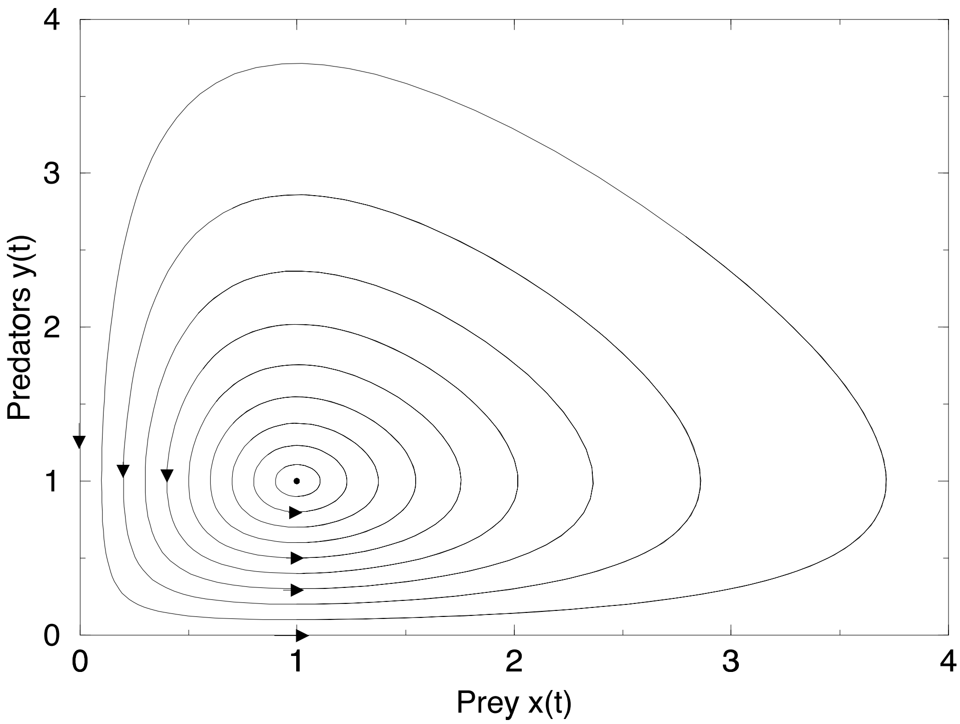


FIG. 2. *Fully nonlinear behavior of the predator–prey system. As with Fig. 1, we have assumed that $a = c = \alpha = 1$ so that the two critical points are the saddle at $(0,0)$ and the center at $(1,1)$. Note the periodic behavior between these two fixed points.*

To get a better idea of the periodic motion, we can calculate the eigenvectors associated with critical point II.

$$\lambda = i\sqrt{ac}: \begin{pmatrix} -i\sqrt{ac} & -c \\ a & -i\sqrt{ac} \end{pmatrix} \vec{v} = 0 \longrightarrow -i\sqrt{ac}v_1 - cv_2 = 0 \longrightarrow \vec{v}^{(1)} = \begin{pmatrix} \sqrt{ac} \\ -ia \end{pmatrix}.$$

Rewriting the eigenvector then gives

$$\vec{w}^{(1)} = \begin{pmatrix} \sqrt{ac} \\ -ia \end{pmatrix} \exp\left(i\sqrt{ac}t\right) = \begin{pmatrix} \sqrt{ac}\cos\sqrt{ac}t \\ a\sin\sqrt{ac}t \end{pmatrix} + i \begin{pmatrix} \sqrt{ac}\sin\sqrt{ac}t \\ -a\cos\sqrt{ac}t \end{pmatrix}.$$

Rewriting our solution in terms of a purely real solution then is easily done by combining the real and imaginary parts to form

$$\vec{w} = c_1 \begin{pmatrix} \sqrt{ac}\cos\sqrt{ac}t \\ a\sin\sqrt{ac}t \end{pmatrix} + c_2 \begin{pmatrix} \sqrt{ac}\sin\sqrt{ac}t \\ -a\cos\sqrt{ac}t \end{pmatrix}.$$

Thus the population of predators ($y(t)$) and prey ($x(t)$) can be calculated explicitly:

$$x(t) = \frac{c}{\alpha} + c_1\sqrt{ac}\cos\sqrt{ac}t + c_2\sqrt{ac}\sin\sqrt{ac}t$$
$$y(t) = \frac{a}{\alpha} + c_1 a\sin\sqrt{ac}t - c_2 a\cos\sqrt{ac}t\,.$$

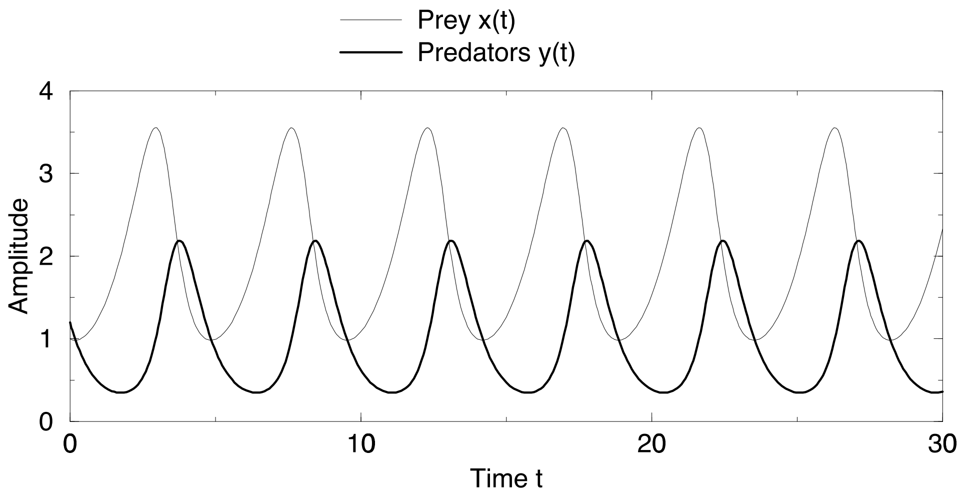


FIG. 3. *Fully nonlinear behavior of the predator–prey system as a function of time. Here $a = \alpha = 1$ and $c = 2$. Note the $\pi/2$ lag between the solutions.*

To simplify this further, we can replace the constants c_1 and c_2 by the two new constants K and ϕ so that

$$x(t) = \frac{c}{\alpha} + \frac{c}{\alpha} K \cos(\sqrt{ac}t + \phi)$$
$$y(t) = \frac{a}{\alpha} + \frac{a}{\alpha}\sqrt{\frac{c}{a}} K \sin(\sqrt{ac}t + \phi)\,.$$

This representation allows us to see explicitly the fact that the populations are $\pi/2$ out of phase. Thus the reaction to changes of population occur (near the critical point) one quarter of a cycle out of phase. This behavior is demonstrated in Fig. 3.

Lecture 7

We begin the discussion of boundary value problems by considering a partial differential equation governing the dynamics of a stretched wire or string. It can be shown from Newton's law that the evolution of the string is given by

$$u_{tt} = c^2 u_{xx}$$

where u is the displacement and c measures the material properties of the string or wire.

We let

$$u = e^{i\lambda t} v(x)$$

so that

$$-\lambda^2 v = c^2 v_{xx}$$

Then

$$v_{xx} + \frac{\lambda^2}{c^2} v = 0$$

From our methods developed earlier, we can easily find solutions to the resulting second-order ODE. These are

$$v = c_1 \sin\left(\frac{\lambda}{c}x\right) + c_2 \cos\left(\frac{\lambda}{c}x\right)$$

If we wanted to make a guitar, we would clamp both ends of our wire so that

$$V(0) = 0 \quad \text{and} \quad V(L) = 0$$

Obviously, $V=0$ satisfies both the governing equation and boundary conditions. However, we are interested in nontrivial solutions only. Thus we impose the boundary conditions on our general solution V. So then

$$\text{at } x=0: \qquad V(0) = 0 = C_2$$

so then

$$V(x) = C_1 \sin(\lambda/c\, x)$$

and then

$$\text{at } x=L \qquad V(L) = 0 = C_1 \sin(\lambda/c\, L)$$

Thus

$$\sin \frac{\lambda}{c} L = 0$$

so that

$$\frac{\lambda}{c} L = \pm n\pi \qquad n = 0, 1, 2, 3, \ldots$$

so then

$$\lambda_n = \pm \frac{n\pi}{L} \cdot c \qquad \text{eigenvalues (infinite number)}$$

The corresponding eigenfunctions are

$$V_n = C_n \sin\left(\frac{\lambda_n}{c} x\right) = C_n \sin\left(n\frac{\pi}{L} x\right)$$

This is markedly different than what we have covered so far. In particular

- There are an infinite number of solutions, not just two!

This raises the equestions

1. What is the general solution?
2. What do you do with an infinite number of solutions?
3. How do you determine the C_n?

The general solution: following the ideas of linear superposition, we can argue that

$$V = C_1 \sin\left(\frac{\pi x}{L}\right) + C_2 \sin\left(\frac{2\pi x}{L}\right) + \cdots + C_n \sin\left(\frac{n\pi x}{L}\right) + \cdots$$

or

$$V = \sum_{n=1}^{\infty} C_n \sin\left(n\pi x/L\right)$$

This begs the question: are the solutions V_n linearly independent or not?

Consider the two eigenfunctions

$$V_n = \sin(n\pi x/L) \quad \text{and} \quad V_m = \sin(m\pi x/L)$$

Then calculate

$$W[V_n, V_m] = \sin\left(\frac{n\pi x}{L}\right) \frac{m\pi}{L} \cdot \cos\left(\frac{m\pi x}{L}\right) - \frac{n\pi}{L} \cos\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right)$$

$$= \frac{\pi}{L}\left[\frac{m}{2}\left(\sin\left[(n+m)\frac{\pi x}{L}\right] + \sin\left[(n-m)\frac{\pi x}{L}\right]\right)\right.$$

$$\left. - \frac{n}{2}\left(\sin\left[(n+m)\frac{\pi x}{L}\right] - \sin\left[(n-m)\frac{\pi x}{L}\right]\right)\right]$$

$$= \frac{\pi}{2L}\left\{ (m-n)\sin\left[(n+m)\frac{\pi x}{L}\right] + (m+n)\sin\left[(n-m)\frac{\pi x}{L}\right]\right\}$$

So as long as

$$n \neq m \quad \Rightarrow \quad W \neq 0$$

and V_n and V_m are linearly independent.

We proceed now to define the concept of a inner product

$$(u, v) = \int_0^L u(x)\, v^*(x)\, dx$$

where $*$ denotes complex conjugation. If we find that

$$(u, v) = 0$$

then u and v are said to be orthogonal, i.e. they are linearly independent.

Going back to our example, we find

$$(v_n, v_m) = \int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx$$

$$= \int_0^L \left[\frac{1}{2}\cos\left((n-m)\frac{\pi x}{L}\right) - \frac{1}{2}\cos\left((n+m)\pi x/L\right)\right] dx$$

$$= \frac{L}{2(n-m)\pi} \sin\frac{(n-m)\pi x}{L} - \frac{L}{2(n+m)\pi} \sin\frac{(n+m)\pi x}{L} \Big|_0^L$$

$$= 0 \qquad \left(\text{note: } \sin[(n-m)\pi] = 0\right)$$

So once again, we find v_n and v_m are orthogonal.

Within this context, we can also define the norm of a function

$$\|u\| = (u, u)^{1/2} = \left(\int_0^L u u^* dx\right)^{1/2}$$

In our case

$$\|v_n\|^2 = \int_0^L \sin(n\pi x/L) \sin(n\pi x/L)\, dx$$

$$= \int_0^L \sin^2(n\pi x/L)\, dx = \int_0^L \left[\frac{1}{2} - \frac{1}{2}\cos(2n\pi x/L)\right] dx$$

$$= x/2 - \frac{L}{4n\pi} \sin(2n\pi x/L) \Big|_0^L = L/2$$

So then

$$\|v_n\| = \sqrt{L/2}$$

We can then construct normalized eigenfunctions such that

$$w_n = \frac{v_n}{\| v_n \|}$$

So that

$$(w_n, w_m) = \delta_{nm}$$

where

$$\delta_{nm} = \begin{cases} 1 & n = m \\ 0 & \text{otherwise} \end{cases}$$

For the example given then

$$w_n = \sqrt{2/L}\, \sin(n\pi x/L) \quad \Rightarrow \quad v(x) = \sum_{n=1}^{\infty} c_n w_n$$

Finally, we note from the initial problem that we could impose an initial condition. So then

$$u(x,0) = e^{i\lambda \cdot 0} v(x) = v(x) = f(x)$$

But recall that

$$V(x) = \sum_{n=1}^{\infty} C_n W_n = \sqrt{2/L} \sum_{n=1}^{\infty} C_n \sin(n\pi x/L)$$

so then

$$f(x) = V(x) = \sqrt{2/L} \sum_{n=1}^{\infty} C_n \sin(n\pi x/L)$$

Imposing the initial condition allows us to determine the C_n. This is the only remaining unknown. So then consider

$$(V_m, f(x)) = \left(\sqrt{2/L} \sum_{n=1}^{\infty} C_n \sin(n\pi x/L),\ V_m\right)$$

$$= \sqrt{2/L}\, C_n \left(\sum_{n=1}^{\infty} \sin(n\pi x/L),\ V_m\right)$$

$$= \sqrt{2/L}\, C_n\, \delta_{nm}$$

$$= \sqrt{2/L}\, C_m$$

so

$$C_m = \sqrt{L/2}\, (V_m, f(x))$$

so that

$$V(x) = \sqrt{2/L} \sum_{n=1}^{\infty} C_n \sin(n\pi x/L) = \sqrt{2/L} \sum_{n=1}^{\infty} \sqrt{\tfrac{L}{2}}\, (V_m, f(x)) \sin(n\pi x/L)$$

$$V(x) = \sum_{n=1}^{\infty} (V_m, f(x)) \sin(n\pi x/L)$$

This is also called a Fourier Series.

Lecture 8

Linear Operator Theory and Functional Analysis

Before proceeding further with a discussion on eigenfunctions, we note some important connections between linear algebra and our present function space. In particular, we note the following analogous ideas.

	Vector Space	Function Space
eigenvalue problem	$A\vec{x} = \lambda\vec{x}$	$Lu = \lambda u$
inner product orthogonality	$\vec{x}\cdot\vec{y} = 0$	$(u,v) = 0$
norm	$\lVert\vec{x}\rVert = \vec{x}\cdot\vec{x}$	$\lVert u\rVert = (u,u)$

We also note the similarities in solving the following nonhomogeneous problems.

Method 1: Inverse

$$A\vec{x} = \vec{b} \qquad\qquad Lu = f$$

then

$$\vec{x} = A^{-1}\vec{b} \qquad\qquad u = L^{-1}f$$

However, at this point we don't know how to calculate L^{-1}. It turns out that this is the Green's function method.

Method 2: Eigenfunction Expansion

For the matrix system we have

$$A\vec{x} = \vec{b}$$

so consider the eigenvalue problem

$$A\vec{x} = \lambda\vec{x} \quad \longrightarrow \quad \vec{x} = \sum_{n=1}^{\infty} c_n \vec{x}_n \qquad \vec{x}_n \text{ - eigenvectors}$$

now if A is self-adjoint (Hermitian) then $A = (A^T)^*$ and

$$\vec{x}_n \cdot \vec{x}_m = \delta_{nm} \qquad \delta_{nm} = \begin{cases} 1 & n=m \\ 0 & n \neq m \end{cases}$$

so then we let

$$A\vec{x} = \vec{b} \quad \rightarrow \quad A\left(\sum_{n=1}^{\infty} c_n \vec{x}_n\right) = \vec{b}$$

$$\sum_{n=1}^{\infty} c_n A\vec{x}_n = \vec{b}$$

$$\sum_{n=1}^{\infty} c_n \lambda_n \vec{x}_n = \vec{b}$$

now take the inner product with respect to $\vec{x}_m$

$$\sum_{n=1}^{\infty} c_n \lambda_n \vec{x}_n \cdot \vec{x}_m = \vec{b} \cdot \vec{x}_m$$

$$\sum_{n=1}^{\infty} c_n \lambda_n \delta_{nm} = \vec{b} \cdot \vec{x}_m$$

$$c_m \lambda_m = \vec{b} \cdot \vec{x}_m$$

$$c_m = \vec{b} \cdot \vec{x}_m / \lambda_m$$

so

$$\vec{x} = \sum_{n=1}^{\infty} c_n \vec{x}_n = \sum_{m=1}^{\infty} \frac{\vec{b} \cdot \vec{x}_m}{\lambda_m} \vec{x}_m \qquad \text{eigenfunction expansion}$$

Similarly for the function space we have

$$Lu = f$$

with the eigenvalue problem

$$Lu_n = \lambda_n u_n \qquad \text{with} \quad (u_n, u_m) = \delta_{nm}$$

Thus

$$u = \sum_{n=1}^{\infty} c_n u_n$$

Taking inner products we find

$$(u_n, Lu) = (u_n, f)$$

if L is self-adjoint then

$$(u_n, Lu) = (Lu_n, u) = (u_n, f)$$

$$(\lambda_n u_n, u) = (u_n, f)$$

$$\lambda_n \left(u_n, \sum_{m=1}^{\infty} c_m u_m\right) = (u_n, f)$$

$$\lambda_n c_n (u_n, u_n) = (u_n, f)$$

$$\lambda_n c_n = (u_n, f)$$

$$c_n = (u_n, f)/\lambda_n$$

So

$$u = \sum_{n=1}^{\infty} \frac{(u_n, f)}{\lambda_n} u_n$$ eigenfunction expansion

One further important result is what is known as the Fredholm Alternative theorem or soluability

vector space:

$$A\vec{x} = \vec{b}$$

for vectors we consider the adjoint equation, i.e. the null space of the adjoint

$$A^{T*}\vec{y} = \vec{0}$$

Then

$$A\vec{x}\cdot\vec{y} = \vec{x}\cdot A^{T*}\vec{y} = \vec{0} = \vec{b}\cdot\vec{y}$$

and

$$\vec{b}\cdot\vec{y} = 0 \qquad \text{soluability condition}$$

Function space:

$$Lu = f$$

and we consider the null space of the adjoint operator L^*

$$L^* v = 0$$

So

$$(v, Lu) = (v, f)$$

$$(L^* v, u) = (v, f)$$

$$0 = (v, f)$$

then $\quad (v, f) = 0 \qquad$ soluability condition

The Fredholm Alternative theorem suggests that it is quite important to know the adjoint of a given differential operator L. Thus we define the adjoint L^*.

We begin by considering

$$L = a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx} + c(x) \qquad a < x < b$$

with some boundary conditions

$$\begin{aligned} \alpha_1 u(a) + \beta_1 u'(a) &= 0 \\ \alpha_2 u(b) + \beta_2 u'(b) &= 0 \end{aligned} \qquad \text{Robin boundary condition}$$

So then consider

$$\begin{aligned} (v, Lu) &= \int_a^b v\,(a u_{xx} + b u_x + c u)\,dx \\ &= \int_a^b [v(a u_{xx}) + v b u_x + v c u]\,dx \end{aligned}$$

Now we integrate by parts and note

i. $$\begin{aligned} \int_a^b v(a u_{xx})\,dx &= a v u_x\Big|_a^b - \int_a^b (av)_x u_x \\ &= a v u_x\Big|_a^b - (av)_x u\Big|_a^b + \int_a^b (av)_{xx}\cdot u\,dx \end{aligned}$$

ii. $$\int_a^b v b u_x\,dx = u v b\Big|_a^b - \int_a^b (vb)_x \cdot u$$

So then

$$(v, Lu) = avu_x - (av)_x u + uvb \Big|_a^b + \int_a^b [(av)_{xx} u - (bv)_x u + cvu]\, dx$$

$$= J(u,v) + \int_a^b \left((av)_{xx} - (bv)_x + cv \right) u \, dx$$

$$= J(u,v) + (L^* v, u)$$

Thus the Formal Adjoint is

$$L^*[u] = \frac{d^2}{dx^2}(a(x)u) - \frac{d}{dx}[b(x)u] + c(x)u$$

and $J(u,v)$ is know as the bilinear concomitant or conjuct. So then

$$(v, Lu) - (L^* v, u) = J(u,v) \qquad \text{Green's Formula (Lagrange's Identity)}$$

The adjoint of the problem is found by making the conjunct zero. So then

L^* with boundary conditions so that $J(u,v) = 0$ $\Rightarrow$ adjoint problem

An operator L is said to be self-adjoint if the adjoint is the same as the linear operator so that

$$(v, Lu) = (Lv, u) \qquad L \rightarrow \text{self-adjoint}$$

Note: formal self-adjointness only requires $L = L^*$

Example: Calculate the formal adjoint, the adjoint and determine if the following is self-adjoint

$$L = \frac{d^2}{dx^2} - \frac{d}{dx}$$

with

$$u(0) = 0 \quad \text{and} \quad u'(L) = 0$$

To start, we consider

$$(v, Lu) = \int_0^L v\,(u_{xx} - u_x)\,dx \qquad \text{(integrate by parts)}$$

$$= \left[v u_x - v_x u \right]_0^L + \int_0^L v_{xx} \cdot u\,dx - \left[vu \right]_0^L + \int_0^L v_x u\,dx$$

$$= \left[v u_x - v_x u - vu \right]_0^L + \int_0^L (v_{xx} + v_x)\,u\,dx$$

$$= (L^* v, u) + \big[v(L)u_x(L) - v_x(L)u(L) - v(L)u(L) - v(0)u_x(0) + v_x(0)u(0) + v(0)u(0) \big]$$

$$= (L^* v, u) + \big[-v_x(L)u(L) - v(L)u(L) - v(0)u_x(0) \big]$$

To get the adjoint we must take then

$$v(0) = 0 \quad \text{and} \quad v_x(L) + v(L) = 0$$

So then

$$L^* = \frac{d^2}{dx^2} + \frac{d}{dx} \qquad \text{Formal adjoint}$$

$$L^* = \frac{d^2}{dx^2} + \frac{d}{dx} \quad \text{with} \quad \begin{matrix} v(0) = 0 \\ v_x(L) + v(L) = 0 \end{matrix} \qquad \text{adjoint}$$

The operator L is clearly not self-adjoint or formally self-adjoint

Lecture 9

Sturm-Liouville Theory: nonhomogeneous problems

We would like now to establish a general method for solving the nonhomogeneous Sturm-Liouville problem. In particular, by using the method of eigenfunction expansions, we would like to generate a general solution. Thus we consider

$$Ly = -[p(x)y_x]_x + q(x)y$$

and

$$Ly = \mu r(x)y + f(x) \qquad 0 \le x \le L$$

with

$$\alpha_1 y(0) + \beta_1 y'(0) = 0$$
$$\alpha_2 y(L) + \beta_2 y'(L) = 0$$

and we assume the usual about $p(x)$, $q(x)$ and $r(x)$. Note here that μ is a given constant.

To begin with, we know the following about Sturm-Liouville eigenvalue problems.

1. It is self-adjoint
2. The eigenvalues and eigenfunctions are real
3. The eigenvalues are simple
4. Eigenfunctions are orthogonal with respect to $r(x)$
5. Eigenvalues are ordered: $\lambda_1 < \lambda_2 < \lambda_3 < \ldots < \lambda_n < \ldots <$ with $\lambda_n \to \infty$ as $n \to \infty$
6. The eigenfunctions form a complete set, i.e. we can expand any function in terms of the eigenfunctions.

These facts allow us to generate the general solution. We begin by considering the eigenvalue problem associated with our nonhomogeneous problem:

$$Ly = \lambda r(x) y$$

with boundary conditions. We assume we can find the eigenvalues and eigenfunctions of this problem

eigenvalues: $\lambda_1 < \lambda_2 < \lambda_3 < \dots < \lambda_n < \dots$

eigenfunctions: $\phi_1, \phi_2, \phi_3, \dots, \phi_n, \dots$

where we assume the eigenfunctions are normalized.

