AMATH 568
Winter Quarter 2023
Professor J. Nathan Kutz

HOMEWORK #2
Due: January 18, 2023

1. Consider the nonhomogeneous problems of Problem 1 and 2: $\vec{x}' = \mathbf{A}\vec{x} + \vec{g}(t)$.

(a) Let $\vec{x} = \mathbf{M}\vec{y}$ where the columns of $\mathbf{M}$ are the eigenvectors of the above problems.
(b) Write the equations in terms of $\vec{y}$ and multiply through by $\mathbf{M}^{-1}$.
(c) Show the resulting equation is

$$\vec{y}' = \mathbf{D}\vec{y} + \vec{h}(t)$$

where $\mathbf{D} = \mathbf{M}^{-1}\mathbf{A}\mathbf{M}$ is a diagonal matrix whose diagonal elements are the eigenvalues of the problem considered and $\vec{h}(t) = \mathbf{M}^{-1}\vec{g}(t)$.
(d) Show that this system is now *decoupled* so that each component of $\vec{y}$ can be solved independently of the other components.

2. Given $L = -d^2/dx^2$ find the eigenfunction expansion solution of

$$\frac{d^2y}{dx^2} + 2y = -10\exp(x) \qquad\qquad y(0) = 0, y'(1) = 0$$

3. Given $L = -d^2/dx^2$ find the eigenfunction expansion solution of

$$\frac{d^2y}{dx^2} + 2y = -x \qquad\qquad y(0) = 0, y(1) + y'(1) = 0$$

4. Consider the Sturm-Liouville eigenvalue problem:

$$Lu = -\frac{d}{dx}\left[p(x)\frac{du}{dx}\right] + q(x)u = \lambda\rho(x)u \qquad\qquad 0 < x < L$$

with the boundary conditions

$$\alpha_1 u(0) - \beta_1 u'(0) = 0$$
$$\alpha_2 u(L) - \beta_2 u'(L) = 0$$

and with $p(x) > 0$, $\rho(x) > 0$, and $q(x) \geq 0$ and with $p(x)$, $\rho(x)$, $q(x)$ and $p'(x)$ continuous over $0 < x < L$. With the inner product $(\phi, \psi) = \int_0^L \rho(x)\phi(x)\psi^*(x)dx$, show the following:

(a) L is a self-adjoint operator.
(b) Eigenfunctions corresponding to different eigenvalues are orthogonal, i.e. $(u_n, u_m) = 0$.
(c) Eigenvalues are real, non–negative and eigenfunctions may be chosen to be real valued.
(d) Each eigenvalue is simple, i.e. it only has one eigenfunction. (Hint: recall that for each eigenvalue, there can be at most two linearly independent solutions – calculate the Wronskian of these two solutions and see what it implies.)