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Every complete axiomatizable theory is decidable.

1) Suppose that *T* is a complete theory.
2) Suppose that *S* is a set of axioms of *T*.

To show: that *T* is decidable.

3) Suppose that *T* is inconsistent.
4) Then every sentence of the language of *T* is a theorem of *T*.
5) But the set of sentences of the language of *T* is decidable.
6) Hence, *T* is decidable.
7) Suppose, then, that *T* is consistent and that *A* is a sentence of the language of *T*.

To show: that there is a step-by-step procedure that determines in a finite number of steps whether *A* is or is not a theorem of *T*.

8) Either *A* or –*A* is a theorem of *T* (since *T* is complete).
9) Hence, either *A* or –*A* is a consequence of *S*.
10) Furthermore, by the compactness theorem, either *A* or –*A* is a consequence of some finite subset of *S*, say $\{S_1 \ldots S_n\}$.
11) So either $(S_1 \,\&\, \ldots\, S_n) \rightarrow A$ or $(S_1 \,\&\, \ldots\, S_n) \rightarrow -A$ is a valid sentence of first-order logic.
12) But the set of valid sentences is effectively enumerable.
13) To determine whether *A* is a theorem of *T* consider each sentence of the enumeration in turn:
    a) Is it a conditional? If not, proceed to the next sentence on the list.
    b) If so, is its antecedent or (in case its antecedent is a conjunction) each conjunct of its antecedent a member of *S*? (This can be determined in a finite number of steps since *S* is decidable.) If not, proceed to the next sentence.
    c) If so, is its consequent either *A* or –*A*? If not, proceed to the next sentence.
    d) If the consequent is *A*, then *A* is a theorem of *T*.
    e) If the consequent is –*A*, then (since *T* is consistent) *A* is not a theorem of *T*.
14) Thus, there is a step-by-step procedure that determines in a finite number of steps whether *A* is or is not a theorem of *T*.
15) Hence, *T* is decidable.