We proceed to solve this by our usual eigenfunction expansion. In particular, since our eigenfunctions form a complete set we have

$$y = \phi(x) = \sum_{n=1}^{\infty} b_n \phi_n$$

In addition, we also expand the function f(x) in the eigenfunctions. Actually, due to the r(x), we expand the following

$$\frac{f(x)}{r(x)} = \sum_{n=1}^{\infty} c_n \phi_n$$

We can determine the c_n in this case by taking the inner product of both sides [note: $(u,v) = \int_0^L r(x)\, uv\, dx$]

$$\left(f(x)/r(x), \phi_m \right) = \left(\sum_{n=1}^{\infty} c_n \phi_n, \phi_m \right)$$

$$\int_0^L r(x) \cdot f(x)/r(x)\, \phi_m\, dx = \int_0^L r(x) \left(\sum_{n=1}^{\infty} c_n \phi_n \right) \phi_m\, dx$$

$$\int_0^L f(x)\, \phi_m\, dx = c_m$$

so then

$$c_m = (f, \phi_m)_2 = \int_0^L f(x)\, \phi_m\, dx$$

and $\quad f(x)/r(x) = \sum_{n=1}^{\infty} c_n \phi_n.$

Inserting the expansion for $y = \phi(x)$ and $f(x)/r(x)$ into the nonhomogeneous problem then gives.

$$L\phi = L\left(\sum_{n=1}^{\infty} b_n \phi_n \right) = \sum_{n=1}^{\infty} b_n L\phi_n = \sum_{n=1}^{\infty} b_n \lambda_n r(x) \phi_n$$

$$= \mu r(x)\phi + f(x) = r(x) \cdot \left[\mu \sum_{n=1}^{\infty} b_n \phi_n \right] + r(x) \left[\sum_{n=1}^{\infty} c_n \phi_n \right]$$

So then

$$\sum_{n=1}^{\infty} b_n \lambda_n r(x) \phi_n = r(x) \left[\mu \sum_{n=1}^{\infty} b_n \phi_n + \sum_{n=1}^{\infty} c_n \phi_n \right]$$

or

$$\sum_{n=1}^{\infty} b_n \lambda_n \phi_n = \sum_{n=1}^{\infty} (\mu b_n + c_n) \phi_n$$

Rearranging one final time gives

$$\sum_{n=1}^{\infty} \left[(\lambda_n - \mu) b_n - c_n \right] \phi_n = 0$$

In order for this equality to hold for all x in the interval $0 \leq x \leq L$, we require

$$(\lambda_n - \mu) b_n - c_n = 0 \qquad n = 1, 2, 3, \ldots$$

This gives three different cases

I. $\mu \neq \lambda_n$: $\qquad b_n = \dfrac{c_n}{\lambda_n - \mu} \qquad n = 1, 2, 3, \ldots$

II. $\mu = \lambda_n$: $c_n \neq 0$ $\Rightarrow$ no solution!

III. $\mu = \lambda_n$: $c_n = 0$ $\Rightarrow$ b_n undetermined $\rightarrow$ not unique

The Key is $c_n = 0$ or $c_n \neq 0$, which is a restatement of the Fredholm alternative Theorem.

Back to case I, our solution (general) is then

$$y = \phi(x) = \sum_{n=1}^{\infty} \frac{c_n}{\lambda_n - \mu} \phi_n(x)$$

[note: $c_n = (f, \phi_n)_2$]

Example: Solve the boundary value problem with an eigenfunction expansion

$$y'' + 2y = -x$$

with $y(0)=0$ and $y(1) + y'(1) = 0$.

We rewrite the problem as

$$-y'' = 2y + x$$

To match the Sturm-Liouville form with $p(x)=1$, $q(x)=0$, $r(x)=1$, $\mu=2$, and $f(x)=x$.

The associated eigenvalue problem is

$$y'' + \lambda y = 0$$

with $y(0)=0$ and $y(1)+y'(1)=0$. This has the general solution

$$y = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$$

The first boundary condition gives

$$y(0) = c_1 = 0$$

so that

$$y = c_2 \sin\sqrt{\lambda}x$$

The second boundary condition gives

$$y(1) + y'(1) = c_2 \sin\sqrt{\lambda} + c_2\sqrt{\lambda}\cos\sqrt{\lambda} = 0$$

so that

$$\sin\sqrt{\lambda_n} + \sqrt{\lambda_n}\cos\sqrt{\lambda_n} = 0 \Rightarrow \sqrt{\lambda_n} = -\tan\sqrt{\lambda_n}$$

This is a trancendental equation for the eigenvalues. To find the normalized eigenvalues we note

$$\phi_n = d_n \sin\sqrt{\lambda_n}\, x$$

So that

$$\|\phi\|^2 = \int_0^L \phi_n^2\, dx = d_n^2 \int_0^L \sin^2\sqrt{\lambda_n}\, x\, dx$$

$$= \frac{d_n^2}{2}\int_0^L (1 - \cos 2\sqrt{\lambda_n}\, x)\, dx$$

$$= \frac{d_n^2}{2}(1 + \cos^2\sqrt{\lambda_n}) = 1$$

So then

$$d_n = \left(\frac{2}{1 + \cos^2\sqrt{\lambda_n}}\right)^{1/2}$$

and the eigenfunctions are

$$\phi_n = \frac{\sqrt{2}\,\sin\sqrt{\lambda_n}\, x}{(1 + \cos^2\sqrt{\lambda_n})^{1/2}} \qquad n = 1, 2, 3, \ldots$$

with $\sqrt{\lambda_n} = -\tan\sqrt{\lambda_n}$

From the Sturm-Liouville theory, we know that

$$b_n = \frac{c_n}{\lambda_n - 2}$$

The only thing missing is the c_n coefficients. We can find this from

$$c_n = \int_0^L f(x)\, \phi_n(x)\, dx = d_n \int_0^L x \sin\sqrt{\lambda_n}x\, dx$$

$$= d_n \left(\frac{\sin\sqrt{\lambda_n}}{\sqrt{\lambda_n}} - \frac{\cos\sqrt{\lambda_n}}{\sqrt{\lambda_n}} \right) = d_n \frac{2\sin\sqrt{\lambda_n}}{\lambda_n}$$

So then

$$c_n = \frac{2\sqrt{2}\sin\sqrt{\lambda_n}}{\lambda_n (1+\cos^2\sqrt{\lambda_n})^{1/2}}$$

So $$f(x) = 4\sum_{n=1}^{\infty} \frac{\sin\sqrt{\lambda_n}\sin\sqrt{\lambda_n}x}{\lambda_n(1+\cos^2\sqrt{\lambda_n})}$$

Our eigenfunction expansion solution is then

$$y = \sum_{n=1}^{\infty} \frac{c_n}{\lambda_n - 2}\sin\sqrt{\lambda_n}x$$

or

$$y = 4\sum_{n=1}^{\infty} \frac{\sin\sqrt{\lambda_n}}{\lambda_n(\lambda_n-2)(1+\cos^2\sqrt{\lambda_n})} \sin\sqrt{\lambda_n}x$$

with $\sqrt{\lambda_n} = -\tan\sqrt{\lambda_n}$

Lecture 10

The Dirac Delta function and Green's functions

To begin a discussion on Green's functions, we must first understand how to deal with the Dirac Delta function, which turns out not to be a function at all. Rather, it is what is called a generalized function. The analysis of how to more properly deal with such functions is called Distribution Theory.

To get to the concept of the Delta function, we consider the following equation

$$Lu = f \qquad 0 \leq x \leq L$$

where

$$f(x) = \begin{cases} f_0(x) & x_0 - \xi < x < x_0 + \xi \\ 0 & \text{elsewhere} \end{cases}$$

Thus the forcing $f(x)$ only acts on the differential equation over the interval $x_0 - \xi < x < x_0 + \xi$.

The Impulse of the forcing is defined as

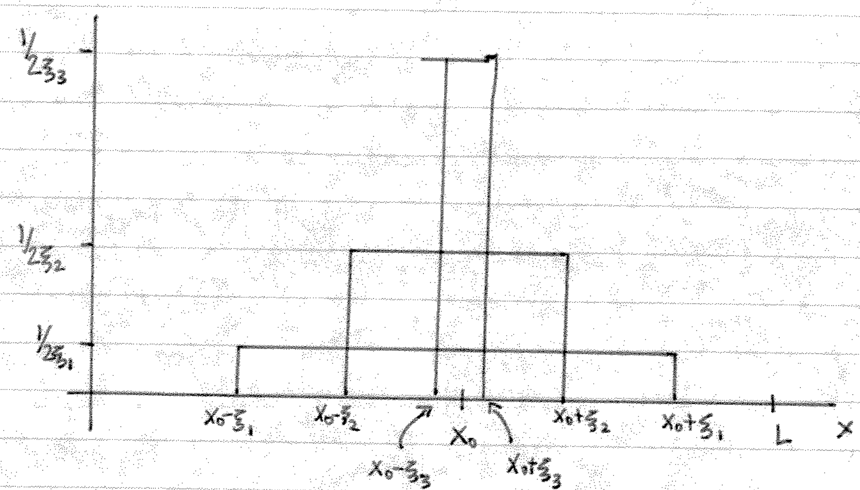
$$I(\xi) = \int_0^L f(x)\,dx = \int_{x_0-\xi}^{x_0+\xi} f_0(x)\,dx$$

As a specific example, consider

$$f(x) = \begin{cases} 1/2\xi & x_0 - \xi < x < x_0 + \xi \\ 0 & \text{elsewhere} \end{cases} \quad \Rightarrow \quad I(\xi) = \int_{x_0-\xi}^{x_0+\xi} \frac{1}{2\xi}\,dx = 1$$

In this case, the impulse is always unity.

To construct the Dirac Delta function (and this is only one way), we consider the limit as $\xi \to 0$. Note that the area under the function, i.e., the impulse remains constant



Thus

$$\delta(x-x_0) = \lim_{\xi \to 0} \begin{cases} 1/2\xi & x_0-\xi < x < x_0+\xi \\ 0 & \text{elsewhere} \end{cases}$$

In this limit then, $\delta(x-x_0)$ is zero everywhere except at x_0. And at x_0, $\delta(x-x_0)$ is infinity, but its impulse is unity.

The delta function therefore only makes sense when integrated. Thus for the above definition of the delta function we find

$$(f(x), \delta(x-x_0)) = \int_0^L f(x)\,\delta(x-x_0)\,dx = \lim_{\xi \to 0} \int_{x_0-\xi}^{x_0+\xi} f(x)\, 1/2\xi \, dx$$

$$= \lim_{\xi \to 0} \frac{1}{2\xi} \int_{x_0-\xi}^{x_0+\xi} f(x)\,dx = \lim_{\xi \to 0} \frac{1}{2\xi} f(\bar{x}) \qquad (x_0-\xi \le \bar{x} \le x_0+\xi)$$

$$= f(x_0) \qquad \text{(sifting property)}$$

On to the Green's Function!

Let's dive right in and go for the kill. This will illustrate the power of the Green's Function method and technique. Recall that in the end we are trying to solve

$$Lu = f \qquad 0 \le x \le L$$

And the Green's function (fundamental solution) does this by calculating L^{-1} so that $u = L^{-1}f$.

The Green's function problem associated with $Lu = f$ is the following

$$L^* G = \delta(x-\xi) \qquad 0 \le x, \xi \le L$$

So then take the inner product

$$(Lu, G) = (u, L^*G) \Rightarrow (f, G) = (u, \delta(x-\xi))$$

and note

$$(u, \delta(x-\xi)) = \int_0^L u(x)\, \delta(x-\xi)\, dx = u(\xi)$$

Thus

$$u(\xi) = (f, G) = \int_0^L f(x)\, G(x, \xi)\, dx$$

and since x and ξ are dummy variables, interchange them

$$\underline{\underline{u(x) = \int_0^L f(\xi)\, G(\xi, x)\, d\xi}}$$

So that $\quad L^{-1}[f] = \int_0^L f(\xi)\, G(\xi, x)\, d\xi$

To illustrate how to use this method and the calculation of the Green's function, we consider the following example.

Example: Solve

$$u_{xx} = f(x) \qquad 0 \le x \le L$$

with $u(0) = 0$
$u_x(L) = 0$

We first note that this problem is Self-adjoint. Therefore $L = L^*$ and the boundary conditions are also the same. The associated Green's function is then calculated from

$$G_{xx} = \delta(x-\xi) \qquad 0 \le x, \xi, \le L$$

with $G(0) = 0$
$G_x(L) = 0$

In order to solve this we note that if we integrate accross ξ from below (ξ^-) to above (ξ^+) we get

$$\int_{\xi^-}^{\xi^+} G_{xx}\,dx = G_x \Big|_{\xi^-}^{\xi^+} = [G_x]_{\xi^-}^{\xi^+} = [G_x]_\xi = \int_{\xi^-}^{\xi^+} \delta(x-\xi) = 1$$

so that

$$[G_x]_\xi = 1$$

Further, we require the solution to be continuous so that

$$[G]_\xi = 0$$

where the brackets denote the jump at $x = \xi$.

We can solve the problem in two parts

I. $x < \xi$: $\quad G_{xx} = 0 \quad$ with $\quad G(0) = 0$

solving for G gives

$$G = Ax + B$$

and applying the condition $G(0)=0 \Rightarrow B=0$ so that

$$G = Ax \qquad x < \xi$$

II. $x > \xi$: $\quad G_{xx} = 0 \quad$ with $\quad G_x(L) = 0$

solving for G again gives

$$G = Cx + D$$

and applying $G_x(L)=0$ yields $C=0$ so that we have

$$G = D$$

So we have two unknowns A and D. However we also have two additional equations.

I. $[G]=0 \Rightarrow G(\xi^+) - G(\xi^-) = D - A\xi = 0$

II. $[G_x]=1 \Rightarrow G_x(\xi^+) - G_x(\xi^-) = 0 - A = 1$

So then

$$A = -1 \quad \text{and} \quad D = A\xi = -\xi$$

To summarize our results, we then have

$$G(x,\xi) = \begin{cases} -x & x<\xi \\ -\xi & x>\xi \end{cases}$$

which is our Green's function or fundamental solution. To solve our original problem, we recall that

$$u(x) = \int_0^L f(\xi)\, G(\xi, x)\, d\xi$$

If, for example, $f(x)=x$ then we would find that

$$\begin{aligned} u(x) &= \int_0^x \xi(-\xi)\,d\xi + \int_x^L \xi\,(-x)\,d\xi \\ &= \int_0^x -\xi^2\,d\xi + \int_x^L -x\,\xi\,d\xi \\ &= -\frac{\xi^3}{3}\Big|_0^x - x\frac{\xi^2}{2}\Big|_x^L \\ &= \frac{-x^3}{3} - x\frac{L^2}{2} + x\frac{x^2}{2} \end{aligned}$$

So then

$$u(x) = \left(\frac{x^2}{2} - \frac{L^2}{2}\right)x - \frac{x^3}{3}$$

You can easily verify that this solution satisfies both the nonhomogeneous problem and the boundary conditions.

Example: Calculate the Green's function for the following problem

$$G_{xx} + k^2 G = \delta(x-\xi) \qquad 0 \le x, \xi \le L$$

with $\quad G_x(0)=0 \quad$ and $\quad G_x(L)=0$

We begin by considering once again the two regions

I. $\quad x \le \xi: \qquad G_{xx} + k^2 G = 0 \qquad$ with $\quad G_x(0)=0$

this gives

$$G = A\cos kx$$

II. $\quad x \ge \xi: \qquad G_{xx} + k^2 G = 0 \qquad$ with $\quad G_x(L)=0$

this gives

$$G = B\cos k(x-L)$$

We apply the jump conditions

$[G]=0: \qquad B\cos k(\xi-L) - A\cos k\xi = 0$

$[G_x]=1: \qquad -kB\sin k(\xi-L) + kA\sin k\xi = 1$

Solving for A and B we find: $\quad A = \dfrac{\cos k(\xi-L)}{k\sin kL}, \quad B = \dfrac{\cos k\xi}{k\sin kL}$

so then

$$G(x,\xi) = \begin{cases} \dfrac{\cos kx \cos k(\xi-L)}{k\sin kL} & x<\xi \\[2ex] \dfrac{\cos k(x-L)\cos k\xi}{k\sin kL} & x>\xi \end{cases}$$

Lecture 11

General Theory for Green's Functions

From the previous lecture, we found that we could calculate the inverse of a given functional equation:

$$Lu = f \quad \Rightarrow \quad u = L^{-1}f$$

This was accomplished with the Green's function. In particular, we solved

$$L^{*}G = \delta(x-\xi) \quad \Rightarrow \quad G(x,\xi)$$

and found

$$u(x) = \int_0^L f(\xi)\, G(\xi, x)\, d\xi$$

So once again the adjoint equation plays a key role. The question should arise concerning the important case of self-adjoint problems, and in particular, Sturm-Liouville theory.

Thus we consider

$$Lu = -[p(x)\,u_x]_x + q(x)\,u = f(x) \qquad 0 \le x \le L$$

Since this operator is self-adjoint, the Green's function satisfies

$$LG = -[p(x)\,G_x]_x + q(x)\,G = \delta(x-\xi) \qquad 0 \le x, \xi \le L$$

with

$$\alpha_1 G(0) + \beta_1 G_x(0) = 0$$
$$\alpha_2 G(L) + \beta_2 G_x(L) = 0$$

As with the previous examples, we must consider the jump of the derivative at $x=\xi$. Recall that we assume G is continuous. Thus

$$[G]_{x=\xi} = 0$$

We then consider

$$LG = -[p(x)G_x]_x + q(x)G = \delta(x-\xi)$$

and integrate accross $x=\xi$. Thus

$$\int_{\xi_-}^{\xi_+} \left(-[p(x)G_x]_x + q(x)G\right)dx = \int_{\xi_-}^{\xi_+} \delta(x-\xi)\,dx$$

so that

$$-[p(x)G_x]_{\xi_-}^{\xi_+} + \int_{\xi_-}^{\xi_+} q(x)G\,dx = 1$$

$$[p(x)G_x]_{x=\xi} = -1$$

or

$$[G_x]_{x=\xi} = -1/p(\xi)$$

So requiring that

$$[G]_{x=\xi} = 0$$

$$[G_x]_{x=\xi} = -1/p(\xi)$$

gives the two conditions necessary to determine the unknown constants.

And if y_1 and y_2 are the solutions for $x<\xi$ and $x>\xi$ respectively, we then find

$$G(x,\xi) = \begin{cases} \dfrac{y_1(x)\,y_2(\xi)}{p(\xi)W(\xi)} & x<\xi \\[2ex] \dfrac{y_1(\xi)\,y_2(x)}{p(\xi)W(\xi)} & x>\xi \end{cases}$$

which is the fundamental solution.

<u>Example:</u> $G_{xx} = \delta(x-\xi)$ $\quad G(0)=0$ and $G_x(L)=0$

so then $\quad x<\xi:\quad G_{xx}=0 \rightarrow G=Ax+B \xrightarrow{\text{B.C.}} G=Ax$

so then $\quad y_1 = x$

$x>\xi:\quad G_{xx}=0 \rightarrow G=Ax+B \xrightarrow{\text{B.C.}} G=B$

so then $\quad y_2 = 1$

I. $W[y_1,y_2] = y_1y_2' - y_1'y_2 = x\cdot 0 - 1\cdot 1 = -1$

II. $p(\xi) = 1$

so then

$$G(x,\xi) = \begin{cases} \dfrac{y_1(x)y_2(\xi)}{p(\xi)W(\xi)} = -x & x<\xi \\[2ex] \dfrac{y_1(\xi)y_2(x)}{p(\xi)W(\xi)} = -\xi & x>\xi \end{cases}$$

Example: $G_{xx} + k^2 G = \delta(x-\xi)$ $\qquad G_x(0) = G_x(L) = 0$

$x < \xi$: $\quad y_1 = \cos kx$

$x > \xi$: $\quad y_2 = \cos k(x-L)$

so then

I. $p(\xi) = 1$

II. $W[y_1,y_2] = y_1 y_2' - y_1' y_2 = \cos kx \cdot -k\sin k(x-L) + k \sin kx \cdot \cos k(x-L)$

$$= k\left[\sin kx \cos k(x-L) - \cos kx \sin k(x-L)\right]$$

$$= k \sin[kx - (kx - kL)] = k \sin kL$$

Thus

$$G(x,\xi) = \begin{cases} \dfrac{y_1(x)\, y_2(\xi)}{p(\xi) W(\xi)} = \dfrac{\cos kx \cos k(\xi - L)}{k \sin kL} & x < \xi \\[2ex] \dfrac{y_1(\xi)\, y_2(x)}{p(\xi) W(\xi)} = \dfrac{\cos k\xi \cos k(x-L)}{k \sin kL} & x > \xi \end{cases}$$

Example: $\dfrac{d}{dr}\left(r \dfrac{dG}{dr}\right) = \delta(r-\rho)$ $\qquad 0 \le r, \rho \le a$

with $G(a) = 0$ and $G(0) = \text{finite}$

This is a singular problem since $r=0$ causes the derivative terms to disappear. The homogeneous problem has the solutions $G = \{1, \ln r\}$ so that

$r < \rho$: $\quad y_1 = 1$ $\qquad \Rightarrow W = 1/r \Rightarrow pW = r \cdot \frac{1}{r} = 1$

$r > \rho$: $\quad y_2 = \ln r - \ln a = \ln(r/a)$

so then

$$G(r,\rho) = \begin{cases} \ln(\rho/a) & r < \rho \\ \ln(r/a) & r > \rho \end{cases}$$

<u>Example:</u> $y'' + 2y = -x$ with $y(0)=0$ and $y(1)+y'(1)=0$

This problem is of Sturm-Liouville type with $p(x)=1$, $q(x)=2$, and $f(x)=-x$. We can therefore consider the self-adjoint problem for the Green's functions.

$$G_{xx} + 2G = \delta(x-\xi) \qquad \begin{aligned} &G(0)=0 \\ &G(1)+G_x(1)=0 \end{aligned}$$

so then

$x<\xi$: $\quad G_{xx} + 2G = 0 \quad$ with $G(0)=0$

$$G = c_1 \sin\sqrt{2}\,x + c_2 \cos\sqrt{2}\,x \Rightarrow G(0)=0 \Rightarrow c_2 = 0$$

so then $\quad y_1 = \sin\sqrt{2}\,x$

$x>\xi$: $\quad G_{xx} + 2G = 0 \quad$ with $G(1)+G_x(1)=0$

$$G = c_1 \sin\sqrt{2}(x-1) + c_2 \cos\sqrt{2}(x-1)$$

and we have the boundary condition

$$G(1) + G_x(1) = c_2 + \sqrt{2}\,c_1 = 0$$

so then $\quad c_2 = -\sqrt{2}\,c_1 \quad$ and

$$y_2 = \sin\sqrt{2}(x-1) - \sqrt{2}\cos\sqrt{2}(x-1)$$

The Wronskian is then

$$W = y_1 y_2' - y_1' y_2 = \sin\sqrt{2}x \left\{\sqrt{2}\cos\sqrt{2}(x-1) + 2\sin\sqrt{2}(x-1)\right\} - \sqrt{2}\cos\sqrt{2}x \left\{\sin\sqrt{2}(x-1) - \sqrt{2}\cos\sqrt{2}(x-1)\right\}$$

This simplies to the following

$$W = \sqrt{2}\left(\sin\sqrt{2}x\cos\sqrt{2}(x-1) - \cos\sqrt{2}x\sin\sqrt{2}(x-1)\right) + 2\left(\sin\sqrt{2}x\sin\sqrt{2}(x-1) + \cos\sqrt{2}x\cos\sqrt{2}(x-1)\right)$$

$$= \sqrt{2}\sin\left[\sqrt{2}x - \sqrt{2}(x-1)\right] + 2\cos\left[\sqrt{2}x - \sqrt{2}(x-1)\right]$$

$$= \sqrt{2}\sin\sqrt{2} + 2\cos\sqrt{2} = \sqrt{2}\left(\sin\sqrt{2} + \sqrt{2}\cos\sqrt{2}\right)$$

We then have (note $p=1$)

$$G(x,\xi) = \begin{cases} \dfrac{\sin\sqrt{2}x\left(\sin\sqrt{2}(\xi-1) - \sqrt{2}\cos\sqrt{2}(\xi-1)\right)}{\sqrt{2}\left(\sin\sqrt{2} + \sqrt{2}\cos\sqrt{2}\right)} & x<\xi \\[2ex] \dfrac{\sin\sqrt{2}\xi\left(\sin\sqrt{2}(x-1) - \sqrt{2}\cos\sqrt{2}(x-1)\right)}{\sqrt{2}\left(\sin\sqrt{2} + \sqrt{2}\cos\sqrt{2}\right)} & x>\xi \end{cases}$$

