1. Consider the nonhomogeneous problems of Problem 1 and 2: \( \vec{x}' = A\vec{x} + \vec{g}(t) \).

(a) Let \( \vec{x} = M\vec{y} \) where the columns of \( M \) are the eigenvectors of the above problems.
(b) Write the equations in terms of \( \vec{y} \) and multiply through by \( M^{-1} \).
(c) Show the resulting equation is 
\[ \vec{y}' = D\vec{y} + \vec{h}(t) \]
where \( D = M^{-1}AM \) is a diagonal matrix whose diagonal elements are the eigenvalues of the problem considered and \( \vec{h}(t) = M^{-1}\vec{g}(t) \).
(d) Show that this system is now decoupled so that each component of \( \vec{y} \) can be solved independently of the other components.

2. Given \( L = -d^2/dx^2 \) find the eigenfunction expansion solution of
\[ \frac{d^2y}{dx^2} + 2y = -10 \exp(x) \quad y(0) = 0, y'(1) = 0 \]

3. Given \( L = -d^2/dx^2 \) find the eigenfunction expansion solution of
\[ \frac{d^2y}{dx^2} + 2y = -x \quad y(0) = 0, y(1) + y'(1) = 0 \]

4. Consider the Sturm-Liouville eigenvalue problem:
\[ Lu = -\frac{d}{dx} \left[ p(x) \frac{du}{dx} \right] + q(x)u = \lambda \rho(x)u \quad 0 < x < L \]
with the boundary conditions
\[ \alpha_1 u(0) - \beta_1 u'(0) = 0 \]
\[ \alpha_2 u(L) - \beta_2 u'(L) = 0 \]
and with \( p(x) > 0, \rho(x) > 0, \) and \( q(x) \geq 0 \) and with \( p(x), \rho(x), q(x) \) and \( p'(x) \) continuous over \( 0 < x < L \). With the inner product \( (\phi, \psi) = \int_0^L \rho(x)\phi(x)\psi(x)dx \), show the following:
(a) \( L \) is a self-adjoint operator.
(b) Eigenfunctions corresponding to different eigenvalues are orthogonal, i.e. \( (u_n, u_m) = 0 \).
(c) Eigenvalues are real, non–negative and eigenfunctions may be chosen to be real valued.
(d) Each eigenvalue is simple, i.e. it only has one eigenfunction. (Hint: recall that for each eigenvalue, there can be at most two linearly independent solutions – calculate the Wronskian of these two solutions and see what it implies.)