To finish the problem, we must calculate the following integral

$$u(x) = \int_0^1 f(\xi)\, G(\xi,x)\, d\xi$$

$$= \frac{\int_0^x -\xi\cdot\sin\sqrt{2}\xi\left(\sin\sqrt{2}(x-1) - \sqrt{2}\cos\sqrt{2}(x-1)\right)d\xi + \int_x^1 -\xi\sin\sqrt{2}x\left(\sin\sqrt{2}(\xi-1) - \sqrt{2}\cos\sqrt{2}(\xi-1)\right)d\xi}{\sqrt{2}\left(\sin\sqrt{2} + \sqrt{2}\cos\sqrt{2}\right)}$$

$$= \frac{-\left[\sin\sqrt{2}(x-1) - \sqrt{2}\cos\sqrt{2}(x-1)\right]\int_0^x \xi\sin\sqrt{2}\xi\, d\xi - \sin\sqrt{2}x\int_x^1 \xi\left(\sin\sqrt{2}(\xi-1) - \sqrt{2}\cos\sqrt{2}(\xi-1)\right)d\xi}{\sqrt{2}\left(\sin\sqrt{2} + \sqrt{2}\cos\sqrt{2}\right)}$$

$$= -\frac{\left(\sin\sqrt{2}(x-1) - \sqrt{2}\cos\sqrt{2}(x-1)\right)}{\sqrt{2}\left(\sin\sqrt{2} + \sqrt{2}\cos\sqrt{2}\right)}\left(\frac{1}{2}\sin\sqrt{2}\xi - \frac{\xi}{\sqrt{2}}\cos\sqrt{2}\xi\,\Big|_0^x\right)$$

$$- \frac{\sin\sqrt{2}x}{\sqrt{2}\left(\sin\sqrt{2} + \sqrt{2}\cos\sqrt{2}\right)}\left(\frac{1}{2}\sin\sqrt{2}(\xi-1) - \frac{\xi}{\sqrt{2}}\cos\sqrt{2}(\xi-1) - \sin\sqrt{2}(\xi-1)\,\Big|_x^1\right)$$

Continuing on we find

$$= \frac{-\left(\sin\sqrt{2}(x-1) - \sqrt{2}\cos\sqrt{2}(x-1)\right)}{\sqrt{2}\,(\sin\sqrt{2} + \sqrt{2}\cos\sqrt{2})}\left(\frac{1}{2}\sin\sqrt{2}x - \frac{x}{\sqrt{2}}\cos\sqrt{2}x\right)$$

$$\frac{-\sin\sqrt{2}x}{\sqrt{2}\,(\sin\sqrt{2} + \sqrt{2}\cos\sqrt{2})}\left(-\frac{1}{\sqrt{2}} - \frac{1}{2}\sin\sqrt{2}(x-1) + \frac{x}{\sqrt{2}}\cos\sqrt{2}(x-1) + \sin\sqrt{2}(x-1)\right)$$

$$= \frac{1}{2\sqrt{2}\,(\sin\sqrt{2} + \sqrt{2}\cos\sqrt{2})}\Big\{\left(\sqrt{2}\cos\sqrt{2}(x-1) - \sin\sqrt{2}(x-1)\right)\left(\sin\sqrt{2}x - \sqrt{2}x\cos\sqrt{2}x\right)$$
$$-\sin\sqrt{2}x\left(-\sqrt{2} + \sin\sqrt{2}(x-1) + \sqrt{2}x\cos\sqrt{2}(x-1)\right)\Big\}$$

$$= \frac{1}{2\sqrt{2}}\,\frac{1}{\sin\sqrt{2} + \sqrt{2}\cos\sqrt{2}}\Big\{\sqrt{2}\sin\sqrt{2}x\cos\sqrt{2}(x-1) - 2x\cos\sqrt{2}x\cos\sqrt{2}(x-1) - 2\sin\sqrt{2}x\sin\sqrt{2}(x-1)$$
$$+\sqrt{2}x\left(\sin\sqrt{2}(x-1)\cos\sqrt{2}x - \sin\sqrt{2}x\cos\sqrt{2}(x-1)\right) + \sqrt{2}\sin\sqrt{2}x\Big\}$$

$$= \frac{1}{2\sqrt{2}}\,\frac{1}{\sin\sqrt{2} + \sqrt{2}\cos\sqrt{2}}\Big\{-\sqrt{2}x\sin\sqrt{2} - 2x\cos\sqrt{2}x\cos\sqrt{2}(x-1) + \sqrt{2}\sin\sqrt{2}x$$
$$+\sqrt{2}\left(\sin\sqrt{2}x\cos\sqrt{2}(x-1) - \sqrt{2}\sin\sqrt{2}x\sin\sqrt{2}(x-1)\right)\Big\}$$

$$= \frac{1}{2\sqrt{2}}\,\frac{1}{\sin\sqrt{2} + \sqrt{2}\cos\sqrt{2}}\Big\{-\sqrt{2}x\left(\sin\sqrt{2} + \sqrt{2}\cos\sqrt{2}\right) + 2x\sin\sqrt{2}x\sin\sqrt{2}(x-1) + \sqrt{2}\sin\sqrt{2}x$$
$$+\sqrt{2}\sin\sqrt{2}x\left(\cos\sqrt{2}(x-1) - \sqrt{2}\sin\sqrt{2}(x-1)\right)\Big\}$$

$$= -\frac{x}{2} + \frac{\sqrt{2}\,\sin\sqrt{2}x}{2\sqrt{2}\,(\sin\sqrt{2} + \sqrt{2}\cos\sqrt{2})}\left[\sqrt{2}x\sin\sqrt{2}(x-1) + 1 + \cos\sqrt{2}(x-1) - \sqrt{2}\sin\sqrt{2}(x-1)\right]$$

And if you keep going, and more importantly, if you've done the algebra right, you get

$$u(x) = \frac{\sin\sqrt{2}x}{\sin\sqrt{2} + \sqrt{2}\cos\sqrt{2}} - \frac{x}{2}$$

So although the integral is a bit brutal, the answer is very clean and elegant and doesn't involve any sums.

Regular Perturbation Methods

At this point, one may develop a false sense of security concerning ODEs. After all, we can now solve both initial and boundary value problems. However, for second order equations, everything has been linear. And practically speaking, most systems are nonlinear or weakly nonlinear. And there is no systematic and easy way to solve nonlinear problems.

Given this fact, we will spend the rest of the quarter learning how to approximate solutions.

We begin with the equations:

$$Lu = \epsilon F(u, u_t, t) \qquad u(0) = A, \; u_t(0) = B$$

$$Lu = \epsilon F(u, u_x, t) \qquad \alpha_1 u(0) + \beta_1 u_x(0) = 0$$
$$\alpha_2 u(L) + \beta_2 u_x(L) = 0$$

where $\epsilon << 1$ and F is now a forcing which depends on u, u_x, and t. Here we assume Lu to be some linear operator.

The basic assumptions:

- We know how to solve $Lu = 0$
- We don't know how to solve $Lu = F(u, u', t)$

So if we treat the forcing as a small perturbation, we may be able to solve the full problem in some approximate fashion. This is accomplished using perturbation theory. We approximate by letting

$$u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots$$

and inserting this into the original equation. Thus we have

$$L(u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots) = \epsilon F(u_0 + \epsilon u_1 + \cdots, u_0' + \epsilon u_1' + \cdots, t)$$

with initial conditions (for instance)

$$\begin{aligned} u(0) &= u_0(0) + \epsilon u_1(0) + \epsilon^2 u_2(0) + \cdots = A \\ u_t(0) &= u_0'(0) + \epsilon u_1'(0) + \epsilon^2 u_2'(0) + \cdots = B \end{aligned}$$

The idea then is to collect each order of ϵ separately. So then

$O(1)$: $\quad Lu_0 = 0$

$O(\epsilon)$: $\quad Lu_1 = F_1(u_0, u_0', t)$

$O(\epsilon^2)$: $\quad Lu_2 = F_2(u_0, u_1, u_0', u_1', t)$

$\vdots$

$O(\epsilon^n)$: $\quad Lu_n = F_n(u_0, u_1, \ldots u_{n-1}, u_0', u_1', \ldots u_{n-1}', t)$

We also have the intial conditions

$$u_0(0) = A \qquad u_n(0) = 0 \qquad n \geq 2$$

$$u_0'(0) = B \qquad u_n'(0) = 0 \qquad n \geq 2$$

By doing this, we can solve at each order of the perturbation expansion since all the equations have been linearized. Thus the hardest part is to evaluate the inhomogeneous ODEs at each order. Our solution is then

$$u = \sum_{n=0}^{\infty} \epsilon^n u_n$$

or if we only keep, for example, the first two terms

$$u = u_0 + \epsilon u_1 + O(\epsilon^2)$$

which is a valid approximation up to some time T. Murdock presents a thorough discussion of the validity of such a series.

Our general aim in perturbation theory is as follows

- identify a part of the problem we can solve analytically
- determine if there are any small terms in the system
- Calculate an approximate solution to whatever order of accuracy desired.

Example: An initial value problem

$$y'' - y = \epsilon y^2 \qquad y(0) = \alpha \text{ and } y'(0) = \beta$$

We approximate the solution by seeking a formal expansion solution:

$$y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \cdots$$

Plugging in gives

$$(y_0 + \epsilon y_1 + \cdots)'' - (y_0 + \epsilon y_1 + \cdots) = \epsilon (y_0 + \epsilon y_1 + \cdots)(y_0 + \epsilon y_1 + \cdots)$$

and

$$y_0(0) + \epsilon y_1(0) + \epsilon^2 y_2(0) + \cdots = \alpha$$
$$y_0'(0) + \epsilon y_1'(0) + \epsilon^2 y_2'(0) + \cdots = \beta$$

so that

$$O(1): \quad Ly_0 = y_0'' - y_0 = 0 \qquad y_0(0) = \alpha, \; y_0'(0) = \beta$$

$$O(\epsilon): \quad Ly_1 = y_1'' - y_1 = y_0^2 \qquad y_1(0) = 0, \; y_1'(0) = 0$$

$$O(\epsilon^2): \quad Ly_2 = y_2'' - y_2 = 2 y_0 y_1 \qquad y_2(0) = 0, \; y_2'(0) = 0$$

and so on. $L = d^2/dx^2 - 1$ is the linearized operator. Note that it appears at every order. Solving we find

$$y_0 = (\alpha + \beta) e^t/2 + (\alpha - \beta) e^{-t}/2$$

$$y_1 = (\alpha^2 - 2\alpha\beta - 2\beta^2) e^t/6 + (\alpha^2 + 2\alpha\beta - 2\beta^2) e^{-t}/6 + (\alpha+\beta)^2 e^{2t}/12 + (\alpha-\beta)^2 e^{-2t}/12 - (\alpha^2 - \beta^2)/2$$

and

$$y = y_0 + \epsilon y_1 + O(\epsilon^2)$$

This example shows clearly how messy things can get.

<u>Example:</u> A boundary value problem

$$y'' - y = \epsilon y^2 \qquad y(0) = \alpha \text{ and } y(1) = \gamma$$

Once again, we seek the formal solution expansion

$$y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \cdots$$

Plugging in gives the same as the last example with the boundaries

$$y(0) = y_0(0) + \epsilon y_1(0) + \cdots = \alpha$$
$$y(1) = y_0(1) + \epsilon y_1(1) + \cdots = \gamma$$

so then

$$O(1): \quad Ly_0 = y_0'' - y_0 = 0 \qquad y_0(0) = \alpha, \; y_0(1) = \gamma$$

$$O(\epsilon): \quad Ly_1 = y_1'' - y_1 = y_0^2 \qquad y_1(0) = 0, \; y_1(1) = 0$$

$$O(\epsilon^2): \quad Ly_2 = y_2'' - y_2 = 2y_0 y_1 \qquad y_2(0) = 0, \; y_2(1) = 0$$

which can be solved at each order to yield

$$y_0 = \left(\frac{\alpha - \gamma e}{1 - e^2}\right) e^x - e\left(\frac{\alpha e - \gamma}{1 - e^2}\right) e^{-x}$$

The nonhomogeneous equation for y_1 can be solved via the Green's function method or undetermined coefficients. This yields

$$y_1 = A + Be^x + Ce^{-x} + De^{2x} + Ee^{-2x}$$

where A, B, C, D and E can be determined with way too much effort. Then

$$y = y_0 + \epsilon y_1 + O(\epsilon^2)$$

<u>Example:</u> An eigenvalue problem

$$y'' + \lambda y = \epsilon f(y) \qquad y(0) = y(1) = 0$$

Now, we expand not only y, but also λ since $f(y)$ may alter its value

$$y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \cdots$$

$$\lambda = \lambda_0 + \epsilon \lambda_1 + \epsilon^2 \lambda_2 + \cdots$$

So then at leading order and the first correction

$O(1)$ $$y_0'' + \lambda_0 y_0 = 0 \qquad y_0(0) = y_0(1) = 0$$

$O(\epsilon)$ $$y_1'' + \lambda_0 y_1 = f(y_0) - \lambda_1 y_0 \qquad y_1(0) = y_1(1) = 0$$

Solving the $O(1)$ problem gives

$\lambda_0 = n^2\pi^2$: $$y_0 = A \sin n\pi x$$

And at next order with the n^{th} term

$$y_1'' + n^2\pi^2 y_1 = f(A \sin n\pi x) - \lambda_1 A \sin n\pi x$$

we can expand $f = \sum_{m=1}^{\infty} a_m \sin m\pi x$ so that

$$y_1'' + n^2\pi^2 y_1 = (a_n - \lambda_1 A) \sin n\pi x + \sum_{\substack{m=1 \\ m \neq n}}^{\infty} a_m \sin m\pi x$$

The inhomogeneous term $\sin n\pi x$ is resonant and generates a term which does not satisfy the boundary conditions. Therefore we must take

$$\lambda_1 = a_n / A$$

and

$$y_1(x) = c \sin n\pi x + \sum_{n \neq m} \frac{a_m}{(n^2 - m^2)\pi^2} \sin m\pi x$$

The Poincare-Lindstedt Method

We begin this section by considering oscillatory phenomena and the limitations of a regular perturbation expansion.

As a simple example we consider

$$y'' + (1+\epsilon)^2 y = 0 \qquad y(0)=\alpha \text{ and } y'(0)=0$$

This can be easily solved by our standard methods and yields the solution

$$y = \alpha \cos (1+\epsilon)t$$

Thus the solution has the period $T = \frac{2\pi}{1+\epsilon}$ and frequency $\nu(\epsilon) = 1+\epsilon$.

What does a regular expansion yield? We then let

$$y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \dots$$

So then

$$O(1): \quad y_0'' + y_0 = 0 \qquad y_0(0)=\alpha,\ y_0'(0)=0$$

$$O(\epsilon): \quad y_1'' + y_1 = -2y_0 \qquad y_1(0) = y_1'(0) = 0$$

$$O(\epsilon^2): \quad y_2'' + y_2 = -2y_1 - y_0 \qquad y_2(0) = y_2'(0) = 0$$

The leading order problem is easily solved and yields

$$y_0 = \alpha \cos t$$

We then need to solve

$$y_1'' + y_1 = -2y_0 = -2\alpha \cos t \qquad y_1(0) = y_1'(0) = 0$$

The homogeneous solution is

$$y_1 = c_1 \cos t + c_2 \sin t$$

and we guess a particular solution

$$y_p = A\, t \sin t$$

So then $y_p' = A \sin t + A t \cos t$ and $y_p'' = 2A \cos t - A t \sin t$. This gives

$$2A \cos t - At \sin t + At \sin t = -2\alpha \cos t$$

so then $A = -\alpha$ and

$$y = y_0 + \epsilon y_1 + O(\epsilon^2)$$

$$y = \alpha \cos t - \epsilon \alpha t \sin t + O(\epsilon^2)$$

So if $t \ll 1$, then everything is fine. But as $t \sim O(1/\epsilon)$, then $y_0 \sim y_1$ and the perturbation approximation fails to hold.

The $t \sin t$ term is called a secular term or spurious term. It limits the validity of the perturbation expansion.

The basic problem: the regular perturbation expansion does not allow for corrections in the period (or frequency).

We remedy this situation with the Poincare-Lindstedt method. Thus we introduce the strained time

$$\tau = \omega(\epsilon)t = (\omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots)t$$

which allows for frequency corrections. We then consider

$$y'' + k^2 y = \epsilon f(y) \qquad y(0)=\alpha \text{ and } y'(0)=0$$

Note that $y_t = y_\tau \tau_t = \omega y_\tau \rightarrow y_{tt} = \omega^2 y_{\tau\tau}$ and $y'(0) = \omega y_\tau(0) = 0$. So then

$$\omega^2 y_{\tau\tau} + k^2 y = \epsilon f(y) \qquad y(0)=\alpha \text{ and } y_\tau(0)=0$$

Then we expand

$$y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \cdots$$
$$\omega = \omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots$$

So then at $O(1)$, $\omega_0^2 y_{0\tau\tau} + k^2 y_0 = 0$. And if we want 2π periodic solutions, we take $\omega_0 = k$ so that

$$O(1): \quad y_{0\tau\tau} + y_0 = 0 \qquad y_0(0)=\alpha,\ y_{0\tau}(0)=0$$

$$O(\epsilon): \quad y_{1\tau\tau} + y_1 = f(y_0)/k^2 - 2\omega_1 y_{0\tau\tau}/k \qquad y_1(0) = y_{1\tau}(0) = 0$$

And so forth. An example will serve to illustrate the method nicely.

<u>Example</u>: The Van der Pol Oscillator

$$y'' + \epsilon (y^2-1)y' + y = 0$$

We will use the Poincore-Lindstedt method to solve this problem. However, we note the following things

- if $y > 1$ then $y^2-1 > 1$ and the damping coefficient is positive leading to damping.
- if $y < 1$ then $y^2-1 < 1$ and the damping coefficient is negative leading to growth.

Intuitively then, we can expect the existence of what is called a limit cycle. We use the Poincore-Lindstedt method to construct it. We begin by defining the strained time

$$\tau = \omega(\epsilon) t$$

so that

$$\omega^2 y_{\tau\tau} + \epsilon (y^2-1)\, \omega y_\tau + y = 0$$

and then we expand

$$y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \cdots$$

$$\omega = \omega_0 + \epsilon \omega_1 + \epsilon^2 \omega_2 + \cdots$$

And further, we don't know the correct initial conditions. So then let

$$y(0) = a_0 + \epsilon a_1 + \epsilon^2 a_2 + \cdots$$
$$y'(0) = 0$$

And we determine the a_i which give the limit cycle.

Plugging in and collecting powers we find

$O(1)$: $\quad \omega_0^2 y_{0\tau\tau} + y_0 = 0 \qquad y_0(0) = a_0 \qquad y_0'(0) = 0$

This problem is easily solved and yields the solution

$$y_0 = a_0 \cos(\tau/\omega_0)$$

To construct a 2π periodic solution we require $\omega_0 = 1$ so that

$$y_0 = a_0 \cos \tau$$

$O(\epsilon)$:
$$\begin{aligned} y_{1\tau\tau} + y_1 &= -2\omega_1 y_{0\tau\tau} - (y_0^2 - 1)\, y_{0\tau} \\ &= 2\omega_1 a_0 \cos\tau + (a_0^2\cos^2\tau - 1)\, a_0 \sin\tau \\ &= 2a_0\omega_1 \cos\tau + a_0\left(\frac{a_0^2}{4} - 1\right)\sin\tau + \frac{a_0^3}{4}\sin 3\tau = F_1 \end{aligned}$$

To get rid of secular growth terms, we must require the $\cos\tau$ and $\sin\tau$ terms to vanish. We then have (note, we don't want $a_0 = 0$)

$$2a_0\omega_1 = 0 \quad \Rightarrow \quad \omega_1 = 0$$

$$a_0\left(a_0^2/4 - 1\right) = 0 \quad \Rightarrow \quad a_0 = 2$$

Alternatively, we could apply Fredholm-Alternative. We note the adjoint equation is $V_{\tau\tau} + V = 0$ which produces both $\sin\tau$ and $\cos\tau$ solutions. Thus

$$(F_1, \cos\tau) = 0 \quad \Rightarrow \quad \omega_1 = 0$$
$$(F_1, \sin\tau) = 0 \quad \Rightarrow \quad a_0 = 2$$

$O(\epsilon^2)$ $$Y_{2\tau\tau} + Y_2 = -2\omega_2 Y_{0\tau\tau} - \omega_1^2 Y_{1\tau\tau} - (Y_0^2 - 1) Y_{1\tau}$$

$$- \omega_1 (Y_0^2 - 1) Y_{0\tau} - 2 Y_0 Y_{0\tau} Y_1$$

but since $\omega_1 = 0$

$$Y_{2\tau\tau} + Y_2 = -2\omega_2 Y_{0\tau\tau} - (Y_0^2 - 1) Y_{1\tau} - 2 Y_0 Y_1 Y_{0\tau}$$

$$= 4\omega_2 \cos\tau + \left(\tfrac{1}{4} - \cos^2\tau\right)(3\cos\tau - 3\cos 3\tau - 4a_1 \sin\tau)$$

$$-2\sin\tau\cos\tau\,(3\sin\tau - \sin 3\tau + 4a_1\cos\tau) = F_2$$

Using a little trig identity magic, we then find

$$F_2 = \left(4\omega_2 - \tfrac{7}{4}\right)\cos\tau - 2a_1 \sin\tau + \cos 3\tau - a_1 \sin 3\tau - \tfrac{5}{4}\cos 5\tau$$

Either by applying solvability (Fredholm) or just simply getting rid of secular growth terms we find

$$4\omega_2 - 7/4 = 0 \quad \Rightarrow \quad \omega_2 = 7/16$$

$$2a_1 = 0 \quad \Rightarrow \quad a_1 = 0$$

Our solution is thus

$$Y(0) = 2 + O(\epsilon^2)$$
$$\tau = (1 + \epsilon^2 7/16 + O(\epsilon^4))\,t$$

and

$$Y = 2\cos[(1 + 7\epsilon^2/16)\,t] + \epsilon\left[\tfrac{3}{4}\sin[(1+7\epsilon^2/16)\,t] - \tfrac{1}{4}\sin 3[(1+7\epsilon^2/16)\,t]\right] + O(\epsilon^2)$$

Forced Nonlinear Oscillations: Duffings Equation

We begin by considering the origins of the damped and forced Duffings equation. The beginning point is the nonlinear equation for a pendulum which can be written

$$y'' + \sin y = 0$$

If we add some forcing of strength γ and frequency ω and in addition allow for dissipational or frictional forces we find

$$\underbrace{y'' + \delta y'}_{\text{damping}} + \sin y = \underbrace{\gamma \cos \omega t}_{\text{harmonic driving}}$$

We also assume the following initial conditions

$$y(0) = \alpha \quad \text{and} \quad y'(0) = \beta$$

Thus this problem contains six parameters: $\delta, \gamma, \omega, \alpha, \beta$ and the sixth which will correspond to the nonlinear strength. In particular, the typical thing to do is assume $y \ll 1$ and then

$$\sin y \approx y$$

But if y is not so small, then

$$\sin y \approx y - y^3/3! + \cdots = y - \frac{y^3}{3!} + O(y^5)$$

Thus we introduce a cubic term to the fray, i.e. the Duffing term.

Linear theory: no damping

We begin by considering the simplest case, that of no nonlinearity or damping. Thus

$$y'' + y = r \cos \omega t$$

Of particular interest is when $\omega = 1$ and we have resonant forcing. Thus if we take

$$y'' + y = r \cos t \qquad y(0) = y'(0) = 0$$

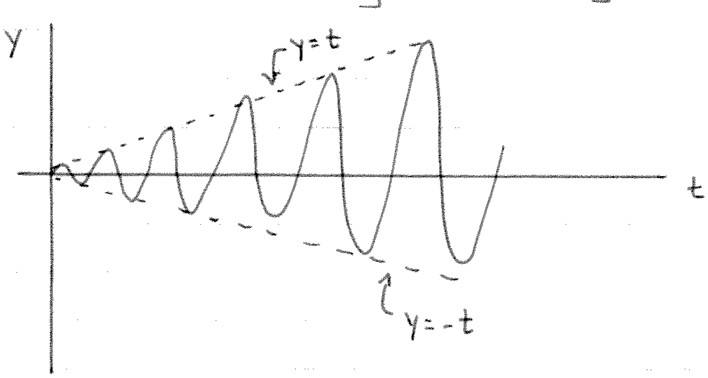
Then we can guess $y_p = At \sin t$ and since $y_p'' = 2A \cos t - At \sin t$ then

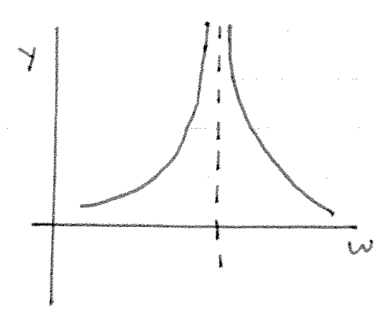
$$(2A \cos t - At \sin t) + At \sin t = r \cos t$$

so then $2A = r$ and $A = r/2$. Our total solution is then $y = c_1 \cos t + c_2 \sin t + r/2\, t \sin t$. Thus if $y(0) = 0 = c_1$ and $y'(0) = c_2 = 0$ so then

$$y = \frac{r}{2} t \sin t$$

This suggests that our solutions grow without bound if harmonically (resonantly) forced.





Linear Theory: damping

We now simply add damping to our previous calculations. Thus

$$y'' + \delta y' + y = \gamma \cos \omega t$$

If the damping is small, then the homogeneous solutions are

$$y = c_1 e^{-\delta/2\, t} \cos \sqrt{1-\delta^2/4}\, t + c_2 e^{-\delta/2\, t} \sin \sqrt{1-\delta^2/4}\, t$$

And we can find the particular solution by guessing $y_p = A\cos\omega t + B\sin\omega t = A\cos t + B\sin t$ (at resonance). Thus

$$y = c_1 e^{-\delta/2\, t} \cos\sqrt{1-\delta^2/4}\, t + c_2 e^{-\delta/2\, t} \sin\sqrt{1-\delta^2/4}\, t + \frac{\delta\omega\gamma}{(1-\omega^2)^2+\delta^2\omega^2}\sin\omega t + \frac{\gamma(1-\omega^2)}{(1-\omega^2)^2+\delta^2\omega^2}\cos\omega t$$

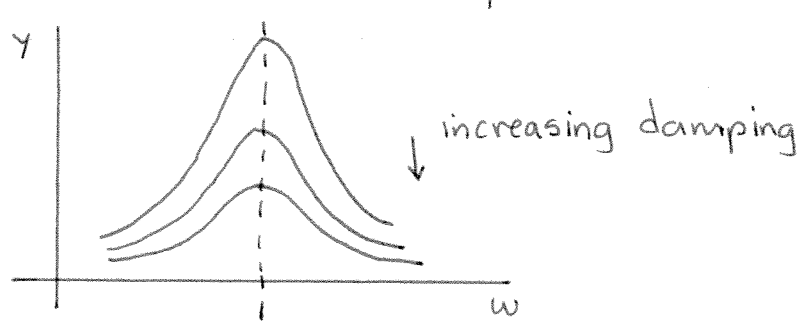
and inserting the initial conditions: (as $t\to\infty$, homogeneous drops out!)

$$y = \frac{\delta\omega\gamma}{(1-\omega^2)+\delta^2\omega^2}\sin\omega t + \frac{\gamma(1-\omega^2)}{(1-\omega^2)^2+\delta^2\omega^2}\cos\omega t$$

Which can be rewritten as

$$y = \frac{\gamma}{(1-\omega^2)^2+\delta^2\omega^2}\left((1-\omega^2)\cos\omega t + \omega\delta\sin\omega t\right) \qquad \text{as } t\to\infty$$

Note now there is no blow-up of solutions.



Nonlinear Theory: no damping

we now consider the effect of nonlinearity without damping

$$y'' + y + \kappa y^3 = r \cos \omega t$$

We do Poincare-Lindstedt

$$\tau = \omega t$$

so then

$$y_{\tau\tau} + \frac{1}{\omega^2} y + \frac{\kappa}{\omega^2} y^3 = \frac{r}{\omega^2} \cos \tau$$

or rewritting

$$y_{\tau\tau} + y = \epsilon \left(\Gamma \cos \tau - \beta y + y^3 \right)$$

where $\frac{r}{\omega^2} = \epsilon \Gamma$, $1/\omega^2 = 1 + \epsilon\beta$, and $\epsilon = -\kappa/\omega^2$. Note that we have only kept the next order term in the expansion since we only get the first correction. Expanding $y = y_0 + \epsilon y_1 + \cdots$

$O(1)$: $\quad y_{0\tau\tau} + y_0 = 0 \quad \Rightarrow \quad y_0 = a_0 \cos \tau + b_0 \sin \tau$

$O(\epsilon)$: $\quad y_{1\tau\tau} + y_1 = \Gamma \cos \tau - \beta y_0 + y_0^3$

$$= \left\{ \Gamma - \beta a_0 + \tfrac{3}{4} a_0 (a_0^2 + b_0^2) \right\} \cos \tau + \left\{ b_0 \left(-\beta + \tfrac{3}{4}(a_0^2 + b_0^2) \right) \right\} \sin \tau$$
$$+ \tfrac{a_0}{4} (a_0^2 - 3b_0^2) \cos 3\tau + \tfrac{b_0}{4} (3a_0^2 - b_0^2) \sin 3\tau$$

Removing secular terms: $\quad \Gamma = a_0 \left[\beta - \tfrac{3}{4}(a_0^2 + b_0^2) \right], \quad b_0 \left[\beta - \tfrac{3}{4}(a_0^2 + b_0^2) \right] = 0$

so then

$$b_0 = 0 \quad \text{and} \quad a_0 \left(\beta - \tfrac{3}{4} a_0^2 \right) = \Gamma$$

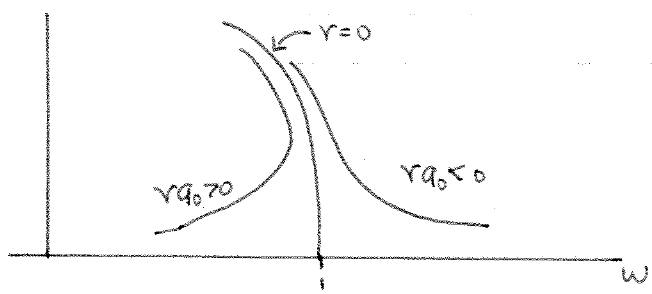
After manipulating back into the original variables, we find

$$\omega^2 = 1 + \frac{3}{4} K a_0^2 - \frac{r}{a_0} \qquad (a_0 \neq 0)$$

or if $r = 0$, then

$$a_0 = 0 \quad \text{or} \quad \pm\sqrt{4(\omega^2 - 1)/3K}$$

In total then, since a_0 measures the response as a function of frequency, we find



Thus solutions still blow up! But only along the curve $r=0$. For any given ω, there is a finite solution. So nonlinearity acts to inhibit growth just as the dissipation does.

Nonlinear Theory: damping

We start now with

$$y'' + \delta y' + y + ky^3 = r \cos \omega t$$

By Poincare-Lindstedt, we can once again rewrite this as

$$y_{\tau\tau} + y = \epsilon \left(\Gamma \cos \tau - \beta y + y^3 - \Delta y_\tau \right)$$

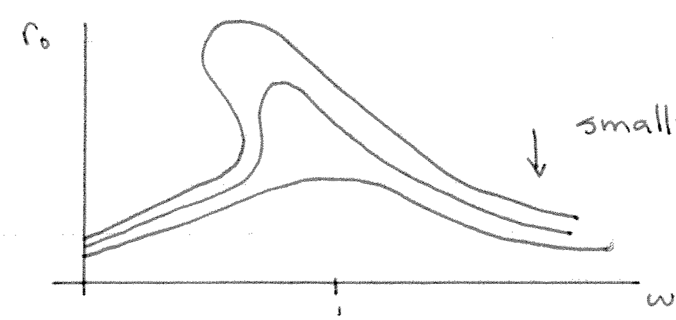
where Γ, β, Δ, etc. are suitably redefined leading order corrections. With $y_0 = a_0 \cos \tau + b_0 \sin \tau$, solvability at next order gives

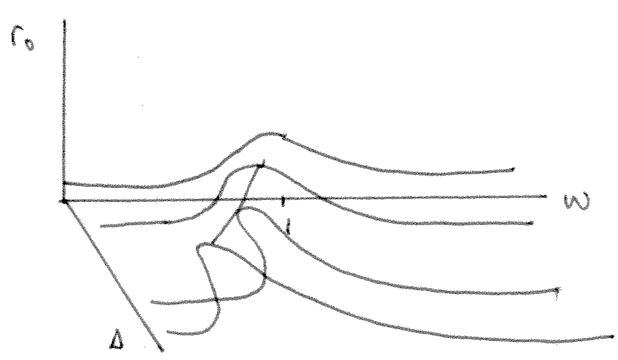
$$\Delta b_0 + a_0 \left(\beta - \tfrac{3}{4}(a_0^2 + b_0^2) \right) = \Gamma$$

$$\Delta a_0 - b_0 \left(\beta - \tfrac{3}{4}(a_0^2 + b_0^2) \right) = 0$$

Defining $r_0 = (a_0^2 + b_0^2)^{1/2}$ then gives

$$r_0^2 \left[\Delta^2 + \left(\beta - \tfrac{3}{4} r_0^2 \right)^2 \right] = \Gamma^2$$





Cusp catastrophe!

Lecture 15

Multiple Scales

Although the Poincare-Lindstedt method is a very powerful technique, it is limited in scope to periodic solutions only. Thus, for the Van der Pol equation, we were only able to find the limit cycle. The method of multiple scales removes some of the limitations.

We begin the study of multiple scales by considering the weakly-damped pendulum problem

$$y'' + 2\epsilon y' + (1+\epsilon)y = 0 \qquad y(0)=\alpha, \; y'(0)=0$$

The exact solution to this problem is

$$y = \alpha e^{-\epsilon t} \cos\sqrt{1+\epsilon-\epsilon^2}\, t + \frac{\epsilon\alpha}{\sqrt{1+\epsilon-\epsilon^2}} e^{-\epsilon t} \sin\sqrt{1+\epsilon-\epsilon^2}\, t$$

which can be found by elementary methods.

The all important observations of this solution

1. The damping occurs on the time scale ϵt
2. The frequency shifts also occur on times ϵt
3. The leading order behavior occurs on scale t (i.e. the oscillatory behavior)

Just as in the Poincare-Lindstedt method, we then define a new time to pick up the "slow time" dynamics of the damping and frequency shifts. The new slow time is

$$\tau = \epsilon t$$

Thus our solution is

$$y = y(t, \tau)$$

that is, a function of both t and τ which will be considered independent variables.

Let us then try to solve this problem using this "multiple scale" (i.e. t and τ) perturbation technique. We first note, however, that the chain rule gives

i. $$y' = y_t + y_\tau \tau_t = y_t + \epsilon y_\tau$$

ii. $$y'' = (y_t + \epsilon y_\tau)_t = y_{tt} + y_{t\tau}\tau_t + \epsilon y_{\tau t} + \epsilon y_{\tau\tau}\tau_t$$

$$= y_{tt} + 2\epsilon y_{t\tau} + \epsilon^2 y_{\tau\tau}$$

We are now ready to expand our solution as

$$y = y_0(t,\tau) + \epsilon y_1(t,\tau) + \epsilon^2 y_2(t,\tau) + \dots$$

and collect powers of epsilon.

Plugging in we find

$O(1)$: $\quad y_{0_{tt}} + y_0 = 0 \qquad y_0(0,0) = \alpha, \quad y_{0_t}(0,0) = 0$

$O(\epsilon)$: $\quad y_{1_{tt}} + y_1 = -[y_0 + 2y_{0_t} + 2y_{0_{t\tau}}]$

with $\quad y_1(0,0) = 0 \quad$ and $\quad y_{1_t}(0,0) = -y_{0_\tau}(0,0)$

$O(\epsilon^2)$: $\quad y_{2_{tt}} + y_2 = -[y_1 + 2y_{1_t} + 2y_{1_{t\tau}} + 2y_{0_\tau} + y_{0_{\tau\tau}}]$

with $\quad y_2(0,0) = 0 \quad$ and $\quad y_{2_t}(0,0) = -y_{1_\tau}(0,0)$

At leading order then, we find

$$y_0 = A(\tau)\cos t + B(\tau)\sin t$$

where $A(0) = \alpha$ and $B(0) = 0$. Thus we no longer have "constants", but rather functions of our slow-time τ. At the next order we find

$$y_{1_{tt}} + y_1 = -[y_0 + 2y_{0_t} + 2y_{0_{t\tau}}]$$

$$= -(A + 2B + 2B_\tau)\cos t + (-B + 2A + 2A_\tau)\sin t$$

Fredholm-Alternative requires us to get rid of the secular growth terms. Thus we must have

$$A_\tau = \tfrac{1}{2}B - A$$

$$B_\tau = -B - \tfrac{1}{2}A \qquad \text{with} \quad A(0) = \alpha, \; B(0) = 0$$

We must now solve for A and B which satisfy a 2×2 system of equations. We can multiply the top equation by $2A$ and bottom by $2B$ to find

$$2AA_\tau = AB - 2A^2$$

$$2BB_\tau = -2B^2 - AB$$

Adding together we find

$$2AA_\tau + 2BB_\tau = -2\,(A^2+B^2)$$

$$(A^2)_\tau + (B^2)_\tau = -2\,(A^2+B^2)$$

$$(A^2+B^2)_\tau = -2\,(A^2+B^2)$$

Thus we can solve for A^2+B^2

$$A^2+B^2 = ce^{-2\tau}$$

At $\tau=0$: $\quad A(0)^2 + B^2(0) = \alpha^2 + 0^2 = \alpha^2 = c$ so then

$$A^2+B^2 = \alpha^2 e^{-2\tau} = (\alpha e^{-\tau})^2$$

So then if

$$\begin{aligned} A(\tau) &= \alpha e^{-\tau} \cos \xi(\tau) \\ B(\tau) &= \alpha e^{-\tau} \sin \xi(\tau) \end{aligned} \quad \Rightarrow \quad A^2+B^2 = (\alpha e^{-\tau})^2$$

We simply now need to determine the $\xi(\tau)$.

Plug $A = \alpha e^{-\tau} \cos \xi(\tau)$ into the $A_\tau = \frac{1}{2}B - A$ equation.
so then

$$2A_\tau + 2A - B = 2\left[-\alpha e^{-\tau}\cos\xi - \alpha \xi_\tau e^{-\tau}\sin\xi\right]$$

$$+ 2\alpha e^{-\tau}\cos\xi - \alpha e^{-\tau}\sin\xi$$

$$= -\alpha\left(2\xi_\tau + 1\right)\sin\xi = 0$$

so then

$$\xi_\tau = -\tfrac{1}{2}$$

$$\xi = -\tau/2$$

and then

$$y_0 = A(\tau)\cos t + B(\tau)\sin t$$

which gives

$$y_0(t,\tau) = \alpha e^{-\tau}\left\{\cos\tfrac{\tau}{2}\cos t - \sin\tfrac{\tau}{2}\sin t\right\}$$

or

$$\underline{\underline{y_0(t,\tau) = \alpha e^{-\tau}\cos\left(t + \tau/2\right)}}$$

so then $y = y_0(t,\tau) + O(\epsilon)$.

A comparison is in order. How well did the multiple scale calculation work. Recall that the exact solution was

$$y = \alpha e^{-\epsilon t} \cos \sqrt{1+\epsilon-\epsilon^2}\, t \; + \underbrace{\frac{\epsilon \alpha}{\sqrt{1+\epsilon-\epsilon^2}} e^{-\epsilon t} \sin \sqrt{1+\epsilon-\epsilon^2}\, t}_{O(\epsilon)}$$

Thus

$$y = \alpha e^{-\epsilon t} \cos \sqrt{1+\epsilon-\epsilon^2}\, t \quad + O(\epsilon)$$

Now we note that

$$(1+\epsilon-\epsilon^2)^{1/2} = 1 + \frac{\epsilon}{2} - \frac{5}{8}\epsilon^2 + \dots$$

Thus

$$y = \alpha e^{-\epsilon t} \cos (1+\epsilon/2)\, t \quad + O(\epsilon)$$

Recall that

$$y_0 = \alpha e^{-\tau} \cos (t + \tau/2) \quad + O(\epsilon)$$

$$= \alpha e^{-\epsilon t} \cos (1+\epsilon/2) t \quad + O(\epsilon)$$

Thus the exact solution and our approximation are identical at leading order!

Lecture 16

Applications of Multiple Scales: The Van der Pol Equation

We now apply the method of multiple scales to the Van der Pol equation. Our hope now is to do much better than either the regular or Poincare-Lindstedt expansion problems.

Therefore we consider

$$y'' + \epsilon(y^2-1)y' + y = 0$$

with

$$y(0)=\alpha \quad \text{and} \quad y'(0)=0$$

We now define the slow-scale

$$\tau = \epsilon t$$

Thus with $y' = y_t + \epsilon y_\tau$ and $y'' = y_{tt} + 2\epsilon y_{t\tau} + \epsilon^2 y_{\tau\tau}$ we have

$$(y_{tt} + 2\epsilon y_{t\tau} + \epsilon^2 y_{\tau\tau}) + \epsilon(y^2-1)(y_t + \epsilon y_\tau) + y = 0$$

We then expand in powers of ϵ.

$$y = y_0(t,\tau) + \epsilon y_1(t,\tau) + \epsilon^2 y_2(t,\tau) + \cdots$$

Recall further that $y'(0)=0 \Rightarrow \left. y_t + \epsilon y_\tau \right|_{\substack{\text{at} \\ t=\tau=0}} = 0$

Collecting powers we find

$O(1)$: $\quad y_{0_{tt}} + y_0 = 0 \qquad y_0(0,0) = \alpha\ , \quad y_{0_t}(0,0) = 0$

$O(\epsilon)$: $\quad y_{1_{tt}} + y_1 = -\left[2y_{0_{t\tau}} + (y_0^2 - 1)\, y_{0_t}\right] \qquad y_1(0,0) = 0\ , \quad y_{1_t}(0,0) = -y_{0_\tau}(0,0)$

$O(\epsilon^2)$: $\quad y_{2_{tt}} + y_2 = -\left[2y_{1_{t\tau}} + (y_0^2 - 1)\, y_{1_t} + 2y_1 y_0 y_{0_t} + (y_0^2 - 1)\, y_{0_\tau} + y_{0_{\tau\tau}}\right]$

$y_2(0,0) = 0\ , \quad y_{2_t}(0,0) = -y_{1_\tau}(0,0)$

We can now solve at each order

$O(1)$: $\quad y_{0_{tt}} + y_0 = 0 \qquad y_0(0,0) = \alpha\ , \quad y_{0_t}(0,0) = 0$

$$y_0 = A(\tau)\cos t + B(\tau)\sin t$$

The initial conditions then give

$$y_0(0,0) = A(0)\cos 0 + B(0)\sin 0 = A(0) = \alpha$$

$$y_{0_t}(0,0) = -A(0)\sin 0 + B(0)\cos 0 = B(0) = 0$$

So then

$$y_0(t,\tau) = A(\tau)\cos t + B(\tau)\sin t$$

where $\quad A(0) = \alpha$
$\quad B(0) = 0$

At next order we find

$$O(\epsilon): \qquad y_{1\,tt} + y_1 = -\left[2y_{0\,t\tau} + (y_0^2 - 1)\,y_{0t}\right] \qquad y_1(0,0) = 0\ ,\ \ y_{1t}(0,0) = -y_{0\tau}(0,0)$$

so then

$$2y_{0\,t\tau} + (y_0^2-1)\,y_{0t} = -2A_\tau \sin t + 2B_\tau \cos t - (-A\sin t + B\cos t)$$

$$+ (A^2\cos^2 t + 2AB\sin t\cos t + B^2\sin^2 t)(-A\sin t + B\cos t)$$

$$= -2A_\tau \sin t + 2B_\tau \cos t - A^3(1-\sin^2 t)\sin t$$

$$-2A^2B(1-\cos^2 t)\cos t - AB^2\sin^3 t + A^2B\cos^3 t$$

$$+2AB^2(1-\sin^2 t)\sin t + B^3(1-\cos^2 t)\cos t + A\sin t - B\cos t$$

$$= \sin t\,(-2A_\tau - A^3 + 2AB^2 + A) + \sin^3 t\,(A^3 - AB^2 - 2AB^2)$$

$$+\cos t\,(2B_\tau - 2A^2B + B^3 - B) + \cos^3 t\,(2A^2B + A^2B - B^3)$$

Now recalling that $\sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$, $\cos^3 x = \frac{3}{4}\cos x + \frac{1}{4}\cos 3x$
we then find

$$= \sin t\,(-2A_\tau - A^3 + 2AB^2 + 3/4\,A^3 - 9/4\,AB^2 + A) + {}^{-}1/4\,\sin 3t\,(A^3 - 3AB^2)$$

$$+\cos t\,(2B_\tau + B^3 - 2A^2B + \tfrac{9}{4}A^2B - \tfrac{3}{4}B^3 - B) + 1/4\,\cos 3t\,(3A^2B - B^3)$$

$$= \sin t\,(-2A_\tau - 1/4\,A^3 - 1/4\,AB^2 + A) - 1/4\,\sin 3t\,(A^3 - 3AB^2)$$

$$+\cos t\,(2B_\tau + 1/4\,B^3 + 1/4\,A^2B - B) - 1/4\,\cos 3t\,(B^3 - 3A^2B)$$

After all that simplification, we now can set the secular growth terms to zero.

Thus we have

$$2A_\tau - A + \frac{1}{4}A^3 + \frac{1}{4}AB^2 = 0$$

$$2B_\tau - B + \frac{1}{4}B^3 + \frac{1}{4}A^2B = 0$$

we multiphy the top equation by A and the second by B so that

$$-A^2 + 2AA_\tau + \frac{1}{4}A^4 + \frac{1}{4}A^2B^2 = 0$$

$$-B^2 + 2BB_\tau + \frac{1}{4}B^4 + \frac{1}{4}A^2B^2 = 0$$

Adding these equations to gether we find

$$2AA_\tau + 2BB_\tau + \frac{1}{4}\left(A^4 + 2A^2B^2 + B^4\right) - (A^2+B^2) = 0$$

or upon rewritting

$$(A^2)_\tau + (B^2)_\tau + \frac{1}{4}(A^2+B^2)^2 - (A^2+B^2) = 0$$

So then letting $\rho = A^2+B^2$ we have

$$\rho_\tau + \frac{1}{4}\rho^2 - \rho = 0$$

We can solve this first order ODE via separation. Thus we have

$$\rho_\tau + \frac{1}{4}\rho^2 - \rho = 0 \quad \Rightarrow \quad \frac{d\rho}{\rho - \rho^2/4} = d\tau$$

Integrating both sides gives

$$\ln \frac{\rho}{1 - \rho/4} = \tau + c$$

or

$$\frac{\rho}{1-\rho/4} = c\, e^{\tau}$$

Recall that our initial condition gives $\rho(0) = A(0)^2 + B(0)^2 = \alpha^2$. Thus at $\tau = 0$ we find

$$\tau = 0: \quad \frac{\rho(0)}{1-\rho(0)/4} = \frac{\alpha^2}{1-\alpha^2/4} = c \quad \Rightarrow \quad c = \frac{4\alpha^2}{4-\alpha^2}$$

Thus

$$\frac{4\rho(\tau)}{4-\rho(\tau)} = \frac{4\alpha^2}{4-\alpha^2}\, e^{\tau}$$

Solving for $\rho(\tau)$ from here gives

$$\rho(\tau) = \frac{4\alpha^2}{\alpha^2 + (4-\alpha^2)\, e^{-\tau}}$$

From the equation $2A_\tau - A + \frac{1}{4}A^3 + \frac{1}{4}AB^2 = 0$ and the fact that $B^2 = \rho - A^2$ we find

$$A_\tau - \tfrac{1}{2}A + \tfrac{\rho}{4}A = 0 \quad \Rightarrow \quad \frac{dA}{A} = \int \tfrac{1}{2}\left(1 - \rho/4\right) d\tau$$

then

$$\ln A = \frac{1}{8}\int\left(4 - \frac{4\alpha^2}{\alpha^2 + (4-\alpha^2)e^{-\tau}}\right) d\tau + c$$

$$= \frac{1}{2}\int \frac{(4-\alpha^2)e^{-\tau}}{\alpha^2 + (4-\alpha^2)e^{-\tau}}\, d\tau + c$$

$$= \frac{(4-\alpha^2)}{2}\,\frac{1}{-(4-\alpha^2)}\,\ln\left(\alpha^2 + (4-\alpha^2)e^{-\tau}\right) + c$$

$$= \ln\left(\alpha^2 + (4-\alpha^2)e^{-\tau}\right)^{-1/2} + c$$

So then

$$A(\tau) = \frac{c}{\left(\alpha^2 + (4-\alpha^2)e^{-\tau}\right)^{1/2}}$$

and with $A(0) = \alpha$, this gives $\alpha = \frac{c}{(4)^{1/2}} \Rightarrow c = (4)^{1/2}\alpha$. so then

$$A(\tau) = \frac{2\alpha}{\left(\alpha^2 + (4-\alpha^2)e^{-\tau}\right)^{1/2}}$$

and then $B = (\rho - A^2)^{1/2}$ so that

$$B(\tau) = 0$$

Then

$$y = y_0(t,\tau) + O(\epsilon) = A(\tau)\cos t + O(\epsilon)$$

and

$$y = \frac{2\alpha}{\sqrt{\alpha^2 + (4-\alpha^2)e^{-\tau}}}\cos t + O(\epsilon)$$

So then as $t \to \infty$, $y \to 2\cos t$ which is the leading order limit cycle.

Lecture 17

Recap and Introduction to Boundary Layer Theory

Thus far, we have developed three perturbation techniques: Regular, Poincare-Lindstedt, and multiple scales. It is good at this point to recap their usefulness.

I. Regular Perturbation Expansion

expansion: $y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \dots$

appropriate problems
- boundary value problems
- eigenvalue problems
- always a good first attempt.

II. Poincare-Lindstedt Expansion

expansion: $y = y_0(\tau) + \epsilon y_1(\tau) + \epsilon^2 y_2(\tau) + \dots$
$\tau = \omega t = (\omega_0 + \epsilon \omega_1 + \epsilon^2 \omega_2 + \dots)t$

appropriate problems
- oscillatory phenomena only
- great for limit cycles
- frequency corrections are easy.

III. Multiple-Scale Expansion

expansion: $y = y_0(t,\tau) + \epsilon y_1(t,\tau) + \epsilon^2 y_2(t,\tau) + \dots$
$\tau = \epsilon t$ "slow time"

appropriate problems
- oscillations and growth/decay
- versatile
- frequency and amplitude corrections
- most ubiquitous

After all this perturbation analysis, one may get the idea that all the mathematical tools are now in place to generically approach any problem.... Au Contrair Meuseur!

To illustrate the breakdown of our methods, we consider the following simple equation.

$$\epsilon y'' + y' + y = 0$$

with $y(0)=0$ and $y(L)=A$.

Since it appears to be a standard perturbation problem with boundary conditions, we attempt a regular expansion.

$$y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \cdots$$

So that

$$O(1): \quad y_0' + y_0 = 0 \qquad y_0(0)=0, \; y_0(L)=A$$

$$O(\epsilon): \quad y_1' + y_1 = -y_0'' \qquad y_1(0)=0, \; y_1(L)=0$$

The leading order problem is easily solved and yields the general solution

$$y_0 = c e^{-x}$$

Applying the boundaries: $x=0$: $y_0(0) = c = 0 \Rightarrow c=0$

$x=L$: $y_0(L) = ce^{-L} = A \Rightarrow c = Ae^{L}$

Thus there seems to be a contradiction.

What went wrong? How could all this hyped-up machinery fail on such an easy problem.

The answer: the problem is <u>singular</u>! (this means...)

1. as $\epsilon \to 0$: the highest derivative vanishes.

2. Recall that the highest derivative determines the number of linearly independent solutions

So for our problem

1. It is second order: we expect two linearly independent solutions.

2. We also then expect two constants to appear

3. The constants are evaluated from two boundary conditions.

But in the perturbation method

1. The problem becomes first-order

2. So now we expect one solution and one constant

3. But we still have two boundary conditions.

The consequence: we get an overdetermined system which leads to contradiction.

Boundary Layers

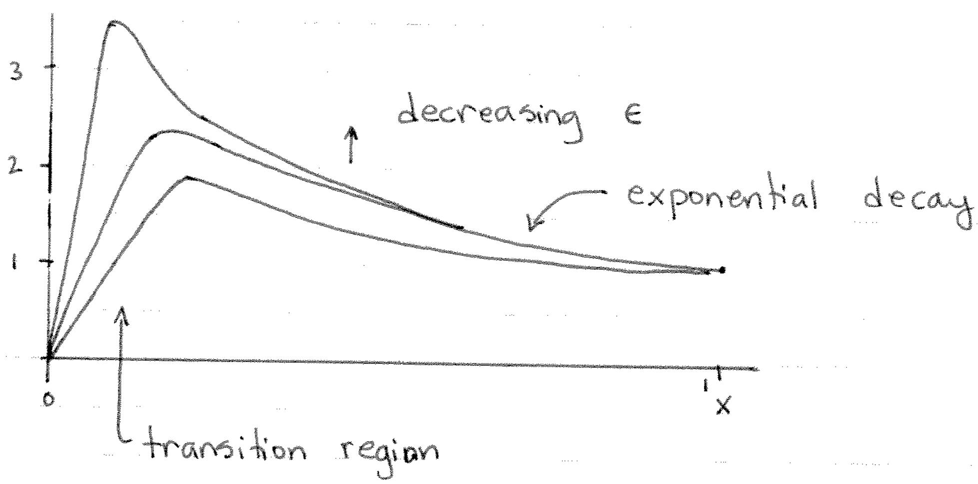
Now consider the problem

$$\epsilon y'' + (1+\epsilon) y' + y = 0 \qquad y(0) = 0, \; y(1) = 1$$

which has the exact solution

$$y(x) = \frac{e^{-x} - e^{-x/\epsilon}}{e^{-1} - e^{-1/\epsilon}}$$

Plotting this gives



This suggests the existence of two regions which are distinct.

1. The "outer region": taking up most of the interval for $0 < x < 1$.

2. The "inner region": taking up a narrow slice near $x = 0$. This is also known as the boundary layer or transition region.

To solve this problem, we break it up just as the problem suggests.

I. Outer problem : regular expansion with $y(1)=1$ $\quad (\delta << x \leq 1)$

$$y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \dots$$

Then plug in and collect powers of ϵ

$O(1)$: $\quad y_0' + y_0 = 0 \qquad y_0(1) = 1$

$O(\epsilon)$: $\quad y_1' + y_1 = -y_0'' - y_0' \qquad y_1(1) = 0$

Solving at leading order gives

$$y_0 = e^{1-x}$$

Continuing solving gives $y_1 = y_2 = y_3 = \dots = 0$.
Thus concludes the outer problem

II. Inner problem: $\quad (0 \leq x < \delta << 1)$

To do this problem, we define a new variable which stretches the boundary region. Thus we let

$$\xi = \frac{x}{\epsilon}$$

Then $y_x = y_\xi \xi_x = \frac{1}{\epsilon} y_\xi$ and $y_{xx} = \frac{1}{\epsilon^2} y_\xi$. This gives the new equation

$$y_{\xi\xi} + (1+\epsilon) y_\xi + \epsilon y = 0$$

with $\quad y(x)\Big|_{x=0} = y(\epsilon \xi)\Big|_{\xi=0} = y(0) = 0$

We now expand: $y = y_0(\xi) + \epsilon y_1(\xi) + \epsilon^2 y_2(\xi) + \dots$

$O(1)$: $\quad y_{0\xi\xi} + y_{0\xi} = 0 \qquad y_0(0) = 0$

$O(\epsilon)$: $\quad y_{1\xi\xi} + y_{1\xi} = -y_{0\xi} - y_0 \qquad y_1(0) = 0$

Solving at leading order gives

$$y_0 = A_0(1 - e^{-\xi})$$

where A_0 is some undetermined constant.

III. Matching: constructing the uniform solution

outer solution: $\quad y_{out} = e^{1-x} \qquad \delta << x \leq 1$
inner solution: $\quad y_{in} = A_0(1 - e^{-\xi}) + O(\epsilon) \qquad 0 \leq x < \delta << 1$

To match, we require that: $\lim_{x \to 0} y_{out} = \lim_{\xi \to \infty} y_{in}$

Thus $\quad \lim_{x \to 0} y_{out} = \lim_{x \to 0} e^{1-x} = \lim_{x \to 0} e(1 - x + x^2/2! + \dots) = e$

$\lim_{\xi \to \infty} y_{in} = \lim_{\xi \to 0} A_0(1-e^{-\xi}) = \lim_{\xi \to 0} A_0 = A_0$

So then

$$A_0 = e$$

The uniform solution is defined as $y_{unif} = y_{in} + y_{out} - y_{match}$.
So then

$$y_{unif}(x) = e(1 - e^{-x/\epsilon}) + e^{1-x} - e$$

which gives

$$\underline{\underline{y_{unif}(x) = e^{1-x} - e^{1-x/\epsilon}}}$$

which matches the leading order behavior of the exact solution!

Lecture 18

Boundary Layers

In the last example, we knew how to determine the location of the boundary layer by simply knowing the exact solution. However, in general we need a more constructive way to determine the location. Thus we consider the second-order linear equation

$$\epsilon y'' + b(x) y' + c(x) y = 0 \qquad 0 \leq x \leq 1$$

$$y(0) = A \qquad \text{and} \qquad y(1) = B$$

and we assume $b(x) \neq 0$ for $0 \leq x \leq 1$

I. Outer solution

We find the outer solution by simply expanding

$$y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \cdots$$

which gives

$$O(1): \qquad b(x) y_0' + c(x) y_0 = 0$$

$$O(\epsilon): \qquad b(x) y_1' + c(x) y_1 = -y_0''$$

We neglect the boundary conditions at this point since we do not know if a layer will form at $x=0$ or $x=1$.

The leading order problem can be easily solved. In fact, we find

$$y_0' + \frac{c(x)}{b(x)} y_0 = 0$$

so that

$$y_0 = k \exp\left[\int c(x)/b(x) \, dx \right]$$

where k is some constant.

II. Inner Solution

Not only do we need to know the location of the boundary layer, we also need to know its thickness. Thus we define a new "stretching" variable

$$\xi = \frac{x}{\delta}$$

so that

$$y_x = y_\xi \xi_x = \frac{1}{\delta} y_\xi$$

$$y_{xx} = \frac{1}{\delta^2} y_{\xi\xi}$$

where $\delta << 1$

Thus we end up with

$$\epsilon \cdot \frac{1}{\delta^2} y_{\xi\xi} + \frac{b(\delta\xi)}{\delta} y_\xi + c(\delta\xi) y = 0$$

and we would like to determine $\delta(\epsilon)$: that is, the thickness of the boundary layer. There are three possibilities.

1. $\epsilon \gg \delta$: then $y_{\xi\xi} = 0$

   this yields : $y = D\xi + E$

   Although this has two arbitrary constants as desired (for satisfying one boundary condition and matching), the solution blows-up in the matching region : $\lim_{\xi\to\infty} y = \lim_{\xi\to\infty} D\xi + E = \infty$.

   We can thus rule out $\epsilon \gg \delta$.

2. $\epsilon \ll \delta$: then $b(\delta\xi) y_\xi = 0$

   this yields : $y = D$

   This constant must satisfy the boundary condition as well as matching. Thus we have an overdetermined system.

   We can thus rule out $\epsilon \ll \delta$.

3. $\epsilon \sim \delta$: The distinguished limit

letting $\epsilon = \delta$ gives : $Y_{\xi\xi} + b(\delta\xi) Y_\xi = 0$

This yields a solution which has two constants of integration which allows for matching and satisfying the boundary conditions. Further, solutions are bounded.

The above procedure above to determine $\delta(\epsilon)$ is called method of dominant balance or method of distinguished limits. What we found is that

$$\xi = \frac{x}{\epsilon}$$

So that

$$Y_{\xi\xi} + b(\epsilon\xi) Y_\xi + \epsilon\, C(\epsilon\xi) y = 0$$

We can now expand: $y = y_0(\xi) + \epsilon\, y_1(\xi) + \epsilon^2 y_2(\xi) + \dots$

$O(1)$: $\quad y_{0\xi\xi} + b(\epsilon\xi) y_{0\xi} = 0$

$O(\epsilon)$: $\quad y_{1\xi\xi} + b(\epsilon\xi) y_{1\xi} = -C(\epsilon\xi) y_0$

At leading order then we have

$$y_{0_{\xi\xi}} + b(\epsilon\xi)\, y_{0_{\xi}} = 0$$

but we note that

$$b_0 = b(\epsilon\xi) = \begin{cases} b(0) + O(\epsilon) & \text{if boundary layer is at } x=0 \\ b(1) + O(\epsilon) & \text{if boundary layer is at } x=1 \end{cases}$$

Thus

$$y_{0_{\xi\xi}} + b_0\, y_{0_{\xi}} = 0$$

and

$$y_0 = D e^{-b_0 \xi} + E$$

III. Matching

At this point we have

$$y_{out} = K e^{\int c(x)/b(x)\, dx}$$

$$y_{in} = D e^{-b_0 \xi} + E$$

and matching requires

layer at $x=0$: $\lim_{\xi \to \infty} y_{in} = \lim_{x \to 0} y_{out}$

layer at $x=1$: $\lim_{\xi \to -\infty} y_{in} = \lim_{x \to 1} y_{out}$

We can now determine the boundary layer location in a straightforward fashion

Case 1: $b(x)>0 \Rightarrow b_0>0$

then at

$$x=0: \quad \lim_{\xi\to\infty} y_{in} = \lim_{\xi\to\infty} De^{-b_0\xi}+E = E$$

$$x=1: \quad \lim_{\xi\to-\infty} y_{in} = \lim_{\xi\to-\infty} De^{-b_0\xi}+E = \infty \qquad \text{(unbounded)}$$

Thus if $b(x)>0$, boundary layer occurs at $x=0$

Case 2: $b(x)<0 \Rightarrow b_0 = -|b_0|<0$

then at

$$x=0: \quad \lim_{\xi\to\infty} y_{in} = \lim_{\xi\to\infty} De^{|b_0|\xi}+E = \infty \qquad \text{(unbounded)}$$

$$x=1: \quad \lim_{\xi\to-\infty} y_{in} = \lim_{\xi\to-\infty} De^{|b_0|\xi}+E = E$$

Thus if $b(x)<0$, boundary layer occurs at $x=1$

For $b(x)>0$ then, we find: $E = K \lim_{x\to 0} e^{\int c(x)/b(x)\,dx}$ and the additional conditions $y_{out}(1)=B$ and $y_{in}(0)=A$. Thus K, E, and D are determined and

$$y_{unif} = y_{in}+y_{out}-y_{match} = De^{-b_0\xi}+E+Ke^{\int c(x)/b(x)\,dx} - E$$

and

$$\underline{\underline{y_{unif} = De^{-b_0\xi}+Ke^{\int \frac{c(x)}{b(x)}dx}}}$$

Lecture 19

Initial Layers and Limit Cycles in Singular Limits

With the concept of boundary layers now well in hand. We turn to an interesting problem which is very familiar: the Rayleigh oscillator. But now with a twist: it's singular.

$$\epsilon y'' + \left[-y' + \tfrac{1}{3}(y')^3\right] + y = 0$$

So now the ϵ appears in front of the highest derivative. Thus the nonlinear damping/growth term dominates the behavior.

Letting $z = \frac{dy}{dt}$, we can rewrite this as a system

$$\frac{dy}{dt} = z$$

$$\epsilon \frac{dz}{dt} = z - \frac{1}{3}z^3 - y$$

Dividing the bottom equation by the top equation gives the following

$$\epsilon \frac{dz}{dy} = \frac{z - \frac{1}{3}z^3 - y}{z}$$

So now we treat $z = \frac{dy}{dt}$ as a function of y.

I. Leading order: outer solution

The outer solution can be found by a regular expansion

$$z = z_0 + \epsilon z_1 + \epsilon^2 z_2 + \ldots$$

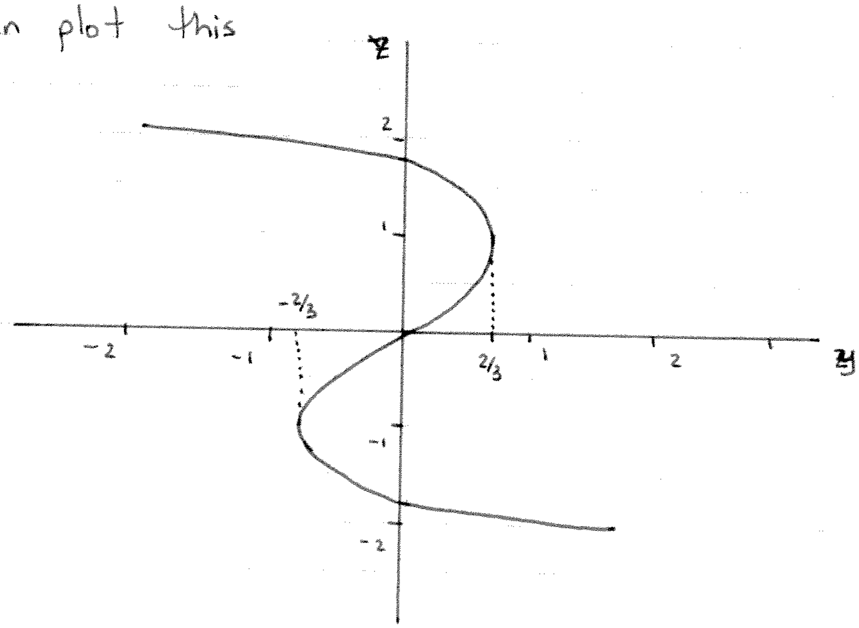
which gives

$$O(1): \qquad \frac{z_0 - \frac{1}{3} z_0^3 - y}{z_0} = 0$$

This then gives

$$z_0 - \frac{1}{3} z_0^3 = y$$

We can plot this



Note that for $|y| < 2/3$, there are three possible values for z_0.

What if we are not exactly on the leading order curve? To find out, we go back to the original equations:

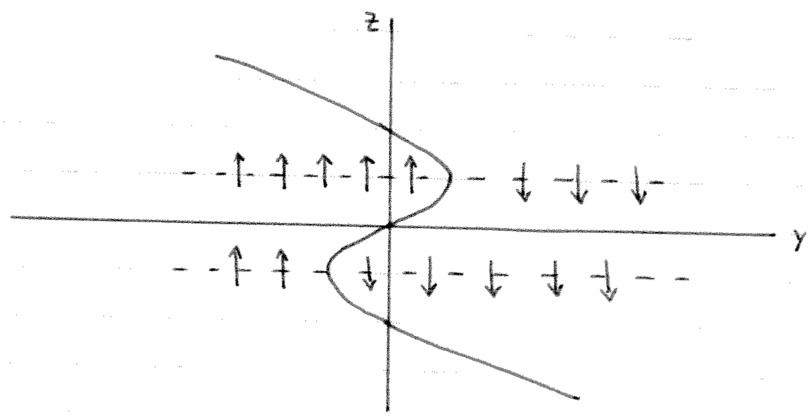
$$\frac{dy}{dt} = z \qquad \epsilon \frac{dz}{dt} = z - \frac{1}{3}z^3 - y$$

Thus we consider

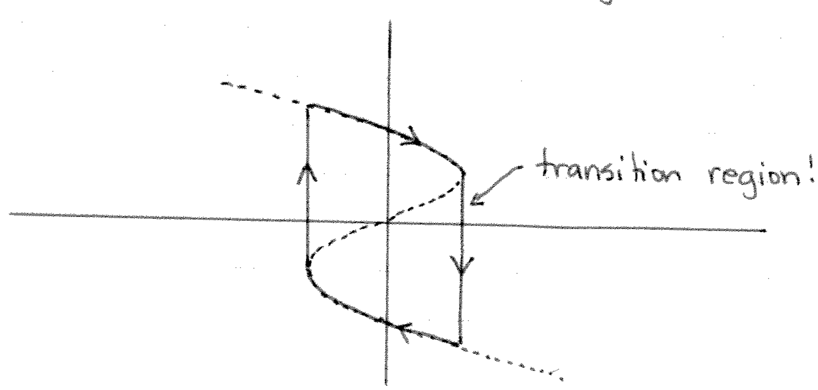
at $z=1$: $\quad \epsilon \frac{dz}{dt} = \frac{2}{3} - y = \begin{cases} >0 & \text{if } \quad y < 2/3 \\ <0 & \text{if } \quad y > 2/3 \end{cases}$

at $z=-1$: $\quad \epsilon \frac{dz}{dt} = -\frac{2}{3} - y = \begin{cases} >0 & y < -2/3 \\ <0 & y > -2/3 \end{cases}$

So our plot then becomes



So then, our solution (limit cycle) might look like



II. Transition Region: inner expansion

The transition region suggests the following inner layer expansion

$$\xi = \frac{y - 2/3}{\epsilon}$$

Thus $z_y = z_\xi \xi_y = \frac{1}{\epsilon} z_\xi$, $z_{yy} = \frac{1}{\epsilon^2} z_{\xi\xi}$, and $y = \frac{2}{3} + \epsilon\xi$.
Plugging in gives

$$\frac{dz}{d\xi} = \frac{z - \frac{1}{3}z^3 - 2/3 - \epsilon\xi}{z}$$

Expanding once again gives

$$z = z_0(\xi) + \epsilon z_1(\xi) + \epsilon^2 z_2(\xi) + \dots$$

So that

$$O(1): \qquad \frac{dz_0}{d\xi} = \frac{z_0 - z_0^3/3 - 2/3}{z_0}$$

This can be solved by separation

$$\int \frac{z_0 \, dz_0}{z_0 - z_0^3/3 - 2/3} = \int d\xi$$

which yields (a little faith please)

$$-\frac{2}{9} \ln |z_0 + 2| + \frac{2}{9} \ln |z_0 - 1| - \frac{1}{3(z_0 - 1)} = -\frac{1}{3}(\xi + c)$$

where c is a constant of integration.

III. Matching

Top right corner:

In this case, our outer solution approaches the following

$$\text{outer:} \quad (y, z_0) = \left(\tfrac{2}{3}^-, 1\right)$$

The inner solution already matches this since

$$\text{inner:} \quad \lim_{\xi \to -\infty} (\xi, z_0) = \lim_{\xi \to -\infty} (-\infty, 1)$$

since $\frac{-1}{z_0 - 1} \to -\infty$ or $\ln|z-1| \to -\infty$ as $\xi \to -\infty$, with $z = 1$

Bottom right corner:

Now our solution approaches the following

$$\text{outer:} \quad (y, z_0) = \left(\tfrac{2}{3}^-, -2\right)$$

And the inner solution matches this since

$$\text{inner:} \quad \lim_{\xi \to -\infty} (\xi, z_0) = \lim_{\xi \to -\infty} (-\infty, -2)$$

where in this case $\ln|z_0 + 2| \to -\infty$ as $\xi \to -\infty$ with $z = -2$.

Thus matching is complete at leading order and the constant c remains undetermined. We can similary expand on the left transition layer with $\xi = (y + 2/3)/\epsilon$ to get similar results.

Lecture 20

WKB Theory and applications.

To make a second order differential equation unsoluable, we can do two things: add nonlinearity or make it non-constant coefficient. The perturbation methods dealt with to this point can handle either of these cases. The WKB method (Wentzel-Kramers-Brillouin) deals only with linear problems, but with non-constant coefficients. Further, it is extremely effective in handling singular problems.

A large number of problems are characterized by exponential behavior: either real or imaginary. Therefore, it is somewhat natural to seek a solution of the form.

$$y(x) \sim A(x) e^{\frac{S(x)}{\delta}} \qquad \delta \ll 1$$

where δ is a small parameter (note: this is for singular problems)

Thus

$S(x)$ — "phase"

$A(x)$ — "amplitude"

and this is merely an amplitude-phase decomposition.

We begin by considering the simple problem

$$\epsilon^2 y'' = Q(x) y \qquad\qquad Q(x) \neq 0$$

which is singular.

The formal WKB expansion assumes

$$y \sim \exp\left[\frac{S(x)}{\delta}\right] = \exp\left[\frac{S_0(x) + \delta S_1(x) + \delta^2 S_2(x) + \dots}{\delta}\right]$$

In order to use this, we plug it into our equation. Thus we require the following

$$y' = \left(\frac{S_0'(x) + \delta S_1'(x) + \dots}{\delta}\right) \exp\left[\frac{S_0(x) + \delta S_1(x) + \dots}{\delta}\right]$$

$$y'' = \left[\frac{(S_0'(x) + \delta S_1'(x) + \dots)^2}{\delta^2} + \frac{S_0''(x) + \delta S_1''(x) + \dots}{\delta}\right] \exp\left[\frac{S_0(x) + \delta S_1(x) + \dots}{\delta}\right]$$

Plugging into the equation and dividing off the exponential terms yields.

$$\epsilon^2 \left[\frac{S_0'^2(x)}{\delta^2} + \frac{2 S_0'(x) S_1'(x)}{\delta} + \dots + \frac{S_0''(x)}{\delta} + \dots\right] = Q(x)$$

The largest term on the left must balance the $Q(x) \sim O(1)$. Thus

$$\frac{\epsilon^2}{\delta^2} S_0'^2(x) \sim O(1) \implies \frac{\epsilon^2}{\delta^2} \sim O(1) \implies \delta = \epsilon$$

Using $S=\epsilon$ then gives

$$S_0'^2(x) + 2\epsilon S_0'(x)S_1'(x) + \cdots + \epsilon S_0''(x) + \cdots = Q(x)$$

We now collect powers of ϵ.

$O(1)$ $\quad S_0'^2(x) = Q(x)$ $\quad$ (eikonal equation)

$O(\epsilon)$ $\quad S_0''(x) + 2S_0'(x)S_1'(x) = 0$ $\quad$ (transport equation)

$\vdots$

$O(\epsilon^n)$ $\quad S_{n-1}''(x) + 2S_{n-1}'(x)S_n'(x) + \sum_{j=1}^{n-1} S_j' S_{n-j}' = 0 \qquad n \geq 2$

Solving the eikonal equation gives the leading order "phase"

$$S_0'^2(x) = Q(x)$$

$$S_0'(x) = \pm\sqrt{Q(x)}$$

$$S_0(x) = \pm\int^x \sqrt{Q(x)}\, dx$$

At next order we have the transport equation

$$S_0''(x) + 2S_0'(x)S_1'(x) = 0$$

or

$$S_1'(x) = -\frac{S_0''(x)}{2S_0'(x)}$$

We note that we can rewrite this as

$$S_1'(x) = -\frac{S_0''(x)}{2S_0'(x)} = -\frac{1}{2}\left[\ln S_0'(x)\right]'$$

integrating both sides gives

$$S_1(x) = -\frac{1}{2}\ln S_0'(x)$$

$$= -\frac{1}{2}\ln \pm\sqrt{Q(x)}$$

$$= -\frac{1}{2}\ln \pm Q(x)^{1/2}$$

$$= -\frac{1}{4}\ln Q(x)$$

So then we have

$$S_0 = \pm\int^x \sqrt{Q(\sigma)}\,d\sigma$$

$$S_1 = -\frac{1}{4}\ln Q(x)$$

And since

$$y \sim \exp\left[\frac{S_0(x) + \epsilon S_1(x) + \dots}{\epsilon}\right].$$

$$= \exp\left[\frac{\pm\int^x \sqrt{Q(\sigma)}\,d\sigma - \epsilon\frac{1}{4}\ln Q(x)}{\epsilon} + \dots\right]$$

$$\approx \exp\left[\pm\frac{1}{\epsilon}\int^x \sqrt{Q(\sigma)}\,d\sigma\right]\exp\left[-\frac{1}{4}\ln Q(x)\right]$$

And noting that

$$\exp\left[-\tfrac{1}{4}\ln Q(x)\right] = \exp\left[\ln Q(x)^{-1/4}\right] = Q(x)^{-1/4}$$

we then find

$$y(x) \approx c_1 Q(x)^{-1/4} \exp\left[\frac{1}{\epsilon}\int^x \sqrt{Q(\sigma)}\,d\sigma\right] + c_2 Q(x)^{-1/4} \exp\left[\frac{-1}{\epsilon}\int^x \sqrt{Q(\sigma)}\,d\sigma\right]$$

This is the leading order WKB approximation to the singular problem and differs from the exact solution by $O(\epsilon)$.

<u>Boundary layer theory revisited</u>: we now consider the problem

$$\epsilon y'' + b(x) y' + c(x) y = 0$$

with $y(0) = A$ and $y(1) = B$ where for now we take $b(x) > 0$ for $x \in [0,1]$

$$y = \exp\left[\frac{S_0(x) + \epsilon S_1(x) + \epsilon^2 S_2(x) + \cdots}{\epsilon}\right]$$

Plugging in gives

$$\epsilon\left[\frac{S_0'^2(x)}{\epsilon^2} + \frac{2S_0'(x)S_1'(x)}{\epsilon} + \cdots + \frac{S_0''(x)}{\epsilon} + \cdots\right]$$
$$+ b(x)\left[\frac{S_0'(x)}{\epsilon} + S_1'(x) + \cdots\right] + c(x) = 0$$

Collecting powers gives

$O(1/\epsilon)$: $$S_0'^2(x) + b(x)S_0'(x) = 0$$

$O(1)$: $$2S_0'(x)S_1'(x) + S_0''(x) + S_1'(x)\cdot b(x) + c(x) = 0$$

The leading order problem yields

$$S_0'^2(x) + b(x)S_0'(x) = S_0'(x)\left[S_0'(x) + b(x)\right] = 0$$

so that

$$S_0'(x) = 0 \qquad \text{or} \qquad S_0'(x) = -b(x)$$

<u>Case 1</u>: $S_0'(x) = 0 \Rightarrow S_0 = K = \text{constant}$

At next order we then find

$$S_1'(x)\,b(x) + c(x) = 0$$

$$S_1'(x) = -c(x)/b(x)$$

$$S_1(x) = -\int_0^x c(x)/b(x)\,dx$$

So then

$$y = \exp\left[\frac{S_0(x)}{\epsilon} + S_1(x)\right] = \exp\left[\frac{K}{\epsilon}\right]\exp\left[-\int_0^x \frac{c(\sigma)}{b(\sigma)}d\sigma\right]$$

$$y_1(x) = c_1 \exp\left[-\int_0^x \frac{c(\sigma)}{b(\sigma)}d\sigma\right] \qquad \text{outer solution}$$

Case 2: $S_0'(x) = -b(x) \Rightarrow S_0(x) = -\int_0^x b(\sigma)\,d\sigma$

At next order we then find

$$-2b(x)S_1'(x) + -b'(x) + S_1'(x)\,b(x) + c(x) = 0$$

$$-b(x)S_1'(x) - b'(x) + c(x) = 0$$

$$b(x)S_1'(x) + b'(x) = c(x)$$

so then

$$S_1'(x) = \frac{c(x)}{b(x)} - \frac{b'(x)}{b(x)} = \frac{c(x)}{b(x)} - [\ln b(x)]'$$

integrate

$$S_1(x) = \int_0^x \frac{c(\sigma)}{b(\sigma)}\,d\sigma - \ln b(x)$$

so then

$$y = \exp\left[\frac{S_0(x)}{\epsilon} + S_1(x)\right] = \exp\left[-\frac{1}{\epsilon}\int_0^x b(\sigma)\,d\sigma\right] \exp\left[\int_0^x \frac{c(\sigma)}{b(\sigma)}\,d\sigma - \ln b(x)\right]$$

$$y_2(x) = c_2 \frac{1}{b(x)} \exp\left[\int_0^x \frac{c(\sigma)}{b(\sigma)}\,d\sigma - \frac{1}{\epsilon}\int_0^x b(\sigma)\,d\sigma\right] \qquad \text{inner solution}$$

Thus

$$y(x) = c_1 \exp\left[-\int_0^x \frac{c(\sigma)}{b(\sigma)}\,d\sigma\right] + \frac{c_2}{b(x)} \exp\left[-\frac{1}{\epsilon}\int_0^x b(\sigma)\,d\sigma + \int_0^x \frac{c(\sigma)}{b(\sigma)}\,d\sigma\right]$$

where c_1 and c_2 are determined by the boundary conditions.

Conditions for validity of the WKB approximation

formal WKB expansion:

$$y(x) \sim \exp\left[\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n(x)\right], \quad \delta \to 0.$$

In order that the WKB expansion is valid, it is necessary that $\frac{1}{\delta}\sum \delta^n S_n$ is an asymptotic series in δ as $\delta \to 0$:

$$\begin{aligned} \delta S_1 &\ll S_0, \\ \delta S_2 &\ll S_1, \qquad \delta \to 0. \\ &\vdots \\ \delta S_{n+1} &\ll S_n. \end{aligned}$$

If the series is truncated at the term $\delta^{N-1} S_N(x)$ then the next term must be small compared to 1.

$$\delta^N S_{N+1} \ll 1, \quad \delta \to 0.$$

Thus $y \sim e^{\frac{1}{\delta}S_0}$ is never a good approximation,
we need at least $y \sim e^{\frac{1}{\delta}S_0 + S_1}$ (physical optics approximation)

Example

Airy equation: $y'' = xy$

This is a Schrödinger equation ($\epsilon^2 y'' = Q(x)y$) with $\epsilon = 1$ and $Q(x) = x$.

The solutions for the Schrödinger equation are:

$$S_0(x) = \pm \int^x \sqrt{Q(t)}\, dt$$

$$S_1(x) = -\frac{1}{4} \ln Q(x)$$

$$S_2(x) = \pm \int^x \left[\frac{Q''}{8Q^{3/2}} - \frac{5(Q')^2}{32\, Q^{5/2}} \right] dt$$

$$S_3(x) = -\frac{Q''}{16Q^2} + \frac{5(Q')^2}{64Q^3}$$

$$\vdots$$

So we get: $S_0 = \pm \frac{2}{3} x^{3/2}$, $S_1 = -\frac{1}{4} \ln x$, $S_2 = \pm \frac{5}{48} x^{-3/2}$

$$\epsilon S_2 \ll S_1 \ll \frac{1}{\epsilon} S_0, \quad \text{for } \epsilon = 1.$$

$$\epsilon S_2 \ll 1.$$

This holds for $x \to \infty$. We get the leading order behaviour:

$$\underline{y \sim c_\pm x^{-1/4} e^{\pm \frac{2}{3} x^{3/2}}}, \quad x \to \infty.$$

<u>parabolic cylinder equation:</u> $y'' = \left(\frac{1}{4}x^2 - \nu - \frac{1}{2}\right) y.$

Schrödinger equation with $\epsilon = 1$, $Q(x) = \frac{1}{4}x^2 - \nu - \frac{1}{2}$.

$$S_0 = \pm \int^x \sqrt{Q(t)}\, dt = \pm \int^x \frac{t}{2} \left(1 - \frac{4\nu + 2}{t^2}\right)^{1/2} dt$$

as $x \to \infty$:

$$\sim \pm \int^x \frac{t}{2} \left(1 - \frac{2\nu + 1}{t^2}\right) dt$$

$$\sim \pm \left[\frac{x^2}{4} - \left(\nu + \frac{1}{2}\right) \ln x \right] = \pm \left[\frac{x^2}{4} + \ln x^{-(\nu + \frac{1}{2})} \right]$$

$$S_1 = -\frac{1}{4} \ln Q \sim -\frac{1}{4} \ln \left(\frac{1}{4} x^2\right) = \ln \left(\frac{1}{4} x^{-1/2}\right), \quad x \to \infty$$

$\epsilon S_2 \ll S_1 \ll \frac{1}{\epsilon} S_0$ and $\epsilon S_2 \ll 1$ for $\epsilon = 1$, $x \to \infty$.

$$\Rightarrow \quad y \sim \begin{cases} c_+ x^{-\nu-1} e^{x^2/4} \\ c_- x^{\nu} e^{-x^2/4} \end{cases} \qquad x \to \infty.$$

Violation of validity criteria: $\quad y'' = \left(\frac{\ln x}{x}\right)^2 y$

$$\epsilon = 1, \quad Q(x) = \left(\frac{\ln x}{x}\right)^2$$

$$S_0 = \pm \frac{1}{2} (\ln x)^2$$

$$S_1 = \frac{1}{2} \ln x - \frac{1}{2} \ln(\ln x)$$

$$S_2 = \pm \frac{1}{8} \ln(\ln x) \pm \frac{3}{16} (\ln x)^{-2}$$

$$S_2 \ll S_1 \ll S_0 \quad \text{for} \quad x \to \infty,$$

but $S_2 \gg 1$ for $x \to \infty$.

Let's try the next higher term:

$$S_3 = \frac{3}{16} (\ln x)^{-4} - \frac{1}{16} (\ln x)^{-2}.$$

We find: $S_3 \ll S_2$ and $S_3 \ll 1$ for $x \to \infty$!

The asymptotic leading behaviour needs the first three terms:

$$y \sim e^{S_0 + S_1 + S_2} = c_{\pm} \cdot e^{\pm \frac{1}{2} (\ln x)^2} \cdot x^{1/2} \cdot (\ln x)^{-1/2 \pm 1/8}, \quad x \to \infty$$

Eigenvalue problems with WKB

$$y''(x) + E\,Q(x)\,y(x) = 0 \quad , \quad Q(x) > 0 \;, \; y(0) = y(\pi) = 0 .$$

large eigenvalues: $E \to \infty$, $E = \frac{1}{\epsilon} \Rightarrow \epsilon y'' + Q(x) y = 0$.

$$S_0 = \pm \int^x \sqrt{-E Q(t)}\, dt = \pm i \sqrt{E} \int^x \sqrt{Q(t)}\, dt$$

$$S_1 = -\frac{1}{4} \ln Q(x)$$

$$y \sim A\, Q^{-1/4}(x) \sin\left[\sqrt{E} \int_0^x \sqrt{Q(t)}\, dt\right] + B\, Q^{-1/4}(x) \cos\left[\sqrt{E} \int_0^x \sqrt{Q(t)}\, dt\right]$$

Boundary conditions:

$y(0) = 0 \Rightarrow B = 0$.

$y(\pi) = 0 \Rightarrow Q^{-1/4}(\pi) \sin\left[\sqrt{E} \int_0^\pi \sqrt{Q(t)}\, dt\right] = 0$.

$$\sqrt{E} \int_0^\pi \sqrt{Q(t)}\, dt = n\pi$$

eigenvalues: $$\boxed{E_n = \left[\frac{n\pi}{\int_0^\pi \sqrt{Q(t)}\, dt}\right]^2 , \quad n \to \infty .}$$

eigenfunctions: $$\boxed{y \sim A\, Q^{-1/4}(x) \sin\left[\sqrt{E} \int_0^x \sqrt{Q(t)}\, dt\right]}$$

Example $\quad Q(x) = (x+\pi)^4$

$$E_n \sim \frac{9n^2}{49\pi^4} ,$$

$$y_n(x) \sim \sqrt{\frac{6}{7\pi^3}}\; \frac{\sin\left[n(x^3 + 3x^2\pi + 3\pi^2 x)/7\pi^2\right]}{\pi + x} \qquad n \to \infty .$$

Table 10.1 A comparison of the exact eigenvalues E_n of the Sturm-Liouville problem $y''(x) + E(x + \pi)^4 y(x) = 0$ $[y(0) = y(\pi) = 0]$ with the leading-order WKB prediction [see (10.1.34)] for these eigenvalues $E_n \sim 9n^2/49\pi^2$ $(n \to \infty)$

As expected, this prediction becomes more accurate as n increases. The relative error is defined as (approximate − exact)/(exact)

n	E_n(WKB)	E_n(exact)	Relative error, %
1	0.00188559	0.00174401	8.1
2	0.00754235	0.00734865	2.6
3	0.0169703	0.0167524	1.3
4	0.0301694	0.0299383	0.77
5	0.0471397	0.0469006	0.51
10	0.188559	0.188305	0.13
20	0.754235	0.753977	0.035
40	3.01694	3.01668	0.009

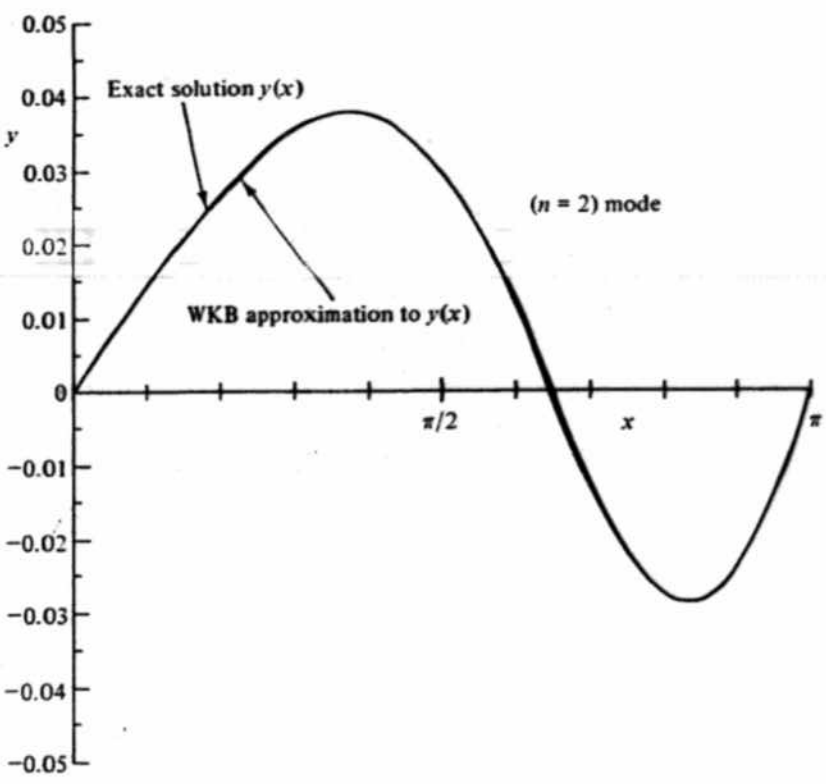


Figure 10.3 Same as in Fig. 10.2 except that $n = 2$. The exact eigenfunction and the WKB approximation are almost indistinguishable.

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Thus, the eigenfunctions are

$$y_n(x) \sim \left(\int_0^\pi \frac{\sqrt{Q(t)}}{2}\,dt\right)^{-1/2} Q^{-1/4}(x) \sin\left[n\pi \frac{\int_0^x \sqrt{Q(t)}\,dt}{\int_0^\pi \sqrt{Q(t)}\,dt}\right], \qquad n \to \infty. \tag{10.1.33}$$

Note that if $Q(x) \equiv 1$, then the right side of (10.1.33) reduces to $\sqrt{2/\pi}\,\sin(nx)$, which is the exact solution to $y'' + y = 0$ $[y(0) = y(\pi) = 0]$.

To demonstrate the accuracy of our results, we choose $Q(x) = (x + \pi)^4$. Then the approximate eigenvalues and eigenfunctions are given by

$$E_n \sim \frac{9n^2}{49\pi^4}, \qquad n \to \infty, \tag{10.1.34}$$

and

$$y_n(x) \sim \sqrt{\frac{6}{7\pi^3}}\,\frac{\sin\left[n(x^3 + 3x^2\pi + 3\pi^2 x)/7\pi^2\right]}{(\pi + x)}, \qquad n \to \infty. \tag{10.1.35}$$

We have checked these results numerically by computer. The comparisons between the approximate analytical and the computer solutions are given in Table 10.1 and Figs. 10.2 and 10.3.

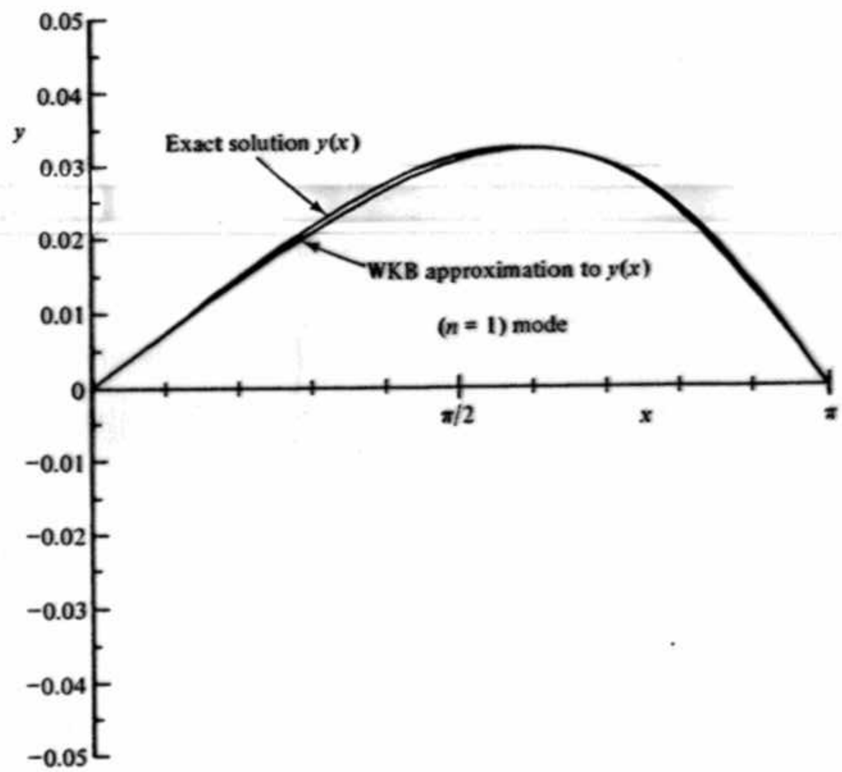


Figure 10.2 Comparison of the exact solution to $y''(x) + E_n(x + \pi)^4 y(x) = 0$ $[y(0) = y(\pi) = 0]$, with the WKB approximation to this solution as given in (10.1.35) for the lowest $(n = 1)$ mode. Although WKB becomes exact as $n \to \infty$, this plot shows that even when $n = 1$ the WKB approximation is extraordinarily accurate.

Elementary Bifurcation Theory

One of the very interesting things which can happen in nonlinear differential equations is that solutions can change as a parameter is adjusted in the problem. This "bifurcation" behavior is of great interest in many fields.

We begin by considering the simple example

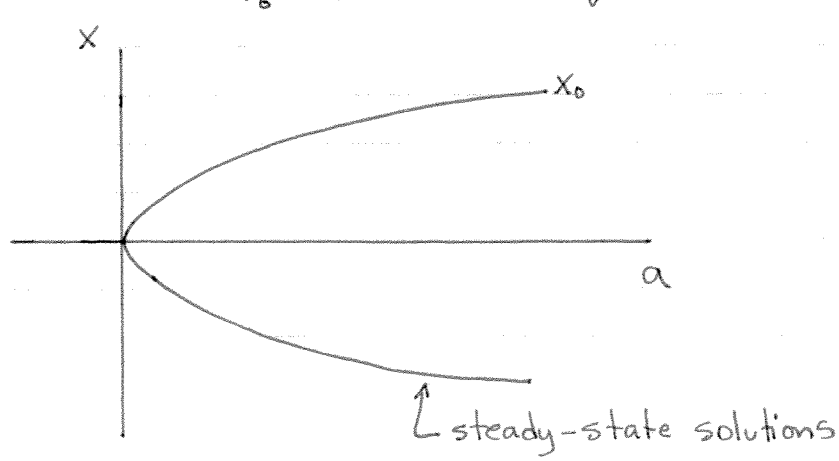
$$\frac{dx}{dt} = a - x^2$$

where a is some constant. We can easily solve this problem by separation. However, we will approach this nonlinear problem with qualitative methods. Thus we consider the following

Equilibrium: $a - x^2 = 0$

The equilibrium solutions are then

$$x_0 = \pm\sqrt{a} \qquad \text{(equilibrium)}$$



The question naturally arises: what if I'm not on an equilibrium solution? What is the dynamics then? To explore this, we do linear stability.

Thus we let

$$x = x_0 + \tilde{x} \qquad \text{where} \quad \tilde{x} \ll 1$$

Then plugging in gives

$$(x_0 + \tilde{x})_t = a - (x_0 + \tilde{x})^2$$

$$\tilde{x}_t = a - x_0^2 - 2x_0\tilde{x} - \tilde{x}^2$$

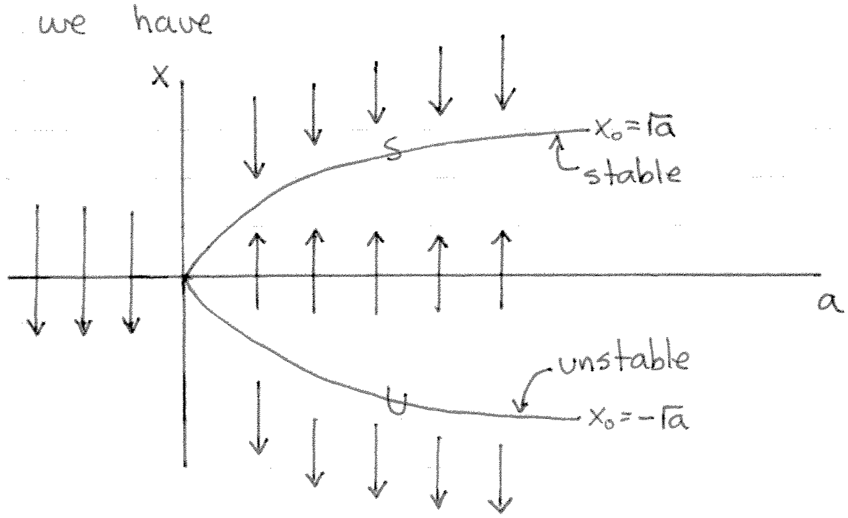
Note that $x_0^2 = a$ and $\tilde{x}^2$ is much smaller than the $\tilde{x}$ terms so that

$$\tilde{x}_t = -2x_0\tilde{x}$$

whose solution is

$$\tilde{x} = \tilde{x}(0)\, e^{-2x_0 t}$$

Now then if $x_0 = \sqrt{a}$ then $\tilde{x} = \tilde{x}(0)\, e^{-2\sqrt{a}t} \to 0$ as $t \to \infty$ and if $x_0 = -\sqrt{a}$ then $\tilde{x} = \tilde{x}(0)\, e^{2\sqrt{a}t} \to \infty$ as $t \to \infty$.
Thus we have



turning point
(saddle-node bifurcation)
(limit point)

The transcritical bifurcation

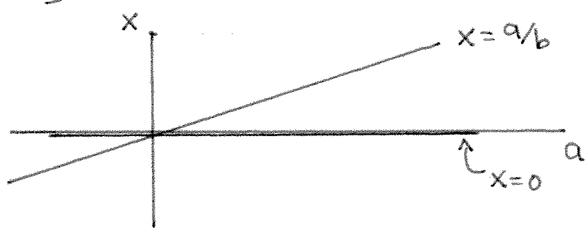
Consider the equation

$$\frac{dx}{dt} = ax - bx^2$$

Equilibrium is achieved when $dx/dt = 0$ so that

$$ax - bx^2 = 0 \quad \rightarrow \quad x_0 = 0,\ a/b$$

Graphically, these solutions are

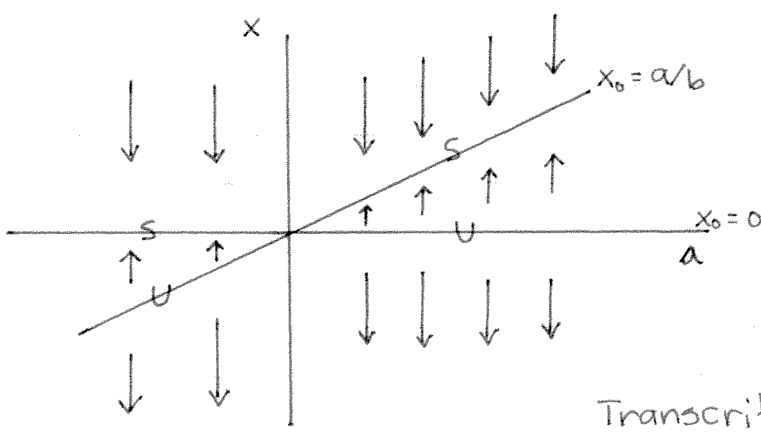


Stability can once again be investigated about each critical point: $x = x_0 + \tilde{x}$

$x_0 = 0$: $\quad \tilde{x}_t = a\tilde{x} - 2bx_0\tilde{x} = a\tilde{x} \Rightarrow \tilde{x} = \tilde{x}(0)e^{at} \rightarrow \begin{cases} +\infty \\ 0 \end{cases}$ as $t \to \infty$ for $a \gtrless 0$

$x_0 = a/b$: $\quad \tilde{x}_t = a\tilde{x} - 2bx_0\tilde{x} = -a\tilde{x} \Rightarrow \tilde{x} = \tilde{x}(0)e^{-at} \rightarrow \begin{cases} 0 \\ \infty \end{cases}$ as $t \to \infty$ for $a \gtrless 0$

The dynamics then produces



Transcritical bifurcation

The Pitchfork Bifurcation

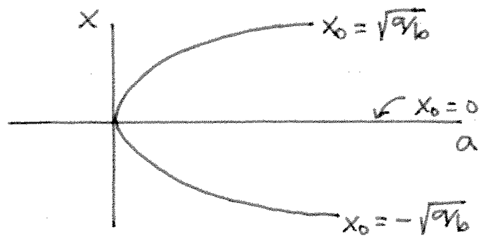
Consider the equation

$$\frac{dx}{dt} = ax - bx^3$$

Equilibrium is achieved for $dx/dt = 0$ so that

$$ax - bx^3 = 0 \quad \rightarrow \quad x_0 = 0, \pm\sqrt{a/b}$$

Graphically

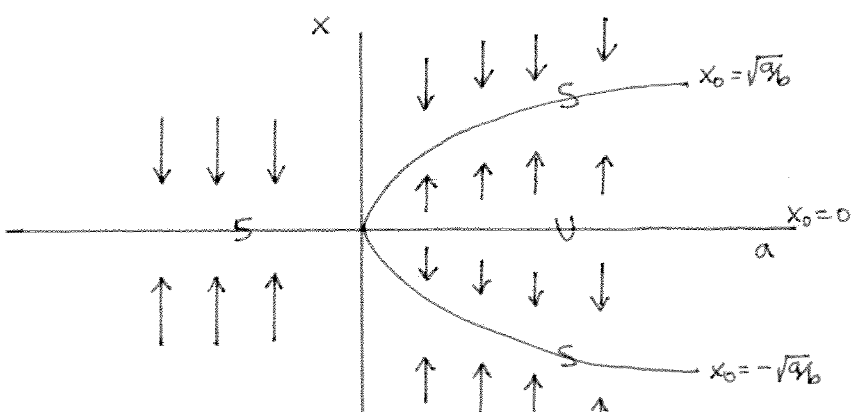


Stability is determined from linearization: $x = x_0 + \tilde{x}$

$x_0 = 0$: $\quad \tilde{x}_t = a\tilde{x} - 3bx_0^2\tilde{x} = a\tilde{x} \Rightarrow \tilde{x} = \tilde{x}(0)e^{at} \rightarrow \begin{cases} \infty \\ 0 \end{cases}$ as $t \to \infty$ for $\begin{cases} a > 0 \\ a < 0 \end{cases}$

$$\left.\begin{array}{ll} x_0 = \sqrt{a/b}: & \tilde{x}_t = a\tilde{x} - 3bx_0^2\tilde{x} = -2a\tilde{x} \Rightarrow \tilde{x} = \tilde{x}(0)e^{-2at} \\ x_0 = -\sqrt{a/b}: & \tilde{x}_t = a\tilde{x} - 3bx_0^2\tilde{x} = -2a\tilde{x} \Rightarrow \tilde{x} = \tilde{x}(0)e^{-2at} \end{array}\right\} \rightarrow \begin{cases} 0 \\ \infty \end{cases} \text{ as } t \to \infty \text{ for } \begin{cases} a > 0 \\ a < 0 \end{cases}$$

The dynamics then produces



Pitchfork (Super-critical) bifurcation

The Hopf Bifurcation

Consider the system

$$\frac{dx}{dt} = -y + (a - x^2 - y^2)x$$

$$\frac{dy}{dt} = x + (a - x^2 - y^2)y$$

with a real constant a. In this example, a steady-state solution bifurcates to a time-periodic solution. Equilibrium solutions require $x' = y' = 0$ so that

$$-y + x(a - x^2 - y^2) = 0$$

$$x + y(a - x^2 - y^2) = 0$$

Obviously, the origin is a critical point: $(x_0, y_0) = (0,0)$. And in fact, that is the only one. We now linearize

$$x = x_0 + \tilde{x}$$
$$y = y_0 + \tilde{y}$$

so that

$$\frac{d\tilde{x}}{dt} = -\tilde{y} + a\tilde{x}$$

$$\frac{d\tilde{y}}{dt} = \tilde{x} + a\tilde{y}$$

letting $\vec{x} = \begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix}$

$$\vec{x}' = \begin{pmatrix} a & -1 \\ 1 & a \end{pmatrix}\vec{x}$$

This is an eigenvalue problem if we let $\vec{x} = \vec{v} e^{\lambda t}$

$$\begin{pmatrix} a-\lambda & -1 \\ 1 & a-\lambda \end{pmatrix} \vec{v} = 0$$

The eigenvalues are

$$(a-\lambda)(a-\lambda) + 1 = 0$$

$$\lambda^2 - 2a\lambda + (1+a^2) = 0$$

$$\lambda = a \pm i$$

Thus

$$\vec{x} = c_1 \vec{v}^{(1)} e^{(a+i)t} + c_2 \vec{v}^{(2)} e^{(a-i)t}$$

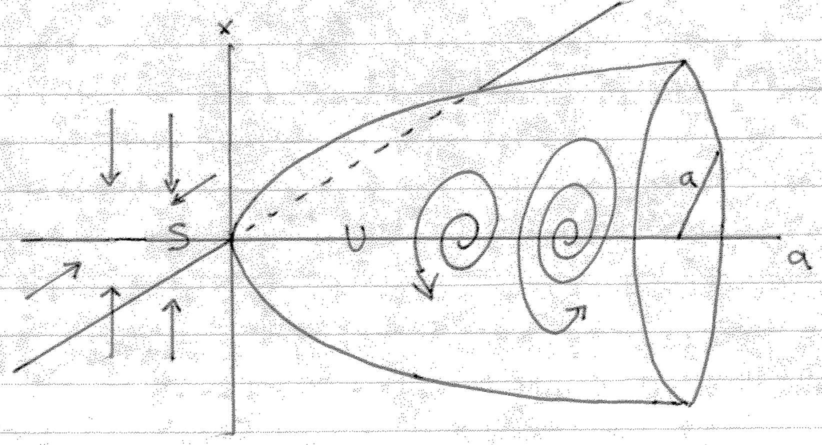
and $\vec{x} \to 0$ for $t \to \infty$ if $a<0$, $\vec{x} \to \infty$ for $t \to \infty$ if $a>0$. This problem can be solved explicitly by letting $x = r\cos\theta$, $y = r\sin\theta$ $(r \geq 0)$. This gives

$$\frac{dr}{dt} = r(a-r^2) \quad \text{and} \quad \frac{d\theta}{dt} = 1$$

so then

$$r^2 = \begin{cases} \dfrac{ar_0^2}{r_0^2 + (a-r_0^2)e^{-2at}} & a \neq 0 \\[2ex] \dfrac{r_0^2}{1+2r_0^2 t} & a = 0 \end{cases} \qquad \theta = t + \theta_0$$

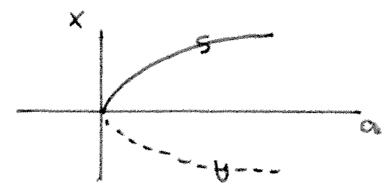
Hopf Bifurcation



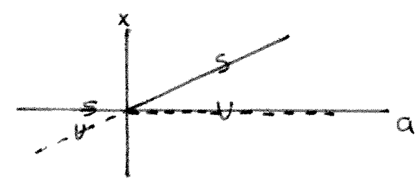
Bifurcation Theory: Normal Forms and Imperfections

In the preceeding lecture, we considered four types of bifurcations. These were

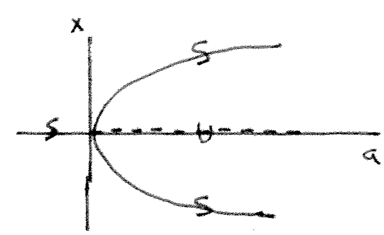
1. $x' = a - x^2$ (saddle-node)



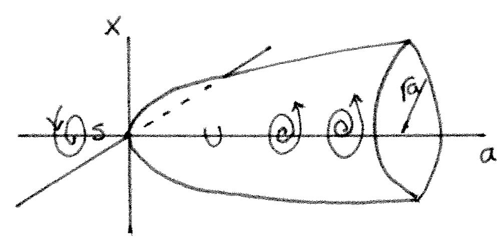
2. $x' = ax - x^2$ (transcritical)



3. $x' = ax - x^3$ (pitchfork)



4. Hopf Bifurcation



These four bifurcations are prototypical examples of the kinds of instabilities which can arise in physically realizable systems.

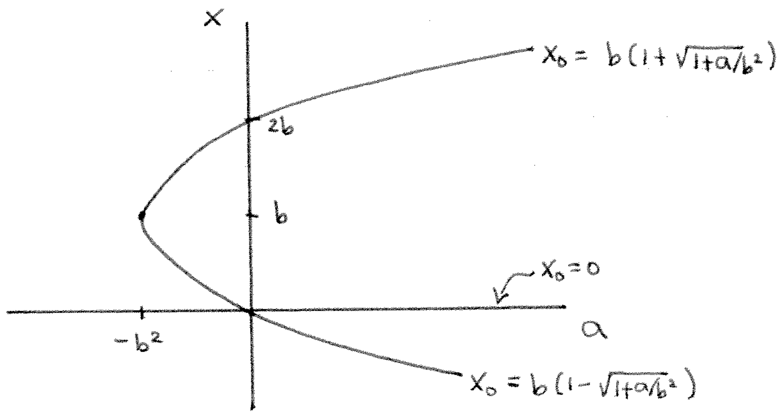
Classifying a given problem and its bifurcation structure is often referred to as a normal form reduction, or alternatively, as the universal unfolding of the bifurcation.

As an example, consider the following problem

$$\frac{dx}{dt} = -x\,(x^2 - 2bx - a)$$

Equilibrium solutions are

$$x\,(x^2 - 2bx - a) = 0 \quad \Rightarrow \quad x_0 = 0,\; b\left(1 \pm \sqrt{1 + a/b^2}\right)$$



Stability is then determined by linearization: $x = x_0 + \tilde{x}$

$$\tilde{x}' = -(x_0^2 - 2bx_0 - a)\tilde{x} \; - x_0\left(2x_0\tilde{x} - 2b\tilde{x}\right)$$

$x_0 = 0$: $\quad \tilde{x}' = a\tilde{x} \qquad \tilde{x} = \tilde{x}(0)\,e^{at} \quad \underset{t\to\infty}{\Rightarrow} \begin{cases} \tilde{x} \to \infty & a > 0 \quad \text{unstable} \\ \tilde{x} \to 0 & a < 0 \quad \text{stable} \end{cases}$

$x_0 = b(1+\sqrt{1+a/b^2})$: $\quad \tilde{x}' = -2\,(x_0^2 - bx_0)\tilde{x} = \underbrace{-2x_0(x_0 - b)}_{>0}\tilde{x} \;\Rightarrow\; \tilde{x} = \tilde{x}(0)\,e^{-2x_0(x_0-b)t} \underset{t\to\infty}{\Rightarrow} \{\, 0 \text{ stabl}$

$x_0 = b(1-\sqrt{1+a/b^2})$: $\quad \tilde{x}' = -2x_0(x_0 - b)\tilde{x} = \begin{cases} \tilde{x} = \tilde{x}(0)\,e^{|2x_0(x_0-b)|t} & -b^2 < a < 0 \\ \tilde{x} = \tilde{x}(0)\,e^{-|2x_0(x_0-b)|t} & a > 0 \end{cases} \underset{t\to\infty}{\Rightarrow} \begin{cases} \infty & \text{unstabl} \\ 0 & \text{stable} \end{cases}$

But what kind of bifurcations are these?

let $y = x - x_0$ $\quad\quad x_0 = $ equilibrium

Then

$$x' = f(x)$$

gives

$$y' = f(y + x_0)$$

$$= f(x_0) + y\, f'(x_0) + \frac{y^2}{2!} f''(x_0) + \frac{y^3}{3!} f'''(x_0) + \dots$$

by a Taylor series expansion. For our case we have

$$x' = -x(x^2 - 2bx - a) = f(x)$$

so then

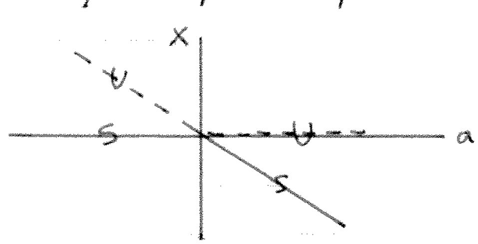
$$f(x) = -x^3 + 2bx^2 + ax$$
$$f'(x) = -3x^2 + 4bx + a$$
$$f''(x) = -6x + 4b$$
$$f'''(x) = -6$$
$$f''''(x) = 0$$

near $x_0 = 0$: $\quad y' = 0 + y \cdot a + \frac{y^2}{2} \cdot 4b + \frac{y^3}{6} \cdot -6 + 0$

$$y' = ay + 2by^2 - y^3$$

now for $y \ll 1$ then

$$y' = ay + 2by^2 \quad \Rightarrow \text{transcritical bifurcation}$$



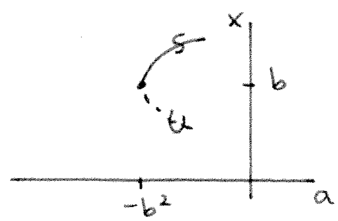
near $x_0 = b$, $a = -b^2$

$$y' = 0 + y \cdot 0 + \frac{y^2}{2} \cdot -2b + \frac{y^3}{6} \cdot -6 + 0$$

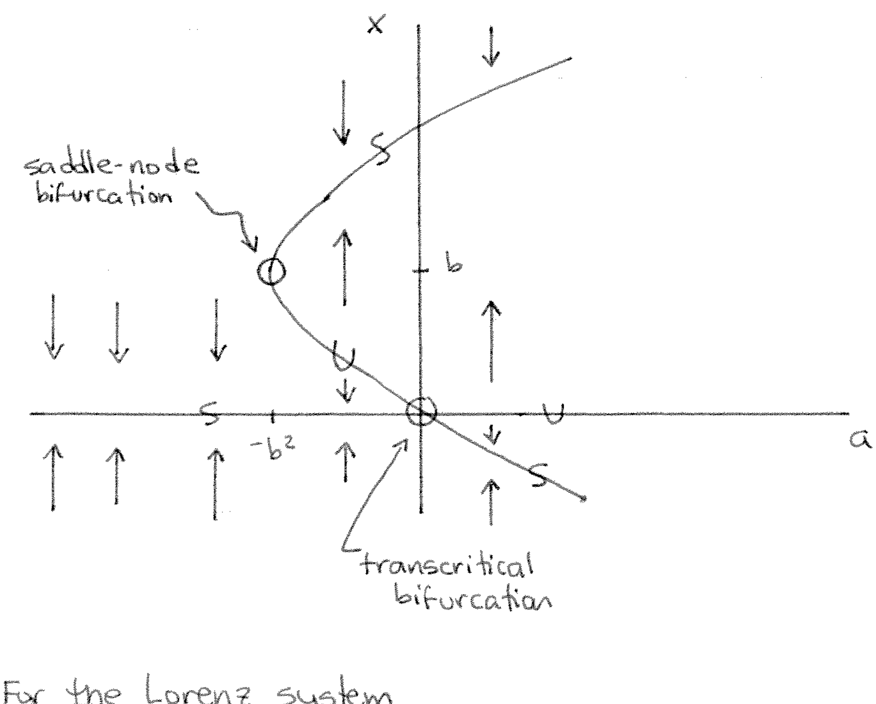
$$y' = -by^2 - y^3$$

and for $y \ll 1$ then

$$y' = -by^2 \Rightarrow \text{saddle node bifurcation}$$



So then the normal forms for the two bifurcation points reduce to the standard forms of the transcritical and saddle node bifurcations.



Note: For the Lorenz system

$$\vec{x}' = \begin{pmatrix} \sigma(y-x) \\ rx - y - zx \\ -bz + xy \end{pmatrix} \Rightarrow \begin{array}{l} \vec{x} = \epsilon \vec{x}_1 + \epsilon^2 \vec{x}_2 + \dots \\ \tau = \epsilon^2 t \\ \epsilon = \sqrt{r-1} \\ A = (1\ 1\ 0)\,\vec{x}_{2c} \end{array} \Rightarrow A_\tau = \frac{\sigma}{1+\sigma}\left(A - A^3/b\right)$$

pitchfork - normal form

Imperfect Bifurcations

What about the structural stability of the canonical bifurcations? That is, if we perturb the equations, does the bifurcation structure survive.

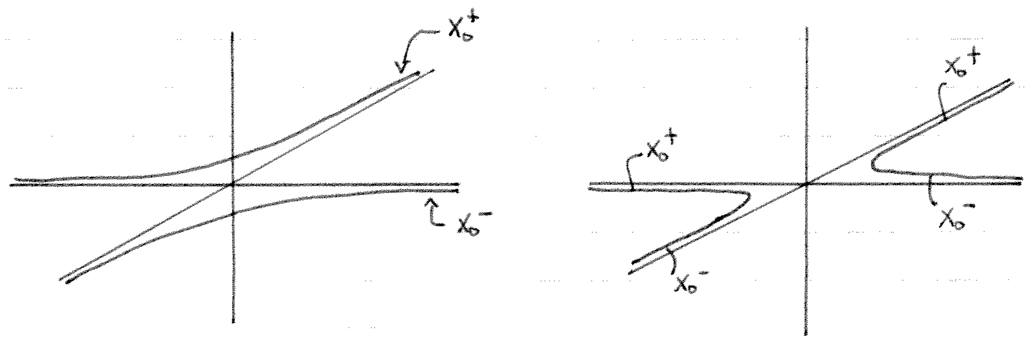
Therefore consider

$$x' = ax - x^2 + \delta$$

where δ is a perturbation to the transcritical problem. The dynamics are then determined as follows

Equilibrium: $\quad ax - x^2 + \delta = 0$

$$x_0 = \frac{a}{2}\left(1 \pm \sqrt{1 + 4\delta/a^2}\right) \qquad (a \neq 0)$$



$\delta > 0$ $\qquad$ $\delta < 0$

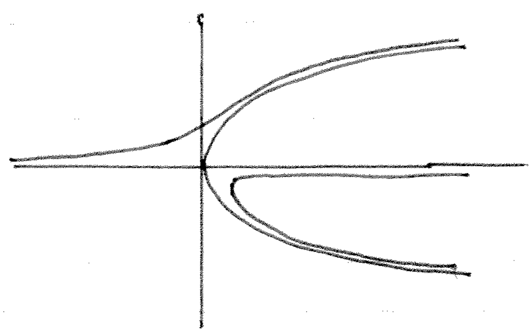
- The bifurcation is destroyed for any $\delta \neq 0$
- The transcritical bifurcation is structurally unstable
- Stability is determined from $x = x_0 + \tilde{x}$

This destruction of the bifurcation structure is also apparent in the pitchfork bifurcation

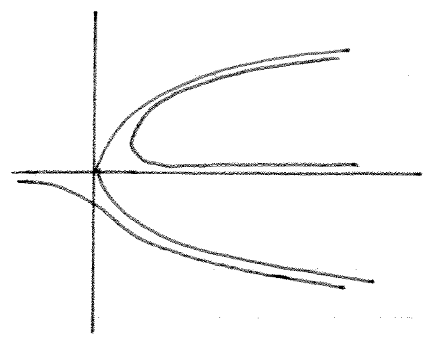
$$x' = ax - x^3 + \delta$$

where δ is some perturbation. Thus we have that

Equilibrium: $ax - x^3 + \delta = 0$

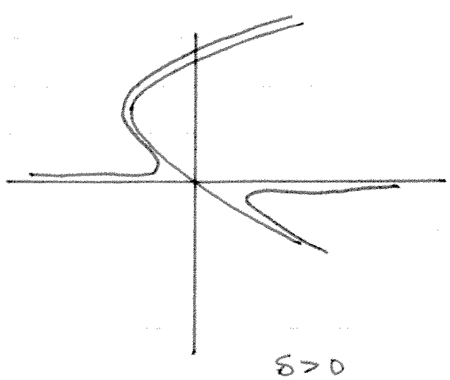


$\delta > 0$

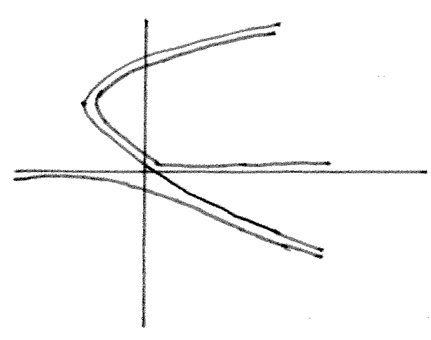


$\delta < 0$

- Pitchfork is destroyed – no bifurcation



$\delta > 0$



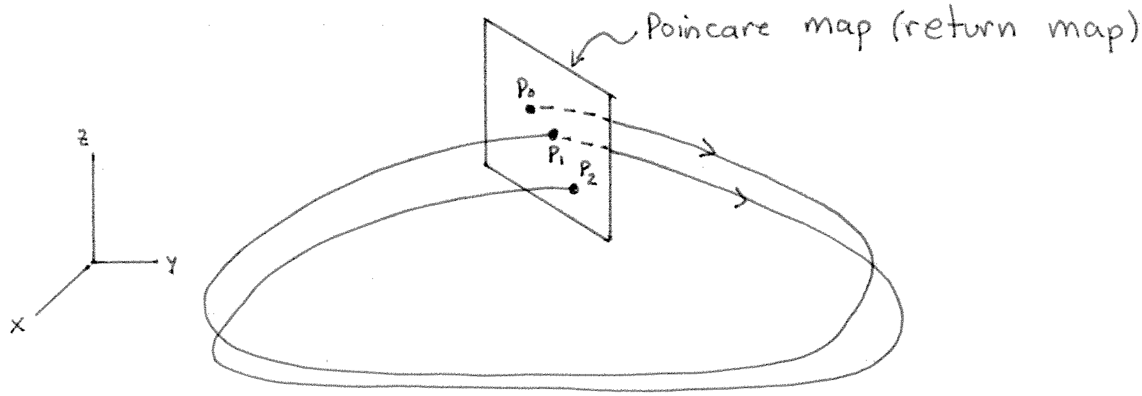
$\delta < 0$

Lecture 26

Stability of Periodic Solutions: Floquet Theory

Up to this point, we have been considering the stability of steady-state solutions. We always do this with linear stability theory. What about periodic solutions? How do we determine stability then?

Graphically, we can think of some solution in an n-dimensional space which is nearly periodic



So for a nearly periodic solution, what we care about is where it crosses the Poincare map. In particular, do we have the following

$$p_1 = \phi(p_0), \quad p_2 = \phi(p_1) \quad \Rightarrow \quad p_{n+1} = \phi(p_n)$$

1. $p_{n+1} = \phi(p_n) \overset{\text{if}}{\Rightarrow} p_n = p_{n+1}$ stable periodic orbit

2. $p_{n+1} = \phi(p_n) \overset{\text{if}}{\Rightarrow} p_n \neq p_{n+1}$, $|p_n - p_{n+1}| > |p_{n-1} - p_n|$ unstable solution

How do we analytically calculate stability criteria?

Floquet Theory

Consider the general second order equation

$$x'' + p_1(t)x' + p_2(t)x = 0 \quad \Rightarrow \quad x = c_1x_1(t) + c_2x_2(t)$$

with periodic coefficients (x_1, x_2 - fundamental solutions)

$$p_1(t) = p_1(t+T)$$
$$p_2(t) = p_2(t+T)$$

We note that for $p_1(t) = 0$, this is called Hill's Equation. Periodicity then implies

$$x''(t+T) + p_1(t+T)x'(t+T) + p_2(t+T)x(t+T) = 0$$

since $p_1(t+T) = p_1(t)$ and $p_2(t+T) = p_2(t)$ we have

$$x''(t+T) + p_1(t)x'(t+T) + p_2(t)x(t+T) = 0$$

Thus $x_1(t+T)$ and $x_2(t+T)$ must also be fundamental solutions. We can always write this new set in terms of the old set. Thus

$$x_1(t+T) = a_{11}x_1(t) + a_{12}x_2(t)$$
$$x_2(t+T) = a_{21}x_1(t) + a_{22}x_2(t)$$

or in matrix form

$$\vec{x}(t+T) = A\vec{x}(t) \qquad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

We can simplify this with the following transformation

$$\vec{x} = B\vec{v}$$

then

$$B\vec{v}(t+T) = AB\vec{v}(t)$$

$$\vec{v}(t+T) = B^{-1}AB\,\vec{v}(t)$$

Now if B is made up of the eigenvectors of A, then $B^{-1}AB$ is a diagonal matrix

$$\vec{v}(t+T) = \Lambda\,\vec{v}(t)$$

where

$$\Lambda = B^{-1}AB = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

and λ_1, λ_2 are the eigenvalues of A. Thus we have the Floquet solutions

$$V_i(t+T) = \lambda_i V_i(t) \qquad i = 1,2$$

which upon applying over n-periods we find

$$V_i(t+nT) = \lambda_i^n V_i(t) \qquad \left(|V_i(t+nT)|^2 = |\lambda_i^n|^2\,|V_i(t)|^2\right)$$

Thus the λ_i^n determine stability since

$$\lim_{\substack{n\to\infty \\ (t\to\infty)}} V_i(t) \Rightarrow \begin{cases} 0 & |\lambda_i| < 1 \\ \infty & |\lambda_i| > 1 \end{cases}$$

The λ_i are called Floquet multipliers. Once you determine them, you can calculate stability. But how do we calculate this stability in practice. Consider once again

$$v_i(t+T) = \lambda_i v_i(t)$$

and multiply by $e^{-\nu_i(t+T)}$. Then we have

$$v_i(t+T)e^{-\nu_i(t+T)} = \lambda_i v_i(t) e^{-\nu_i(t+T)}$$

$$v_i(t+T)e^{-\nu_i(t+T)} = \lambda_i e^{-\nu_i T} \; v_i(t) e^{-\nu_i(t)}$$

Now choose

$$\lambda_i e^{-\nu_i T} = 1 \quad \Rightarrow \quad e^{-\nu_i T} = 1/\lambda_i$$

Then if $\phi = v_i(t) e^{-\nu_i(t)}$ we have

$$\phi(t+T) = \phi(t)$$

and $\phi(t)$ is periodic. Further

$$\nu_i = \frac{1}{T} \ln \lambda_i \qquad \text{(characteristic exponent)}$$

which is unique to within a multiple of $2in\pi/T$. Our periodic solutions are then

$$v_i(t) = e^{\nu_i t} \phi_i(t)$$

where $\phi_i(t+T) = \phi_i(t)$.

How then do we determine these characteristic exponents in practice. Recall that

$$x_1(t+T) = a_{11}x_1(t) + a_{12}x_2(t)$$
$$x_2(t+T) = a_{21}x_1(t) + a_{22}x_2(t)$$

Now choose

$$x_1(0) = 1 \qquad x_1'(0) = 0$$
$$x_2(0) = 0 \qquad x_2'(0) = 1$$

Then at $t=0$

$$x_1(T) = a_{11}x_1(0) + a_{12}x_2(0) \quad \Rightarrow \quad x_1(T) = a_{11}$$
$$x_2(T) = a_{21}x_1(0) + a_{22}x_2(0) \quad \quad x_2(T) = a_{21}$$

and differentiating gives

$$x_1'(T) = a_{11}x_1'(0) + a_{12}x_2'(0) \quad \Rightarrow \quad x_1'(T) = a_{12}$$
$$x_2'(T) = a_{21}x_1'(0) + a_{22}x_2'(0) \quad \quad x_2'(T) = a_{22}$$

Thus

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} x_1(T) & x_1'(T) \\ x_2(T) & x_2'(T) \end{pmatrix}$$

The eigenvalues are calculated to be

$$(x_1(T)-\lambda)(x_2'(T)-\lambda) - x_1'(T)x_2(T) = 0$$

Thus

$$\lambda^2 - 2\left(\frac{1}{2}\left(x_1(T) + x_2'(T)\right)\right)\lambda + \left[x_1(T)x_2'(T) - x_1'(T)x_2(T)\right] = 0$$

where $\Delta(T) = x_1(T)x_2'(T) - x_1'(T)x_2(T)$ is the Wronskian.

But note, since we know the Wronskian is constant, then

$$\Delta(T) = \Delta(0) = x_1(0)\,x_2'(0) - x_1'(0)\,x_2(0) = 1\cdot 1 - 0\cdot 0 = 1$$

So then

$$\lambda^2 - 2\left[\tfrac{1}{2}\left(x_1(T) + x_2'(T)\right)\right]\lambda + 1 = 0$$

$$\lambda^2 - 2\alpha\lambda + 1 = 0$$

where $\alpha = \frac{1}{2}\left(x_1(T) + x_2'(T)\right)$. The eigenvalues are then calculated to be

$$\lambda_{1,2} = \alpha \pm \sqrt{\alpha^2 - 1}$$

with $\lambda_1 \cdot \lambda_2 = 1$. Thus we have the following conclusions

- $|\alpha| > 1 \rightarrow$ one bounded, one unbounded root $\Rightarrow$ unstable
- $|\alpha| < 1 \rightarrow$ complex conjugate pair with unit modulus since $\lambda_1\lambda_2 = 1 \Rightarrow$ stable

Thus since we require $|\alpha| < 1$, this gives

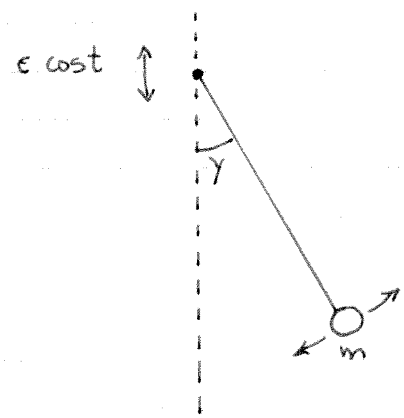
$$|\Gamma| = |x_1(T) + x_2'(T)| < 2$$

where Γ = Floquet discriminant. Thus if we can find the Floquet discrimant to be less than 2 in absolute value, the periodic solution is stable.

Lecture 27

Floquet Theory and the Pendulum

As an application of Floquet theory, we consider the periodically forced pendulum. In particular, suppose we have a pendulum with a periodically oscillating support



The equations of motion ($\Sigma F = ma$) yields the governing equation

$$y'' + (\delta + \epsilon \cos \omega t) \sin y = 0$$

where

δ – measures natural oscillation frequency
ϵ – amplitude of support fluctuations
ω – frequeny of support fluctuations
y – angle away from rest

Note that this is simply Hill's equation if we take $\sin y \sim y$ and $p_1(t) = 0$, $p_2(t) = \delta + \epsilon \cos \omega t$.

In order to get somewhere analytically, we must simplify the sin y term. We consider the two following cases

1. $y \approx 0$: $\sin y \sim y - \frac{y^3}{3!} + \cdots$ (down)

2. $y \approx \pi$: $\sin(y+\pi) = -\sin y \sim -y + \frac{y^3}{3!} + \cdots$ (up)

Thus we have

$$y'' \pm (\delta + \epsilon \cos\omega t)\, y = 0 \qquad \text{Mathieu's Equation}$$

Down Pendulum: $y'' + (\delta + \epsilon \cos\omega t)\, y = 0$

In this case, much can be said concerning the dynamical behavior. In particular, there are well know instabilities which arise if the forcing frequency ($\cos\omega t$) is resonant with the linear oscillations.

A standard multiple-scale expansion with $\epsilon \ll 1$ is easily carried out to determine the growth and frequeng shift due to nonlinearity.

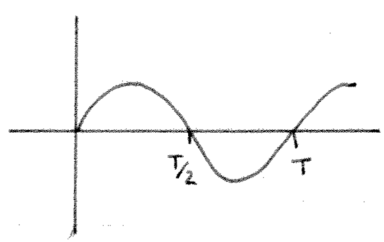
Upward (inverted) Pendulum: $y'' - (\delta + \epsilon \cos \omega t) y = 0$

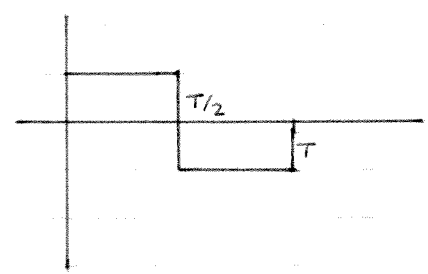
This is the case of interest for this lecture. In particular, we will determine if it is possible to stabilize the pendulum in the upright position, i.e., do periodic solutions become stable in the upright position?

We use Floquet Theory to determine this. First we shift our time by $\pi/2$ so that

$$y'' - (\delta + \epsilon \sin \omega t) y = 0$$

Then we approximate $\sin \omega t$ by a periodic step function





$\omega T = 2\pi \qquad T = 2\pi/\omega$

Thus we have

$$y'' - (\delta + \epsilon) y = 0 \qquad 0 < t < T/2$$

$$y'' - (\delta - \epsilon) y = 0 \qquad T/2 < t < T$$

This then allows for an explicit calculation of the Floquet Discriminant.

Recall that the Floquet Discriminant requires us to calculate

$$\Gamma = x_1(T) + x_2'(T)$$

where $x_1(0)=1,\ x_1'(0)=0$ and $x_2(0)=0,\ x_2'(0)=1$. So then the solution piecewise is

$$0<t<T/2: \qquad y = c_1 e^{\sqrt{\delta+\epsilon}\,t} + c_2 e^{-\sqrt{\delta+\epsilon}\,t}$$

$$T/2<t<T: \qquad y = c_3 e^{\sqrt{\delta-\epsilon}\,t} + c_4 e^{-\sqrt{\delta-\epsilon}\,t}$$

<u>Solution x_1:</u> $x_1(0)=1,\ x_1'(0)=0 \Rightarrow$ $0<t<T/2$: $x_1 = c_1 e^{\sqrt{\delta+\epsilon}\,t} + c_2 e^{-\sqrt{\delta+\epsilon}\,t}$; $T/2<t<T$: $x_1 = c_3 e^{\sqrt{\delta-\epsilon}\,t} + c_4 e^{-\sqrt{\delta-\epsilon}\,t}$

a) $$x_1(0) = c_1 + c_2 = 1$$
$$x_1'(0) = \sqrt{\delta+\epsilon}\,c_1 - \sqrt{\delta+\epsilon}\,c_2 = 0$$

from the second: $c_1 = c_2$

Thus if $c_1 + c_2 = 1 \Rightarrow c_1 = c_2 = 1/2$

$$0<t<T/2: \quad x_1(t) = \cosh\sqrt{\delta+\epsilon}\,t$$

b) $$x_1(T/2) = c_3 e^{\sqrt{\delta-\epsilon}\,T/2} + c_4 e^{-\sqrt{\delta-\epsilon}\,T/2} = \cosh\sqrt{\delta+\epsilon}\,T/2$$
$$x_1'(T/2) = \sqrt{\delta-\epsilon}\,c_3 e^{\sqrt{\delta-\epsilon}\,T/2} - \sqrt{\delta-\epsilon}\,c_4 e^{-\sqrt{\delta-\epsilon}\,T/2} = \sqrt{\delta+\epsilon}\sinh\sqrt{\delta+\epsilon}\,T/2$$

$$c_3 = \tfrac{1}{2} e^{-\sqrt{\delta-\epsilon}\cdot T/2}\left(\cosh\sqrt{\delta+\epsilon}\,T/2 + \sqrt{\delta+\epsilon}/\sqrt{\delta-\epsilon}\,\sinh\sqrt{\delta+\epsilon}\,T/2\right)$$

$$c_4 = \tfrac{1}{2} e^{\sqrt{\delta-\epsilon}\,T/2}\left(\cosh\sqrt{\delta+\epsilon}\,T/2 - \sqrt{\delta+\epsilon}/\sqrt{\delta-\epsilon}\,\sinh\sqrt{\delta+\epsilon}\,T/2\right)$$

Then

$$\frac{T}{2}<t<T: \quad x_1(t) = \cosh\sqrt{\delta+\epsilon}\,\frac{T}{2}\cosh\sqrt{\delta-\epsilon}\,(t-T/2) + \frac{\sqrt{\delta+\epsilon}}{\sqrt{\delta-\epsilon}}\sinh\sqrt{\delta+\epsilon}\,\frac{T}{2}\sinh\sqrt{\delta-\epsilon}\,(t-T/2)$$

$$x_1(T) = \cosh\sqrt{\delta+\epsilon}\,\frac{T}{2}\cosh\sqrt{\delta-\epsilon}\,\frac{T}{2} + \frac{\sqrt{\delta+\epsilon}}{\sqrt{\delta-\epsilon}}\sinh\sqrt{\delta+\epsilon}\,\frac{T}{2}\sinh\sqrt{\delta-\epsilon}\,\frac{T}{2}$$

Solution x_2: $\quad x_2(0)=0,\ x_2'(0)=1 \Rightarrow 0<t<\frac{T}{2} \quad x_2 = C_1 e^{\sqrt{\delta+\epsilon}\,t} + C_2 e^{-\sqrt{\delta+\epsilon}\,t}$

$\frac{T}{2}<t<T \quad x_2 = C_3 e^{+\sqrt{\delta-\epsilon}\,t} + C_4 e^{-\sqrt{\delta-\epsilon}\,t}$

a) $x_2(0) = C_1 + C_2 = 0$

$x_2'(0) = \sqrt{\delta+\epsilon}\,C_1 - \sqrt{\delta+\epsilon}\,C_2 = 1$

Thus $C_1 = -C_2$ and $C_1 - C_2 = 2C_1 = 1/\sqrt{\delta+\epsilon} \quad C_1 = \frac{1}{2\sqrt{\delta+\epsilon}} = -C_2$

$$0<t<T/2: \quad x_2(t) = \frac{1}{2\sqrt{\delta+\epsilon}}\left(e^{\sqrt{\delta+\epsilon}\,t} - e^{-\sqrt{\delta+\epsilon}\,t}\right) = \frac{\sinh\sqrt{\delta+\epsilon}\,t}{\sqrt{\delta+\epsilon}}$$

b) $x_2(T/2) = C_3 e^{\sqrt{\delta-\epsilon}\,T/2} + C_4 e^{-\sqrt{\delta-\epsilon}\,T/2} = \frac{1}{\sqrt{\delta+\epsilon}}\sinh\sqrt{\delta+\epsilon}\,T/2$

$x_2'(T/2) = \sqrt{\delta-\epsilon}\,C_3 e^{\sqrt{\delta-\epsilon}\,T/2} - \sqrt{\delta-\epsilon}\,C_4 e^{-\sqrt{\delta-\epsilon}\,T/2} = \cosh T/2\,\sqrt{\delta+\epsilon}$

$$C_3 = \frac{1}{2}\frac{e^{-\sqrt{\delta-\epsilon}\,T/2}}{\sqrt{\delta-\epsilon}}\left(\cosh\sqrt{\delta+\epsilon}\,T/2 + \sqrt{\delta-\epsilon}/\sqrt{\delta+\epsilon}\,\sinh\sqrt{\delta+\epsilon}\,T/2\right)$$

$$C_4 = \frac{-1}{2}\frac{e^{\sqrt{\delta-\epsilon}\,T/2}}{\sqrt{\delta-\epsilon}}\left(\cosh\sqrt{\delta+\epsilon}\,T/2 - \sqrt{\delta-\epsilon}/\sqrt{\delta+\epsilon}\,\sinh\sqrt{\delta+\epsilon}\,T/2\right)$$

Then

$$x_2(t) = \frac{\cosh\sqrt{\delta+\epsilon}\,\frac{T}{2}\,\sinh\sqrt{\delta-\epsilon}\,(t-\frac{T}{2})}{\sqrt{\delta-\epsilon}} + \frac{\sinh\sqrt{\delta+\epsilon}\,T/2\,\cosh\sqrt{\delta-\epsilon}\,(t-T/2)}{\sqrt{\delta+\epsilon}}$$

$$x_2'(t) = \cosh\sqrt{\delta+\epsilon}\,\frac{T}{2}\,\cosh\sqrt{\delta-\epsilon}\,(t-T/2) + \frac{\sqrt{\delta-\epsilon}}{\sqrt{\delta+\epsilon}}\,\sinh\sqrt{\delta+\epsilon}\,\frac{T}{2}\,\sinh\sqrt{\delta-\epsilon}\,(t-T/2)$$

and

$$x_2'(T) = \cosh\sqrt{\delta+\epsilon}\,\frac{T}{2}\,\cosh\sqrt{\delta-\epsilon}\,\frac{T}{2} + \frac{\sqrt{\delta-\epsilon}}{\sqrt{\delta+\epsilon}}\,\sinh\sqrt{\delta+\epsilon}\,\frac{T}{2}\,\sinh\sqrt{\delta-\epsilon}\,\frac{T}{2}$$

Then

$$\Gamma = x_1(T) + x_2'(T)$$

$$= 2\cosh\sqrt{\delta+\epsilon}\,\tfrac{T}{2}\,\cosh\sqrt{\delta-\epsilon}\,\tfrac{T}{2} + \left(\frac{\sqrt{\delta+\epsilon}}{\sqrt{\delta-\epsilon}} + \frac{\sqrt{\delta-\epsilon}}{\sqrt{\delta+\epsilon}}\right)\sinh\sqrt{\delta+\epsilon}\,\tfrac{T}{2}\,\sinh\sqrt{\delta-\epsilon}\,\tfrac{T}{2}$$

The key now is to rewrite this in terms of our frequency forcing ω. Thus we have

$$T = \frac{2\pi}{\omega}$$

and

$$\Gamma = 2\cosh\sqrt{\delta+\epsilon}\cdot\frac{\pi}{\omega}\,\cosh\sqrt{\delta-\epsilon}\,\frac{\pi}{\omega} + \left(\frac{\sqrt{\delta+\epsilon}}{\sqrt{\delta-\epsilon}} + \frac{\sqrt{\delta-\epsilon}}{\sqrt{\delta+\epsilon}}\right)\sinh\sqrt{\delta+\epsilon}\,\frac{\pi}{\omega}\,\sinh\sqrt{\delta-\epsilon}\,\frac{\pi}{\omega}$$

Note:

1. if $\delta > \epsilon$, then as $\omega \to \infty \Rightarrow \Gamma \to 2$

   Neutral stability

2. if $\delta < \epsilon$

$$\Gamma = 2\cosh\sqrt{\delta+\epsilon}\,\frac{\pi}{\omega}\,\cos\sqrt{\epsilon-\delta}\,\frac{\pi}{\omega} \;\overset{-}{+}\left(\frac{\sqrt{\delta+\epsilon}}{\sqrt{\delta-\epsilon}} + \frac{\sqrt{\delta-\epsilon}}{\sqrt{\delta+\epsilon}}\right)\sinh\sqrt{\delta+\epsilon}\,\frac{\pi}{\omega}\,\sin\sqrt{\epsilon-\delta}\,\frac{\pi}{\omega}$$

in this case $|\Gamma| < 2$ for "windows" in ω. Thus The pendulum can be stabilized in the inverted position.

$\omega \to \infty \qquad \Gamma \to 2\cos\sqrt{\epsilon-\delta}\,\pi/\omega$

$|\Gamma| \le 2$ for all ω sufficiently large!